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Monotone and Generalized Monotone Bifunctions and their Application to Operator Theory

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Περίληψη

Το θέμα της παρούσας διατριβής είναι η μελέτη των μονότονων και γενικευμένα μονότονων "δισυναρτήσεων" (bifunctions), και η χρησιμοποίηση αυτής της μελέτης στη θεωρία των μονότονων και γενικευμένα μονότονων τελεστών σε τοπικά κυρτούς χώρους (συνήθως, χώρους Banach).

Δοθέντος ενός τοπικά κυρτού χώρου X, με τον όρο "τελεστής" εννούμε μια πλειότιμη απεικόνιση από το X στον τοπολογικό δυικό του X^*, δηλαδή μια απεικόνιση $T : X \to 2^{X^*}$, όπου 2^{X^*} είναι το δυναμοσύνολο του X^*. Για κάθε $x \in X$, το $T(x)$ είναι ένα (πιθανώς κενό) υποσύνολο του X^*. Ένας τελεστής T λέγεται μονότονος, αν για κάθε $x, y \in X$ και κάθε $x^* \in T(x)$, $y^* \in T(y)$ ισχύει

$$\langle x^* - y^*, x - y \rangle \geq 0$$

όπου με $\langle x^*, x \rangle$ συμβολίζεται το $x^*(x)$. Η έννοια του μονότονου τελεστή έχει βρει εφαρμογές σε πολλούς κλάδους των μαθηματικών, όπως στη μη γραμμική ανάλυση, στις μερικές διαφορικές εξισώσεις, στη θεωρία διαφορισιμότητας κυρτών συναρτήσεων κλπ. Ειδικώτερα, οι μονότονοι τελεστές έχουν αποδειχθεί ισχυρό εργαλείο στη θεωρία ανισώσεων μεταβολών (variational inequalities), οι οποίες αποτελούν τη βάση πολλών μοντέλων σε φυσικά προβλήματα. Ένας από τους λόγους είναι ότι η κλάση των μονότονων τελεστών περιλαμβάνει τα υποδιαφορικά και τους θετικούς γραμμικούς τελεστές, που συχνά απαντώνται σε τέτοια προβλήματα.

Μια άλλη σημαντική έννοια είναι οι μονότονες δισυναρτήσεις (monotone bifunctions). Αν $C \subseteq X$, μια απεικόνιση $F : C \times C \to \mathbb{R}$ λέγεται μονότονη δισυνάρτηση αν για κάθε $x, y \in C$,

$$F(x, y) + F(y, x) \leq 0.$$

Οι μονότονες δισυναρτήσεις σχετίζονται με το λεγόμενο *πρόβλημα ισορροπίας*, που συνίσταται στην εύρεση ενός σημείου $x_0 \in C$ τέτοιου ώστε

$$\forall y \in C : F(x_0, y) \geq 0.$$

Τα προβλήματα ισορροπίας είχαν μελετηθεί στο παρελθόν σε σχέση με τα θεωρήματα minimax, αλλά ο όρος "πρόβλημα ισορροπίας" χρησιμοποιήθηκε για πρώτη φορά στο σημαντικό άρθρο των Bloom και Oettli [23]. Στο άρθρο αυτό οι συγγραφείς έδειξαν ότι πολλά διαφορετικά μεταξύ τους προβλήματα (ανισώσεις μεταβολών,

μαθηματική βελτιστοποίηση, προβλήματα σταθερού σημείου, προβλήματα "σαγματικού σημείου" (saddle point problems), ισορροπία κατά Nash κλπ) ήταν ειδικές περιπτώσεις του προβλήματος ισορροπίας. Για το λόγο αυτό, πολλοί ερευνητές ασχολήθηκαν με προβλήματα ισορροπίας με μονότονες δισυναρτήσεις (βλέπε [7, 8, 22, 21, 64, 54, 71, 69, 75, 77, 78, 86] και τις αναφορές που περιέχονται σ΄ αυτά).

Στην παρούσα διατριβή θα ασχοληθούμε με τις μονότονες δισυναρτήσεις από άλλη άποψη. Θα επικεντρωθούμε στο συσχετισμό των μονότονων δισυναρτήσεων με τους μονότονους τελεστές. Σε κάθε δισυνάρτηση F θα αντιστοιχήσουμε ένα τελεστή A^F, και σε κάθε τελεστή T θα αντιστοιχήσουμε μια δισυνάρτηση G_T. Μια δισυνάρτηση F θα λέγεται μεγιστικά μονότονη αν ο τελεστής A^F είναι μεγιστικά μονότονος (βλέπε ορισμό στην παράγραφο 1.3). Κύριος σκοπός μας θα είναι η μελέτη μερικών ιδιοτήτων των μονότονων δισυναρτήσεων σε σχέση με αντίστοιχες ιδιότητες των μονότονων τελεστών.

Ένα από τα κύρια αποτελέσματα της διατριβής είναι ότι, κάτω από ασθενείς υποθέσεις, οι μονότονες δισυναρτήσεις είναι τοπικά φραγμένες στο εσωτερικό του πεδίου ορισμού τους. Ως άμεσο αποτέλεσμα, θα συμπεράνουμε τη γνωστή ιδιότητα ότι κάθε μονότονος τελεστής T είναι τοπικά φραγμένος στο εσωτερικό του πεδίου ορισμού του $\operatorname{dom} T = \{x \in X : T(x) \neq \emptyset\}$. Από την άλλη πλευρά, σε αντίθεση με τους μονότονους τελεστές, οι μονότονες δισυναρτήσεις μπορεί να είναι τοπικά φραγμένες και στο σύνορο του πεδίου ορισμού τους· μάλιστα θα δείξουμε ότι αυτό ισχύει πάντοτε όταν το πεδίο ορισμού τους είναι πολυεδρικό.

Επιπλέον θα δείξουμε ότι ένας μονότονος τελεστής T είναι "τοπικά φραγμένος προς τα μέσα"σε κάθε σημείο $x_0 \in \operatorname{dom} T$, ιδιότητα που ανάγεται στη γνωστή ιδιότητα του τοπικά φραγμένου όταν το x_0 ανήκει στο εσωτερικό του $\operatorname{dom} T$.

Οι μονότονοι τελεστές μπορούν να γενικευθούν με πολλούς τρόπους, βλέπε πχ [63] και [74]. Ένας από αυτούς είναι οι λεγόμενοι σ-μονότονοι τελεστές [71], που είναι πλειότιμοι τελεστές $T : X \to 2^{X^*}$ τέτοιοι ώστε για κάθε $x, y \in \operatorname{dom} T$ και $x^* \in T(x)$, $y^* \in T(y)$,

$$\langle x^* - y^*, x - y \rangle \geq -\min\{\sigma(x), \sigma(y)\} \|x - y\|$$

όπου $\sigma : \operatorname{dom} T \to \mathbb{R}_+$ είναι δοσμένη συνάρτηση. Ο T λέγεται προμονότονος (pre-monotone) αν είναι σ-μονότονος για κάποια συνάρτηση σ. Η κλάση των προμονότονων τελεστών περιλαμβάνει πολλούς σημαντικούς τελεστές όπως οι μονότονοι και οι ε-μονότονοι τελεστές. Στην παρούσα διατριβή θα γενικεύσουμε μερικά από τα αποτελέσματα της εργασίας των Iusem, Kassay, Sosa [71] σε απειροδιάστατους χώρους, και επίσης θα εισάγουμε την έννοια της $\sigma-$μονότονης και προμονότονης δισυνάρτησης. Το κύριο αποτέλεσμα είναι ότι οι προμονότονες δισυναρτήσεις είναι τοπικά φραγμένες στο εσωτερικό του πεδίου ορισμού τους, με αντίστοιχο συμπέρασμα για τους προμονότονους τελεστές. Επίσης γενικεύουμε ένα σημαντικό θεώρημα του Libor Veselý. Επιπλέον δείχνουμε ότι, δοθέντων δύο μεγιστικών σ-μονότονων τελεστών S και T, μια αρκετά ασθενής συνθήκη που αφορά τη σχετική θέση των πεδίων ορισμού τους, συνεπάγεται ότι το άθροισμα $T(x) + S(x)$ είναι ασθενώς* κλειστό για κάθε $x \in X$.

Ένα σημαντικό μέρος αυτής της διατριβής αφορά στην εισαγωγή και μελέτη του "μετασχηματισμού Fitzpatrick" μιας δισυνάρτησης. Κατ΄ αρχάς εισάγουμε την έννοιας της κανονικής (normal) δισυνάρτησης και ένα καινούργιο ορισμό μονότονης δισυνάρτησης, που είναι μια μικρή γενίκευση του αντίστοιχου ορισμού των Bloom και Oettli [23], αλλά που επιτρέπει μια καλύτερη αντιστοίχηση των μονότονων δισυναρτήσεων και μονότονων τελεστών. Ένα από τα κύρια χαρακτηριστικά του νέου ορισμού είναι ότι ένας τελεστής T με ασθενώς*-κλειστές τιμές είναι μεγιστικά μονότονος αν και μόνον αν η δισυνάρτηση G_T είναι BO-μεγιστικά μονότονη (βλέπε ορισμό στο τελευταίο κεφάλαιο). Επιπλέον αποδεικνύουμε ότι ο μετασχηματισμός Fitzpatrick της G_T είναι ακριβώς η συνάρτηση Fitzpatrick του T. Επιπλέον, αν μια μονότονη δισυνάρτηση F είναι κυρτή και κάτω ημισυνεχής ως προς τη δεύτερη μεταβλητή της, ο μετασχηματισμός Fitzpatrick μας επιτρέπει να βγάλουμε συμπεράσματα για τη μεγιστική μονοτονία της.

Παρουσιάζουμε τώρα τα περιεχόμενα των διαφόρων κεφαλαίων της διατριβής.

Το κεφάλαιο 1 περιέχει μερικές βασικές έννοιες και αποτελέσματα από την κυρτή ανάλυση, τη συναρτησιακή ανάλυση, τη θεωρία μονότονων τελεστών και τη συνάρτηση Fitzpatrick, προκειμένου να γίνει το κείμενο πιο αυτοδύναμο και να μην ανατρέχει ο αναγνώστης σε άλλες πηγές.

Το κεφάλαιο 2 είναι αφιερωμένο στις μονότονες δισυναρτήσεις. Ορίζουμε τις μεγιστικά μονότονες δισυναρτήσεις και παρουσιάζουμε μερικές αρχικές έννοιες και ιδιότητες. Τα κύρια αποτελέσματα του κεφαλαίου είναι το Θεώρημα 2.9 που δίνει μια ικανή συνθήκη ώστε να ισχύει η ισότητα $A^{G_T} = T$, και το Θεώρημα 2.19 που λέει ότι κάτω από ασθενείς υποθέσεις, μια μονότονη δισυνάρτηση είναι τοπικά φραγμένη σε κάθε σημείο του εσωτερικού του πεδίου ορισμού της. Με τον τρόπο αυτό βρίσκουμε μια εύκολη απόδειξη της αντίστοιχης ιδιότητας για μονότονους τελεστές. Οι προτάσεις 2.32 και 2.33 δείχνουν ότι οι μονότονες δισυναρτήσεις συμπεριφέρονται καλύτερα από τους αντίστοιχους μονότονους τελεστές, αφού μπορούν να είναι τοπικά φραγμένες και στο σύνορο του πεδίου ορισμού τους. Ειδικώτερα δείχνουμε ότι όταν το πεδίο ορισμού είναι τοπικώς πολυεδρικό υποσύνολο του $\mathbb{R}^n$, τότε το τοπικό φράξιμο είναι αυτόματο σε όλο το πεδίο ορισμού. Στο τέλος του κεφαλαίου παρουσιάζουμε μερικά παραδείγματα και αντιπαραδείγματα.

Το κεφάλαιο 3 ασχολείται με τη θεωρία των σ-μονότονων μονότονων τελεστών και δισυναρτήσεων. Εισάγουμε τις έννοιες των σ-μονότονων τελεστών και δισυναρτήσεων σε ένα χώρο Banach, και μελετούμε αρχικά τις στοιχειώδεις ιδιότητές τους. Επίσης, εισάγουμε και μελετούμε τις κλάσεις των προμονότονων τελεστών και δισυναρτήσεων. Στην πρόταση 3.7 αποδεικνύουμε ότι αν ο T είναι σ-μονότονος και η σ άνω ημισυνεχής, τότε ο T έχει ακολουθιακά νορμ$\times$ασθενώς* κλειστό γράφημα. Επιπλέον, το παράδειγμα 3.8 δείχνει ότι η υπόθεση της άνω ημισυνέχειας της σ δε μπορεί να παραληφθεί. Το κύριο αποτέλεσμα είναι το Θεώρημα 3.17 που αποδεικνύει ότι, κάτω από κατάλληλες συνθήκες, οι σ-μονότονες δισυναρτήσεις είναι τοπικά φραγμένες στο εσωτερικό του πεδίου ορισμού τους, πράγμα που επιτρέπει την απόδειξη της αντίστοιχης ιδιότητας για σ-μονότονους τελεστές. Επιπλέον, αποδεικνύουμε μια επέκταση του θεωρήματος του Libor Veselý [92]. Δείχνουμε επίσης ότι, κάτω από μερικές συνθήκες πάνω στο πεδίο ορισμού τους, το άθροισμα των τιμών δύο μεγιστικών σ-μονότονων τελεστών είναι ασθενώς*-κλειστό. Στη συνέχεια αποδεικνύουμε την ύπαρξη λύσης για το πρόβλημα ισορροπίας που ορίζεται σε ένα κυρτό και κλειστό (μη φραγμένο εν γένει) υποσύνολο ενός πεπερασμένης διάστασης χώρου. Το κεφάλαιο τελειώνει με τη σύγκριση του ορισμού του σ-μονότονου τελεστή με άλλους ορισμούς γενικευμένης μονοτονίας που υπάρχουν στη βιβλιογραφία.

Στα τελευταία χρόνια, ένα από τα ισχυρότερα εργαλεία στη μελέτη των μεγιστικά μονότονων τελεστών αποδείχθηκε ότι είναι η συνάρτηση Fitzpatrick. Χάρις σ' αυτήν, πολλά αποτελέσματα της (μερικές φορές ιδιαίτερα δύσκολης) θεωρίας των μεγιστικά μονότονων τελεστών αποδείχθηκαν ευκολώτερα ή και ισχυροποιήθηκαν, χρησιμοποιώντας μεθόδους της κυρτής ανάλυσης. Στο κεφάλαιο 4 δείχνουμε τη στενή σχέση της θεωρίας των μονότονων δισυναρτήσεων με την κυρτή ανάλυση, ορίζοντας το μετασχηματισμό Fitzpatrick φ_F μιας δισυνάρτησης $F : X \times X \to \mathbb{R} \cup \{+\infty, -\infty\}$ ως μια συνάρτηση $\varphi_F : X \times X^* \to \mathbb{R} \cup \{+\infty\}$. Ένα από τα τα σημαντικότερα αποτελέσματα είναι το Θεώρημα 4.11 που δείχνει ότι δοθείσας μιας BO-μεγιστικά μονότονης δισυνάρτησης F, για κάθε $(x, x^*) \in X \times X^*$ ισχύει $\varphi_F(x, x^*) \geq \langle x^*, x \rangle$, ενώ η ισότητα ισχύει αν και μόνον αν $x^* \in A^F(x)$. Επιπλέον στην Πρόταση 4.12 βρίσκουμε μία σχέση μεταξύ του μετασχηματισμού Fitzpatrick και της συνάρτησης Fitzpatrick. Ορίζουμε επίσης τον άνω μετασχηματισμό Fitzpatrick φ^F, δείχνουμε δε ότι μαζί με το μετασχηματισμό Fitzpatrick, αποτελούν ένα ισχυρό εργαλείο. Για παράδειγμα αποδεικνύουμε χρησιμοποιώντας τους ότι όταν ο χώρος είναι ανακλαστικός (reflexive), τότε για κάθε δισυνάρτηση F που είναι κυρτή και κάτω ημισυνεχής ως προς τη δεύτερη μεταβλητή, η F είναι μεγιστικά μονότονη αν και μόνο αν είναι BO-μεγιστικά μονότονη. Στη συνέχεια, θα βρούμε ένα άνω φράγμα για το μετασχηματισμό Fitzpatrick του αθροίσματος δύο δισυναρτήσεων, και θα συνάγουμε μια ανισότητα για το μετασχηματισμό Fitzpatrick που ισχύει όταν η δισυνάρτηση είναι υποαθροιστική (subadditive) ως προς τη δεύτερη μεταβλητή. Επίσης, αποδεικνύουμε μερικά θεωρήματα ύπαρξης λύσης ανισώσεων. Κατόπιν παρουσιάζουμε μερικά παραδείγματα υπολογισμού του μετασχηματισμού Fitzpatrick. Στο τέλος του κεφαλαίου εισάγουμε την έννοια της n-κυκλικά μονότονης και BO-μεγιστικής n-κυκλικά μονότονης δισυνάρτησης. Δείχνουμε τη σχέση που έχουν με τους n-κυκλικά μονότονους τελεστές. Τέλος, γενικεύουμε μερικά αποτελέσματα της παραγράφου 4.3 στην περίπτωση των κυκλικά μονότονων δισυναρτήσεων.

Τα βασικότερα αποτελέσματα των κεφαλαίων 2, 3 και 4, περιέχονται, αντίστοιχα, στις εργασίες [5], [6] και [4]. Για διευκόλυνση του αναγνώστη, στο τέλος της διατριβής υπάρχει ευρετήριο όρων.

Introduction

Our purpose in this thesis is to study and advance in the research area of monotone and generalized monotone operators and bifunctions.

A monotone operator is a set-valued map from a Hausdorff locally compact space X to its topological dual space X^* such that

$$\langle x^* - y^*, x - y \rangle \geq 0$$

for all $x, y \in X$ and $x^* \in T(x)$ and $y^* \in T(y)$ where $\langle x^*, x \rangle = x^*(x)$. Note that when T is single-valued and $X = \mathbb{R}$, then T is nothing else than an increasing map, and this justifies the name "monotone operator". The notion of monotone operator has been found appropriate in various branches of mathematics such as Operator Theory, Partial Differential Equations, Differentiability Theory of Convex Functions, Numerical Analysis and has brought a new life to Nonlinear Functional Analysis and Nonlinear Operator Equations. In particular, monotone operators are a powerful tool to the study of variational inequalities, which are a very useful instrument for constructing mathematical models for several physical and engineering problems. This is because the class of monotone operator includes subdifferentials and continuous positive linear operators, which are usually found in the above mentioned areas.

Generally it is not clear who introduced the notion of monotone operators. Nevertheless, the popular view is that M. Golomb was the first one who introduced this notion in his paper "*Zur Theorie der nichtlinearen Integralgleichungen, Integralgleichungssysteme und allgemeiner Funktionalgleichungen*", Math. 2. 39, 45-75 (1935). For historical discussions and more information we refer to [84] and [121].

Another important notion is the notion of monotone bifunction. If $C \subseteq X$, a function $F : C \times C \to \mathbb{R}$ is called monotone bifunction if for every $x, y \in C$,

$$F(x, y) + F(y, x) \leq 0.$$

Monotone bifunctions are connected to the so-called *equilibrium problem*, which consists in finding $x_0 \in C$ such that

$$\forall y \in C : \; F(x_0, y) \geq 0.$$

Equilibrium problems are related to the minimax problem and were studied by various authors in the past, but the term "equilibrium problem" was introduced in the seminal paper by Blum and Oettli [23]. Blum and Oettli have shown that many important problems (optimization problems, variational inequalities, saddle point problems, fixed point problems, Nash equilibria etc.) can be seen as a particular cases of the equilibrium problem. All these reasons have convinced many mathematicians, after Blum and Oettli's highly influencing paper [23], to start research in this rich and important branch of mathematics, so equilibrium problems were studied in many papers (see [7, 8, 22, 21, 64, 54, 71, 69, 75, 77, 78, 86] and the references therein). Recently, a part of literature has been dedicated to algorithms for finding solutions of equilibrium problems, for example see [69], [54], [75], and [86]. In this thesis we will investigate monotone bifunctions from another standpoint. We will focus on the relation between maximal monotone operators and maximal monotone bifunctions. To each bifunction F we will correspond an operator A^F and for every operator T will correspond a bifunction G_T. A monotone bifunction F will be called maximal monotone if A^F is a maximal monotone operator. We will study some properties of monotone bifunctions in relation with the corresponding property of monotone operators and vice versa.

One of the main results of this thesis is that under weak assumptions, monotone bifunctions are locally bounded in the interior of the convex hull of their domain. As an immediate consequence, one can get the corresponding property for monotone operators. Moreover, in contrast to maximal monotone operators, monotone bifunctions (maximal or not maximal) can also be locally bounded at the boundary of their domain.

We also show that each monotone operator is "inward locally bounded" at every point of the closure of its domain, a property which collapses to ordinary local boundedness at interior points of the domain. Moreover, we derive some properties of cyclically monotone bifunctions.

Monotone operators have been generalized in many ways; see [63] and [74]. One of these generalizations is the so-called σ-monotone operator [71]; a multivalued operator T from X into X^* is called σ-monotone if for all x and y in the domain $\operatorname{dom} T$ of T, and all $x^* \in T(x)$, $y^* \in T(y)$,

$$\langle x^* - y^*,\ x - y\rangle \geq -\min\{\sigma(x), \sigma(y)\}\|x - y\|$$

where $\sigma : \operatorname{dom} T \to \mathbb{R}_+$ is a given function. T is called pre-monotone it is σ-monotone for some σ. Pre-monotone operators include many important classes of operators such as monotone and ε-monotone operators. In this thesis, we extend some results of [71] (which are proved in $\mathbb{R}^n$) to Banach spaces and also introduce the notion of σ-monotone bifunctions. The main result shows that σ-monotone bifunctions are locally bounded in the interior of their domain, which implies that local boundedness of pre-monotone operators. We also state and prove a generalization of the Libor Veselý theorem. Besides, we show that, given two maximal σ-monotone operators T and S, a weak condition on the mutual position of their domains implies that $T(x) + S(x)$ is weak*-closed for every x.

A considerable part of this thesis is devoted to introducing and studying of the "Fitzpatrick transform of a bifunction" and its properties. In fact, we introduce the notion of normal bifunction and a new definition of monotone bifunctions, which is a slight generalization of the original definition given by Blum and Oettli, but which is better suited for relating monotone bifunctions to monotone operators. One of the main features of this new definition is that an operator with weak*-closed convex values is maximal monotone if and only if the corresponding bifunction is BO-maximal monotone. In addition, we show that the Fitzpatrick transform of a maximal monotone bifunction corresponds exactly to the Fitzpatrick function of a maximal monotone operator, in case the bifunction is constructed starting from the operator. Whenever the monotone bifunction is lower semicontinuous and convex with respect to its second variable, the Fitzpatrick transform permits to obtain results on its maximal monotonicity.

We now present a brief outline of the thesis. It consists of four chapters.

Chapter 1 contains some basic knowledge from Convex Analysis and Functional Analysis, the theory of monotone operators and the Fitzpatrick function which allows the study of the proposed material without turning, generally, to other sources.

Chapter 2 is devoted to monotone bifunctions. We define maximal monotonicity of bifunctions, and we present some preliminary definitions, properties and results. A part of our results is inspired by some analogous results from [64]. The main results of this chapter are Theorem 2.9 which provides a sufficient condition under which the equality $A^{G_T} = T$ is true, and Theorem 2.19 which states that under very weak assumptions, local boundedness of monotone bifunctions is automatic at every point of $\operatorname{int} C$. In this way one can obtain an easy proof of the corresponding property of monotone operators. Propositions 2.32 and 2.33 reveal that monotone bifunctions are in some ways better behaved that the underlying monotone operators, since they can be locally bounded even at the boundary of their domain of definition. Besides, we demonstrate that for locally polyhedral domains C in $\mathbb{R}^n$, an automatic local boundedness of bifunctions holds on their whole domain of definition. We also assert that each monotone operator is "inward locally bounded" at every point of the closure of its domain, a property which collapses to ordinary local boundedness at interior points of the domain. At the end of the chapter, we present some noteworthy counterexamples.

Chapter 3 deals with the theory of σ-monotone operators and σ-monotone bifunctions. We introduce the class of σ-monotone and maximal σ-monotone operators in a Banach space, and analyze their properties. We also introduce and study the class of pre-monotone bifunctions which are related to the notion of pre-monotone operators. Proposition 3.7 shows that if T is σ-monotone and σ is upper semicontinuous, then $\operatorname{gr} T$ is sequentially norm$\times$weak*-closed. Moreover, Example 3.8 shows that upper semicontinuity of σ cannot be omitted from the statement of Proposition 3.7. The main Theorem 3.17 shows that, under weak assumptions, σ-monotone bifunctions are locally bounded in the interior of their domain; this allows us to deduce that pre-monotone operators are locally bounded in the interior of their domain. In addition, we state and prove a generalization of the Libor Veselý theorem. We show that also under some conditions on their domain, the sum of the values of two maximal σ-monotone operator is weak*-closed. Afterwards, we confine our attention to finite dimensions and prove the existence of solutions for an equilibrium problem in a (generally unbounded) closed convex subset of an Euclidean space. We conclude this chapter by comparing some types of generalized monotone operators.

The main tool for linking maximal monotone theory to Convex Analysis, is the Fitzpatrick function. In Chapter 4 we point out the connection between bifunctions and Convex Analysis by introducing the notion of Fitzpatrick transform φ_F of a bifunction $F : X \times X \to \mathbb{R} \cup \{+\infty, -\infty\}$ as a function $\varphi_F : X \times X^* \to \mathbb{R} \cup \{+\infty\}$. One of the main results is Theorem 4.11 which proves that given a BO-maximal monotone bifunction F, for every $(x, x^*) \in X \times X^*$ one has $\varphi_F(x, x^*) \geq \langle x^*, x \rangle$; and equality holds if and only if $x^* \in A^F(x)$. Moreover, in Proposition 4.12 we find a link between the Fitzpatrick transform and the Fitzpatrick function. In addition, we define the upper Fitzpatrick transform; we will see that in conjunction with the Fitzpatrick transform, it is very useful in our analysis. In the sequel, by another main theorem we demonstrate that the maximality of A^F and BO-maximality of F are equivalent whenever the space is reflexive, and F is lower semicontinuous and convex with respect to its second variable. Theorem 4.19 characterizes the BO-maximality through some equivalence statements. We find also an upper bound for the Fitzpatrick transform of a sum and then will deduce an inequality for the Fitzpatrick transform when the bifunction is subadditive with respect to its second variable. Besides, we present some existence theorems. Also we collect several examples concerning the Fitzpatrick transform of bifunctions. Thereafter, we introduce the notion of n-cyclically monotone and BO-n-cyclically maximal monotone bifunctions. Also, we will bring forward their relation to n-cyclically monotone operators. We prove a theorem for BO-n-cyclically maximal monotone bifunctions which is similar to the corresponding theorem of Fitzpatrick functions. Subsequently, we generalize some results from Section 4.3 to cyclically monotone bifunctions.

The main results of Chapters 2, 3 and 4 are contained, respectively, in the papers [5], [6] and [4]. For the convenience of the reader, the thesis is supplemented by an index of the main terms.

Chapter 1

Background and Preliminaries

In the first chapter, we present an overview of some main notions and theorems from Functional Analysis and Convex Analysis to prepare the background for the chapters that follow. Also, this chapter provides all basic concepts of monotone and maximal monotone operators to which we refer in the next chapters.

1.1 Functional Analysis Tools

We start this section by collecting the basic aspects of topological vector spaces and locally convex spaces.

Let X be a vector space. A function $p : X \to \mathbb{R}_+$ is called seminorm if it satisfies:

(i) $p(x + y) \leq p(x) + p(y)$ for all $x, y \in X$;

(ii) $p(\lambda x) = |\lambda| p(x)$ for each $x \in X$ and every scalar λ.

Note that from (ii) we conclude that $p(0) = 0$. Also, a seminorm p that satisfies $p(x) = 0$ only if $x = 0$ is called a *norm*. Usually a norm is denoted by $\|\cdot\|$. A normed space is a pair $(X, \|\cdot\|)$, where X is a vector space and $\|\cdot\|$ is a norm on X. A Banach space is a normed space which is complete with respect to the metric defined by the norm.

A *topological vector space* (TVS, from now on) is a vector space X together with a topology so that the addition and scalar product maps i.e.,

- the map of $X \times X \to X$ defined by $(x, y) \longmapsto x + y$,
- the map of $\mathbb{R} \times X \to X$ defined by $(t, y) \longmapsto ty$,

are continuous with respect to this topology.

Let us fix some notation. Assume that X is a vector space. Given $x, y \in X$, $[x, y]$ will be the closed segment

$$[x, y] = \{(1 - t)\, x + ty : t \in [0, 1]\}.$$

Semi-closed and open segments i.e., $[x, y[$, $]x, y]$ and $]x, y[$ are defined analogously. If E and F are nonempty subsets of X we define the sum (Minkowski sum) of E and F by

$$E + F = \{x + y : x \in E, y \in F\}.$$

In case if $\emptyset \neq A \subset \mathbb{R}$, then $AE = \{\alpha x : \alpha \in A, x \in E\}$.

A set $\emptyset \neq E \subset X$ is convex if $[x, y] \subset X$ whenever $x, y \in E$. We set $\mathbb{R}_+ = [0, +\infty)$. A set $\emptyset \neq E \subset X$ is affine if $(1 - t)\, x + ty \in E$ for every $x, y \in E$ and each $t \in \mathbb{R}$. If E is a subset of X, the convex hull of E, denoted by $\operatorname{co} E$, is the intersection of all convex sets that contain E. In fact

$$\begin{aligned} \operatorname{co} E &= \cap \{C \subset X : E \subset C \text{ and } C \text{ is convex}\} \\ &= \left\{ \sum_{i=1}^{n} t_i x_i : n \in \mathbb{N}, t_i \in \mathbb{R}_+, x_i \in E, \sum_{i=1}^{n} t_i = 1 \right\}. \end{aligned}$$

Assume that $\mathcal{P}$ is a family of seminorms on X. Then one can define a topology $\mathcal{T}$ as follows, $G \in \mathcal{T}$ if and only if for each $x_0 \in G$ there are $p_1, ..., p_n$ in $\mathcal{P}$ and $\varepsilon_1, ..., \varepsilon_n > 0$ such that $\cap_{i=1}^{n} \{x \in X : p(x - x_0) < \varepsilon\} \subset G$.

Definition 1.1 *A TVS is called* locally convex space *(LCS, from now on) if its topology is defined by a family of seminorms.*

1.1.1 Baire Category Theorem

Baire's theorem was proved in 1899 by René-Louis Baire in his doctoral thesis (On the Functions of Real Variables) [12]. In late 1920's, Banach and Steinhaus introduced Baire's theorem into Functional Analysis.

Assume that X is a topological space and $\emptyset \neq D \subset X$. Then D is *dense* in X if $\operatorname{cl} D = X$, that is, for every nonempty open subset U of X we have $D \cap U \neq \emptyset$. A subset F of X is called *nowhere dense* in X if the closure of F has empty interior, i.e., $\operatorname{int}(\operatorname{cl}(F)) = \emptyset$. Note that a set F is nowhere dense if and only if its closure is nowhere dense.

A set $E \subseteq X$ is of the *first category* in X or "meager" in X if E is a countable union of nowhere dense subsets of X, i.e., if the complement $X \backslash D$ contains a countable intersection of open dense subsets of X. Obviously, any countable union of first category sets is of the first category.

A subset U of X is of the *second category* in X or "non-meager" in X if U is not of the first category in X. Equivalently if $U \subset \cup_{n=1}^{\infty} F_n$ whenever $F_1, F_2, ...$ are closed sets, then $\operatorname{int} F_n \neq \emptyset$ for some n.

A *Baire space* is a topological space in which nonempty open sets are not meager. For more information about the Baire spaces see [3], [24], [58], [94], [102] and [103].

The following theorem characterizes Baire spaces.

Theorem 1.2 *Let X be a topological space. Then the following statements are equivalent:*

(i) X is a Baire space.

(ii) Every countable intersection of open dense sets is also dense.

(iii) If $X = \cup_{n=1}^{\infty} F_n$ and each F_n is closed, then $\cup_{n=1}^{\infty} \operatorname{int} F_n$ is dense.

See [3, Theorem 3.46] for a proof.

Theorem 1.3 ***(Baire category theorem)*** *A complete metrizable space is a Baire space.*

A proof can be found in [101, Theorem 5.6] or [3, Theorem 3.47].

1.1.2 The Uniform Boundedness Principle

The Banach-Steinhaus theorem is one of the most effective and potent theorems in Functional Analysis, which states that a set of continuous linear transformations that is bounded at each point of a Banach space is bounded uniformly on the unit ball. Roughly speaking, pointwise boundedness implies uniform boundedness. For more information and complete descriptions see [94] and [102].

Let X and Y be TVS. Set

$$\mathcal{L}(X, Y) = \{\text{all linear transformations } f : X \to Y\}$$

and

$$\mathcal{BL}(X, Y) = \{\text{all continuous linear transformations } f : X \to Y\}.$$

Proposition 1.4 *Suppose that X and Y are TVS and $f \in \mathcal{L}(X, Y)$. Then f is continuous on X if (and only if) f is continuous at the origin.*

The following definition is taken from [94].

Definition 1.5 *Let $\mathfrak{F} \subset \mathcal{L}(X, Y)$. The set $\mathfrak{F}$ is called equicontinuous if for each neighborhood V in Y, there is a neighborhood U in X with $f(U) \subseteq V$ for all $f \in \mathfrak{F}$, or equivalently, for each neighborhood V in Y, $\cap_{f\in\mathfrak{F}} f^{-1}(V)$ is a neighborhood in X. When X and Y are normed spaces, then $\mathfrak{F}$ is equicontinuous if and only if there is a constant α with $||f(x)|| < \alpha||x||$ for every $f \in \mathfrak{F}$.*

Assume that Y and Z are normed spaces. For a given $f \in \mathcal{BL}(Y, Z)$, the norm of f is defined by

$$||f|| = \sup\{||f(y)|| : ||y|| \leq 1\} = \inf\{M > 0 : ||f(y)|| \leq M||x||, y \in Y\}.$$

When Y is a Banach space and Z is a normed space, then the uniform boundedness principle theorem has a simple version as follows.

Theorem 1.6 ***(Uniform boundedness principle)*** *Let Y be a Banach space and Z a normed space. If $\Gamma \subset \mathcal{BL}(Y,Z)$ such that for each y in Y,*

$$\sup\{||f(y)|| : f \in \Gamma\} < \infty,$$

then $\sup\{||f|| : f \in \Gamma\} < \infty$.

1.1.3 Hahn-Banach Theorem and Separation Theorem

Suppose that X is a vector space over the scalar field $\mathbb{F}$. The elements of $\mathcal{L}(X,\mathbb{F})$ are called the linear forms or linear functionals. Also, $\mathcal{L}(X,\mathbb{F})$ is called the *algebraic dual* of X. Moreover, when X it a TVS, then $\mathcal{BL}(X,\mathbb{F})$ is called the *topological (continuous) dual* of X and it depends on the topology. We will denote the algebraic dual and topological dual of X by $X'$and X^*, respectively. The maximal proper vector subspaces of X are called *hyperplane*s (through the origin). By the axiom of choice X^* is proper subset of X'. Every hyperplane H of X can be written as the kernel of a linear form see [43, Proposition 5.1]. Assume that H is a hyperplane and $H = \ker f$. If $f \in X^*$ then H is closed otherwise $f \in X' \backslash X^*$ and H is dense in X. In other word, H is closed if and only if f is continuous, and dense if and only if f is discontinuous.

The Hahn-Banach theorem is one of the important and fundamental theorems in Functional Analysis and states that a continuous linear functional on a vector subspace of X has a continuous extension to the whole of X. We select some applications of this theorem, that can be found in any book on Functional Analysis.

Theorem 1.7 ***(Interior separating hyperplane theorem)*** *Let X be a TVS and A, B two disjoint convex subsets. If A is open, then there exist $f \in X^*$ and $\alpha \in \mathbb{R}$ such that for all $x \in A$, $y \in B$ one has $f(x) > \alpha \geq f(y)$.*

Theorem 1.8 ***(Strong separating hyperplane theorem)*** *Let X be a LCS and A, B two disjoint closed convex subsets. If A is compact, then there exist $f \in X^*$ and $\alpha \in \mathbb{R}$ such that for all $y \in B$ one has $\min_{x\in A} f(x) > \alpha > f(y)$.*

Corollary 1.9 ***(Separating points from closed convex sets)*** *Let X be a LCS and A a closed convex subset. If $z \notin A$, then there exist $f \in X^*$ and $\alpha \in \mathbb{R}$ such that for all $y \in B$ one has $f(z) > \alpha > f(y)$.*

From now on, we will usually represent elements of X^* by starred letters such as x^*, and the value of x^* on $x \in X$ by $\langle x^*, x\rangle$.

1.1.4 Weak and Weak*-Topologies

Assume that X is a LCS. The *weak topology*, is the topology defined by the family of seminorms $\{p_{x^*} : x^* \in X^*\}$, where $p_{x^*}(x) = |\langle x^*, x\rangle|$. We will denote it by $\sigma(X, X^*)$ or "w-topology". Also, the *weak*-topology* on X^*, is the topology defined by the seminorms $\{p_x : x \in X\}$ where $p_x(x^*) = |\langle x^*, x\rangle|$. We will denote it by $\sigma(X^*, X)$ or "weak*-topology". Thus a subset G of X is weakly open if and only if for every x_0 in G there is an $\epsilon > 0$ and there are $x_1^*, ..., x_n^*$ in X^* such that

$$\cap_{i=1}^n \{x \in X : |\langle x_i^*, x - x_0 \rangle| < \varepsilon\} \subset G.$$

We note that a net $\{x_i\}$ in X *converges weakly* to some point x_0 in X if $\langle x^*, x_i \rangle \to \langle x^*, x_0 \rangle$ for each $x^* \in X^*$. We will denote this by $x_i \rightharpoonup x_0$ or $x_i \overset{w}{\to} x_0$. In a similar manner, a net $\{x_i^*\}$ in X^* is weak*-convergent to some point x_0^* in X^* if $\langle x_i^*, x \rangle \to \langle x_0^*, x \rangle$ for each $x \in X$. We will denote this by $x_i^* \rightharpoondown x_0^*$ or $x_i \overset{w^*}{\to} x_0^*$.

Proposition 1.10 *A convex subset of X is closed if and only if it is weakly closed.*

See [43, Chapter **V**, Theorem 1.4 and Corollary 1.4] for a proof.

The Alaoglu theorem asserts that the closed unit ball of the dual space of a normed vector space is compact in the weak*- topology [2]. This theorem was extended to separable normed vector spaces by Stefan Banach. Finally, this theorem was generalized by the Bourbaki group to LCS.

Theorem 1.11 *(**Alaoglu theorem**) Suppose that X is a TVS and U is a neighborhood of 0 in X. If*

$$K = \{x^* \in X^* : |\langle x^*, x \rangle| \leq 1 \quad \forall x \in U\},$$

then K is weak-compact.*

See [102, Theorem 3.15]

1.2 Convex Analysis Tools

The purpose of this section is to outline the basic aspects of the Convex Analysis in TVS or LCS. We set as usual $\overline{\mathbb{R}} = \mathbb{R} \cup \{+\infty, -\infty\}$.

1.2.1 Lower Semicontinuous and Convex Functions

Assume that X is real vector space and $f : X \to \overline{\mathbb{R}}$ is a function. Its *domain* (or effective domain) is defined by

$$\operatorname{dom} f = \{x \in X : f(x) < \infty\}.$$

Also, the *epigraph* of f is defined by

$$\operatorname{epi} f = \{(x, r) \in X \times \mathbb{R} : f(x) \leq r\}.$$

The function f is called *proper* if $\operatorname{dom} f \neq \emptyset$ and $f(x) > -\infty$ for each $x \in X$. In addition, f is said to be *convex* when for all $x, y \in X$ and for each $t \in [0,1]$,

$$f((1-t)x + ty) \leq (1-t) f(x) + t f(y).$$

We say that f is *concave* if the function $-f$ is convex and f is *affine* whenever it is both convex and concave.

We recall that a function f is called *quasi-convex* if for each $x, y \in X$ and for every $t \in [0,1]$,

$$f((1-t)x + ty) \leq \max\{f(x), f(y)\}.$$

An alternative way of defining a quasi-convex function f is to require that each sublevel set $S_r(f) = \{x \in X : f(x) \leq r\}$ is a convex set.

A function f is called *quasi-concave* if $-f$ is quasi-convex

The following theorem is known. We refer the reader to [120] for the proof of all results contained in this and the two subsequent subsections.

Theorem 1.12 *Suppose that $f : X \to \overline{\mathbb{R}}$ is a function. Then the following statements are equivalent:*

(i) f is convex;

(ii) $\operatorname{dom} f$ *is convex and*

$$\forall x, y \in \operatorname{dom} f, \quad \forall t \in]0,1[: \ f((1-t)x + ty) \leq (1-t) f(x) + t f(y);$$

(iii) $\forall n \in \mathbb{N}, \quad \forall x_1, ..., x_n \in X, \quad \forall t_1, ..., t_n \in]0,1[, \ t_1 + \cdots + t_n = 1:$

$$f(t_1 x_1 + \cdots + t_n x_n) \leq t_1 f(x_1) + \cdots + t_n f(x_n);$$

(iv) $\operatorname{epi} f$ *is a convex subset of* $X \times \mathbb{R}$.

Suppose that X is a Hausdorff LCS, Λ is a set of indices and $\{f_\alpha\}_{\alpha \in \Lambda}$ functions on X. The *convex hull* of $\{f_\alpha\}_{\alpha \in \Lambda}$ is denoted by

$$\operatorname{conv} \{f_\alpha\}_{\alpha \in \Lambda}.$$

It is the convex hull of the pointwise infimum of the collection see [98, page 37].

Theorem 1.13 *Suppose that X is a Hausdorff LCS, Λ is a set of indices and $\{f_\alpha\}_{\alpha \in \Lambda}$ functions on X. Assume that f is the convex hull of the collection. Then*

$$f(x) = \inf\left\{\sum\nolimits_{\alpha \in \Lambda} \lambda_\alpha f_\alpha(x_\alpha) : \sum\nolimits_{\alpha \in \Lambda} \lambda_\alpha x_\alpha = x\right\}.$$

where the infimum is taken over all representations of x as a convex combination of elements x_α, such that only finitely many coefficients λ_α are nonzero. (The formula is also valid if one actually restricts x_α to lie in $\operatorname{dom} f_\alpha$*.)*

According to the definition of convex hull and the above theorem we have the following fact:

Suppose that X is a Hausdorff LCS and Λ is a set of indices and $\{f_\alpha\}_{\alpha\in\Lambda}$ functions on X. The *concave hull* of $\{f_\alpha\}_{\alpha\in\Lambda}$ is the concave hull of the pointwise supremum of the collection. Let f be the concave hull of the collection. Then

$$f(x) = \sup\left\{\sum\nolimits_{\alpha\in\Lambda} \lambda_\alpha f_\alpha(x_\alpha) : \sum\nolimits_{\alpha\in\Lambda} \lambda_\alpha x_\alpha = x\right\}. \tag{1.1}$$

where the supremum is taken over all representations of x as a concave combination of elements x_α, such that only finitely many coefficients λ_α are nonzero.

Now assume that X is a topological space. A function $f : X \to \overline{\mathbb{R}}$ is called *lower semicontinuous* (briefly, lsc) at $x_0 \in X$ if for each $\varepsilon > 0$ there exists a neighborhood U_{x_0} of x_0 such that $f(x) \geq f(x_0) - \varepsilon$ for all x in U_{x_0}. This can be expressed as $f(x_0) \leq \liminf_{x\to x_0} f(x)$. Also, f is said to be lsc if it is lsc at each point of $\operatorname{dom} f$. Equivalently, f is lsc if and only if $\operatorname{epi} f$ is closed. Note that f is called *upper semicontinuous* (shortly, usc) if $-f$ is lsc.

Proposition 1.14 *Suppose that X is a Hausdorff LCS, Λ is a set of indices and $\{f_\alpha\}_{\alpha\in\Lambda}$ is a collection of convex (lsc) functions on X. Then their pointwise supremum $f = \sup\{f_\alpha : \alpha \in \Lambda\}$ is convex (lsc).*

We point out that the investigation of lsc functions is a particular case of the study of closed convex sets.

Theorem 1.15 *Suppose that X is a Hausdorff LCS and $f : X \to \overline{\mathbb{R}}$ is a function. Then the following statements are equivalent:*
(i) f is convex and lsc;
(ii) f is convex and weakly lsc;
(ii) $\operatorname{epi} f$ *is convex and closed;*
(v) $\operatorname{epi} f$ *is convex and weakly closed.*

It is well-known that if f is convex on $]a, b[$, then it is continuous on $]a, b[$ whenever $a, b \in \mathbb{R}$. The next propositions concern the extension of this result to more general spaces.

Proposition 1.16 *Let f be a proper, lsc and convex function on a Banach space. If* $\operatorname{int}(\operatorname{dom} f) \neq \emptyset$*, then f is continuous on* $\operatorname{int}(\operatorname{dom} f)$.

Proposition 1.17 *Suppose that X is a Hausdorff LCS. If the convex function $f : X \to \overline{\mathbb{R}}$ is bounded above on a neighborhood of a point of its domain, then f is continuous on the interior of its domain. Moreover, if f is not proper then f is identically $-\infty$ on* $\operatorname{int}(\operatorname{dom} f)$.

A function $f : X \to \overline{\mathbb{R}}$ is called *closed* if it is lsc everywhere, or if its epigraph is closed.

Definition 1.18 ***(Closure of a function)*** *The closure (or lsc hull) of a function f is the function* $\operatorname{cl} f : X \to \overline{\mathbb{R}}$ *defined by*

$$\operatorname{cl} f(x) = \liminf_{y \to x} f(y) \text{ or equivalently } \operatorname{epi}(\operatorname{cl} f) = \operatorname{cl}(\operatorname{epi} f).$$

The next proposition gives some properties of the $\operatorname{cl} f(x)$.

Proposition 1.19 *Suppose that $f : X \to \overline{\mathbb{R}}$ is convex. Then*
(i) $\operatorname{cl} f$ *is convex;*
(ii) if $g : X \to \overline{\mathbb{R}}$ is convex, lsc and $g \le f$, then $g \le \operatorname{cl} f$
(iii) $\operatorname{cl} f$ *does not take the value $-\infty$ if and only if f is bounded from below by a continuous affine function;*
(iv) if there exists $x_0 \in X$ such that $\operatorname{cl} f(x_0) = -\infty$ *(in particular if $f(x_0) = -\infty$), then* $\operatorname{cl} f(x) = -\infty$ *for every $x \in \operatorname{dom} \operatorname{cl} f \supset \operatorname{dom} f$.*

1.2.2 Convex Functions and Fenchel Conjugate

In this subsection X and Y are Hausdorff LCS and $f : X \to \overline{\mathbb{R}}$ is a function. The Fenchel conjugate of f is the function $f^* : X^* \to \overline{\mathbb{R}}$ defined by

$$f^*(x^*) = \sup_{x \in X} \{\langle x^*, x\rangle - f(x)\}.$$

Note that if there exists $x_0 \in X$ so that $f(x_0) = -\infty$, then $f^*(x^*) = +\infty$ for each $x^* \in X^*$. Also, $f^*(x^*) = \sup_{x \in \operatorname{dom}} \{\langle x^*, x\rangle - f(x)\}$ whenever f is proper. Assume that g is defined on the dual space X^*, i.e. $g : X^* \to \overline{\mathbb{R}}$ is a function, one also consider its conjugate $g^* : X \to \overline{\mathbb{R}}$ by $g^*(x) = \sup_{x^* \in X^*} \{\langle x^*, x\rangle - g(x^*)\}$. One also consider the biconjugate function f^{**} defined by

$$f^{**}(x) = (f^*)^*(x) = \sup_{x^* \in X^*} \{\langle x^*, x\rangle - f^*(x^*)\}.$$

Suppose that $f, g : X \to \overline{\mathbb{R}}$ are two functions, the infimal convolution [120, page 43] of f and g is defined by

$$(f \Box g)(x) := \inf\{f(y) + g(x - y) : y \in X\}.$$

The next theorem collects some noteworthy properties of conjugate functions.

Theorem 1.20 *Suppose that $f, g : X \to \overline{\mathbb{R}}$, $h : X^* \to \overline{\mathbb{R}}$ and $A \in \mathcal{BL}(X, Y)$.*
(i) f^* is convex and weak-lsc, h^* is lsc and convex;*
(ii) the ***Young-Fenchel inequality****: for all $(x, x^*) \in X \times X^*$*

$$f(x) + f^*(x^*) \ge \langle x^*, x\rangle;$$

(iii) reverse order ruling:

$$f \le g \Rightarrow f^* \ge g^*;$$

(iv) $f^* = (\operatorname{cl} f)^* = (\operatorname{cl}(\operatorname{co} f))^*$ and $f^{**} \le \operatorname{cl}(\operatorname{co} f) \le \operatorname{cl} f \le f$;
(v) $(Af)^* = f^* \circ A^*$;
(vi) $(f \Box g)^* = f^* + g^*$.

The next result is well known.

Proposition 1.21 *Suppose that $f : X \to \overline{\mathbb{R}}$ is lsc and convex. Then f^* is also lsc and convex, and $f^{**} = f$.*

Let us close this subsection by the fundamental result in duality theory:

Proposition 1.22 *Suppose $f : X \to \overline{\mathbb{R}}$ is a function such that* $\operatorname{dom} f \neq \emptyset$.

(i) If $\operatorname{cl}(\operatorname{co} f)$ is proper, then $f^{**} = \operatorname{cl}(\operatorname{co} f)$, otherwise $f^{**} = -\infty$.

(ii) Assume that f is convex. If f is lsc at $x_0 \in \operatorname{dom} f$, then $f(x_0) = f^{**}(x_0)$; moreover, if $f(x_0) \in \mathbb{R}$, then $f^{**} = \operatorname{cl} f$ and $\operatorname{cl} f$ is proper.

Note that according to the previous proposition we always have $f^* = f^{***}$.

1.2.3 The Subdifferential

In this subsection X is Hausdorff LCS and $f : X \to \overline{\mathbb{R}}$ is a function. If $f(x) \in \mathbb{R}$, then the *subdifferential* of f at x is the set $\partial f(x)$ of all $x^* \in X^*$ satisfying

$$\langle x^*, y - x \rangle \leq f(y) - f(x).$$

When $f(x) \notin \mathbb{R}$ we define $\partial f(x) = \emptyset$. We say that f is subdifferentiable at x if $\partial f(x) \neq \emptyset$. Note that ∂f is a set-valued map from X to X^*. Generally, the elements of the subdifferential of f at x are called *subgradients* of f at x.

The following theorem contains some elementary properties of ∂f.

Theorem 1.23 *Let $f : X \to \overline{\mathbb{R}}$ and $x_0 \in X$ be such that $f(x_0) \in \mathbb{R}$. Then:*

(i) $\partial f(x_0)$ is a weak-closed and convex subset (maybe empty) of X^*;*

(ii) if $\partial f(x) \neq \emptyset$, then $\operatorname{cl}(\operatorname{co} f)(x_0) = \operatorname{cl}(f)(x_0) = f(x_0)$ and

$$\partial(\operatorname{cl}(\operatorname{co} f)(x_0)) = \partial(\operatorname{cl}(f)(x_0)) = \partial(f(x_0));$$

(iii) if f is proper, $\operatorname{dom} f$ is a convex set and f is subdifferentiable at each $x \in \operatorname{dom} f$, then f is convex.

One can easily check that equality in the Young-Fenchel inequality holds if and only if $x^* \in \partial f(x)$, i.e.,

$$x^* \in \partial f(x) \Leftrightarrow f(x) + f^*(x^*) = \langle x^*, x \rangle.$$

The following result is due essentially to Ioffe-Tikhomirov and it is a very important calculus rule for the subdifferential of supremum.

Theorem 1.24 *Suppose that $(A, \mathcal{T})$ is a Hausdorff compact topological space and $f_\alpha : X \to \overline{\mathbb{R}}$ is a convex function for every $\alpha \in A$. Consider the function $f := \sup_{\alpha \in A} f_\alpha$ and $F(x) := \{\alpha \in A : f_\alpha(x) = f(x)\}$. Assume that the mapping $A \ni \alpha \longmapsto f_\alpha(x) \in \overline{\mathbb{R}}$ is usc and $x_0 \in \operatorname{dom} f$ is such that f_α is continuous at x_0 for every $\alpha \in A$. Then*

$$\partial f(x_0) = \operatorname{cl} \operatorname{co} \left(\cup_{\alpha \in F(x_0)} \partial f_\alpha(x_0) \right).$$

There are many interesting results and discussions about the different kind of subdifferentials and abstract subdifferential in [62].

From the definition of subdifferential we conclude that if $f, g : X \to \overline{\mathbb{R}}$ are proper, lsc and convex, then $\partial f(x) + \partial g(x) \subset \partial (f + g)(x)$. But the converse is not true in general (even in Banach spaces).

Proposition 1.25 *Suppose that Y is a Banach space and $f, g : Y \to \overline{\mathbb{R}}$ are convex and $0 \in \operatorname{core}(\operatorname{dom} f - \operatorname{dom} g)$. Then*

$$\partial f + \partial g = \partial (f + g).$$

Proof. See [25, Corollary 2.5]. ■

1.2.4 Tangent and Normal Cones

We begin with some basic definitions and results.

In this subsection X is Hausdorff LCS and K is a nonempty subset of X. The function $\iota_K : X \to \mathbb{R} \cup \{+\infty\}$ defined by

$$\iota_K(x) := \begin{cases} 0 & \text{if } x \in K, \\ +\infty & \text{otherwise} \end{cases}$$

is called the *indicator function* of K.

Definition 1.26 *Let $C \subset X$. The* support function *of the set C is the function $\sigma_C : X^* \to \overline{\mathbb{R}}$ defined by*

$$\sigma_C(x^*) = \sup_{c \in C} \langle x^*, c \rangle$$

(recall that $\sup \emptyset = -\infty$).

Evidently if $C \subset X$ is nonempty, then σ_C is lsc and convex and $\sigma_C(0) = 0$. In fact σ_C is sublinear (i.e., subadditive and positively homogeneous). Moreover,

$$\sigma_C = (\iota_C)^*.$$

Note that a nonempty subset C of a real vector space is called a *cone* if $x \in C$ and $\lambda \geq 0$ imply $\lambda x \in C$.

Definition 1.27 *Let X be Hausdorff LCS and K a nonempty subset of X. The normal cone of K at $x \in X$ is the set $N_K(x)$ defined by*

$$N_K(x) = \begin{cases} \{x^* \in X^* : \langle x^*, y - x \rangle \leq 0 \quad \forall y \in K\} & \text{if } x \in K, \\ \emptyset & \text{otherwise.} \end{cases}$$

This defines a set-value map $N_K : X \to 2^{X^*}$. The following proposition is an immediate consequence of the above definition and Theorem 1.23.

Proposition 1.28 *For a nonempty, closed, and convex $K \subset X$, the following statements hold:*

(i) $N_K = \partial \iota_K$;

(ii) $N_K(x)$ *is weak***-closed and convex subset of* X^* *for all* $x \in X$*;*

(iii) $N_K(x)$ *is a cone for all* $x \in K$*.*

For a nonempty subset K of X, the *polar cone* of K is the subset $\overset{\circ}{K}$ of X^* defined by

$$\overset{\circ}{K} = \{x^* \in X^* : \langle x^*, x \rangle \leq 0 \quad \forall x \in K\}.$$

The *antipolar cone* of $F \subset X^*$ is the subset $\overset{\diamond}{F}$ of X defined by

$$\overset{\diamond}{F} = \{x \in X : \langle x^*, x \rangle \leq 0 \quad \forall x^* \in F\}.$$

Also, the *tangent cone* is defined as the antipolar cone of the normal cone and denoted by T_K. More precisely, $T_K : X \to 2^X$ is defined by

$$T_K(x) = \overset{\diamond}{N}_K(x) = \{y \in X : \langle x^*, y \rangle \leq 0 \quad \forall x^* \in N_K(x)\}.$$

Note that when Y is a reflexive Banach space, we have $\overset{\circ}{K} = \overset{\diamond}{K}$. In fact, $\overset{\circ}{K} \subset Y^{**} = Y$.

In order to introduce a convenient characterization of tangent cone, we assume that Z is normed space and S is a nonempty subset of Z.

Definition 1.29 *[72, page 82]* (i) *Let $\bar{x} \in \operatorname{cl} S$ be a given element. A vector $h \in Z$ is called a* tangent vector *to S at $\bar{x}$ if there are a sequence $\{x_n\}$ in S and a sequence $\{\lambda_n\}$ of positive real numbers with*

$$\lim_{n\to\infty} x_n = \bar{x} \qquad \text{and} \qquad h = \lim_{n\to\infty} \lambda_n (x_n - \bar{x}).$$

(ii) *The set $T(S, \bar{x})$ of all tangent vectors to S at $\bar{x}$ is called* sequential Bouligand tangent cone *to S at $\bar{x}$ or* contingent cone *to S at $\bar{x}$.*

By the definition of tangent vectors it follows immediately that the contingent cone is in fact a cone.

The *Clarke tangent cone* to S at $\bar{x} \in \operatorname{cl} S \subset Z$ is defined as the set $T_{Cl}(\bar{x}, S)$ of all vectors $h \in Z$ with the following property: for every sequence $\{x_n\}$ in S with $\lim_{n\to\infty} x_n = \bar{x}$ and every sequence $\{\lambda_n\}$ in $\mathbb{R}$ with $\lambda_n \to 0, \lambda_n > 0$, there is a sequence $\{h_n\}$ in X with $\lim_{n\to\infty} h_n = h$ and $x_n + \lambda_n h_n \in S$ for all $n \in \mathbb{N}$.

It is evident that the Clarke tangent cone $T_{Cl}(\bar{x}, S)$ is always a cone. Note that If $\bar{x} \in S$, then the Clarke tangent cone $T_{Cl}(\bar{x}, S)$ is contained in the contingent cone $T(S, \bar{x})$. The Clarke tangent cone $T_{Cl}(\bar{x}, S)$ is always a closed convex cone [42]. Also, If $\bar{x} \in S$, then the contingent cone is closed and $T_{Cl}(\bar{x}, S) \subset T(S, \bar{x})$ [72, pages 82 and 83].

Proposition 1.30 *Let S be a nonempty subset of a real normed space. If the set S is starshaped with respect to some $\bar{x} \in S$, then*

$$T(S, \bar{x}) = \mathrm{cl}(\mathrm{cone}\, S \setminus \{\bar{x}\}).$$

Proof. See [72, Chapter 4, page 87]. ■

1.3 Monotone Operators

In this section we will focus on monotone and maximal monotone operators and we will point out the connection between subdifferentials of lsc and convex functions and maximal monotone operators. In particular, we are interested in analyzing when the sum of two maximal monotone operators is maximal monotone. Also, we will introduce the Fitzpatrick function and we will observe the connection between maximal monotone operators and convex functions in reflexive and not necessarily reflexive Banach spaces. The basic tools we will use are the Fitzpatrick and Penot functions.

1.3.1 Monotone and Maximal Monotone Operators

Let X be Hausdorff LCS. A multivalued operator from X to X^* is simply a map $T : X \to 2^{X^*}$. The domain, range and graph of T are, respectively, defined by

$$\mathrm{dom}\, T = \{x \in X : T(x) \neq \emptyset\}, \; R(T) = \{x^* \in X^* : \exists x \in X; x^* \in T(x)\},$$

$$\mathrm{gr}\, T = \{(x, x^*) \in X \times X^* : x \in \mathrm{dom}\, T \text{ and } x^* \in T(x)\}.$$

For a given operator T, the inverse operator $T^{-1} : X^* \to 2^{X^{**}}$ is defined by means of its graph:

$$\mathrm{gr}\, T^{-1} := \{(x^*, x) \in X^* \times X^{**} : (x, x^*) \in \mathrm{gr}\, T\}.$$

For two multivalued operators T and S we say that S is an extension of T and write $T \subset S$ if $\mathrm{gr}\, T \subset \mathrm{gr}\, S$.

Definition 1.31 *A set $M \subset X \times X^*$ is*

(i) monotone *if $\langle y^* - x^*, y - x \rangle \geq 0$ whenever $(x, x^*) \in M$ and $(y, y^*) \in M$;*

(ii) strictly monotone *if $\langle y^* - x^*, y - x \rangle > 0$ whenever $(x, x^*) \in M$ and $(y, y^*) \in M$ and $x \neq y$;*

(iii) maximal monotone *if it is monotone and it is not properly included in any other monotone subset of $X \times X^*$. That is, if M_1 is a monotone subset of $X \times X^*$ and $M \subset M_1$, then $M = M_1$.*

We say that an element $(x, x^*) \in X \times X^*$ is *monotonically related* to M if $\langle y^* - x^*, y - x \rangle \geq 0$ for all $(y, y^*) \in M$.

In the next definition, we will formulate the definition of monotone operators in terms of their graphs. We remind first that a finite sequence $x_1, x_2, ..., x_{n+1}$ such that $x_{n+1} = x_1$ is called a *cycle*.

Definition 1.32 *An operator $T : X \to 2^{X^*}$ is called*

(i) monotone *if* $\operatorname{gr} T$ *is monotone;*

(ii) if $\operatorname{gr} T$ *is maximal monotone;*

(iii) cyclically monotone*, if for every cycle $x_1, x_2, ..., x_{n+1} = x_1$ in X and each $x_i^* \in T(x_i)$ for $i = 1, ..., n$,*

$$\sum_{i=1}^{n} \langle x_i^*, x_{i+1} - x_i \rangle \leq 0;$$

(iv) maximal cyclically monotone *if it is cyclically monotone and its graph cannot be enlarged without destroying this property, i.e., whenever T_1 is a cyclically monotone map such that $T \subset T_1$, then $T = T_1$.*

We also say that an operator T is *strictly monotone* if $\operatorname{gr} T$ is strictly monotone.

According to the above definitions, if T is maximal monotone and (x, x^*) in $X \times X^*$ is monotonically related to $\operatorname{gr} T$, then $x \in \operatorname{dom} T$ and $x^* \in T(x)$. By applying the Zorn's lemma, we can extend every monotone operator T to a maximal monotone operator $\tilde{T}$. One can easily check that T is (maximal) monotone if and only if T^{-1} is.

An direct consequence of the definition of maximal monotone operators is the following.

Proposition 1.33 *Let Y be a Banach space. If $T : Y \to 2^{Y^*}$ is maximal monotone, then $T(y)$ is convex and weak*-closed.*

It is straightforward to see that ∂f is cyclically monotone when f is proper, lsc and convex. We borrow the following two theorems from [99].

Theorem 1.34 *[99, Theorem A] Suppose that Y is a Banach space. Then the subdifferential of every proper, lsc and convex function is maximal monotone.*

Theorem 1.35 *[99, Theorem B] Suppose that Y is a Banach space and $T : Y \to 2^{Y^*}$ is an operator. In order that there exist a proper, lsc and convex function f on Y such that $T = \partial f$, it is necessary and sufficient that T be a maximal cyclically monotone operator. Moreover, in this case T determines f uniquely up to an additive constant.*

Proposition 1.36 *Suppose that X is a Hausdorff LCS, and $T : X \to 2^{X^*}$ is cyclically monotone and $(x_0, x_0^*) \in \operatorname{gr} T$. Define $f_T : X \to \overline{\mathbb{R}}$ by*

$$f_T(x) := \sup \left(\langle x_n^*, x - x_n \rangle + \sum_{i=0}^{n-1} \langle x_i^*, x_{i+1} - x_i \rangle \right)$$

where the supremum is taken for all families $(x_i, x_i^*) \in \operatorname{gr} T$, for $n \in \mathbb{N}$ and $i = 1, ..., n$. Then f_T is proper, lsc and convex, $f_T(x_0) = 0$ and $T(x) \subset \partial(f_T(x))$ for each x in X.

Proof. See [120, Proposition 2.4.3] or [98, Theorem 24.8]. ■

The multifunction $\mathcal{J}(\cdot) := \partial(\frac{1}{2}\|\cdot\|^2) : Y \to 2^{Y^*}$ is called the *duality mapping* of Y. The following holds

$$\mathcal{J}(x) = \left\{x^* \in X^* : \langle x^*, x\rangle = \|x\|^2 = \|x^*\|^2\right\}.$$

Note that since $(\frac{1}{2}\|\cdot\|^2)$ is proper, lsc and convex, $\mathcal{J}$ is maximal monotone. When Y is a Hilbert space, then $\mathcal{J} = I$, the identity mapping, and hence is onto. Also it is well known that $\mathcal{J}$ is onto if and only if Y is reflexive (see [40, Theorem 3.4]).

Minty has proved a noteworthy theorem in [85] for Hilbert spaces, which states that T is maximal monotone if and only if $R(T + \mathcal{J}) = Y^*$. Rockafellar extended this result to reflexive Banach spaces for which both $\mathcal{J}$ and $\mathcal{J}^{-1}$ are single-valued, in which case $\|\cdot\|^2$ is differentiable. This result is commonly known as Rockafellar's characterization of maximal monotone operators.

We now give the definition of local boundedness and some results on this notion.

Definition 1.37 *Let X be a Hausdorff LCS and $T : X \to 2^{X^*}$ be an operator, T is called* locally bounded *at x_0 if there exists a neighborhood U of x_0 such that the set*

$$T(U) = \cup\{T(x) : x \in U\}$$

is an equicontinuous subset of X^*.

Note that when Y is a Banach space, then the equicontinuous subsets of Y^* coincide with bounded subsets. In other words, when Y is a Banach space, then an operator T is called locally bounded at $x_0 \in Y$ if there exist $\varepsilon > 0$ and $k > 0$ such that $\|x^*\| \le k$ for all $x^* \in T(x)$ and $x \in B(x_0, \varepsilon)$.

Next theorem is due to Rockafellar and states that monotone operators are locally bounded at each point of the interior of their domain.

Theorem 1.38 *[97] Suppose that Y is a Banach space, $T : Y \to 2^{Y^*}$ is maximal monotone and that* $\operatorname{int}(\operatorname{co}(\operatorname{dom} T))$ *is nonempty. Then* $\operatorname{int}(\operatorname{dom} T) = \operatorname{int}(\operatorname{co}(\operatorname{dom} T))$ *(so* $\operatorname{int}(\operatorname{dom} T)$ *is convex) and T is locally bounded at each point of* $\operatorname{int}(\operatorname{dom} T)$*. Moreover,* $\operatorname{cl}\operatorname{dom} T = \operatorname{cl}(\operatorname{int}(\operatorname{dom} T))$*, hence it is also convex.*

Assume that X is a Hausdorff LCS. Let T and S be two operators on X and $\lambda > 0$. For $x \in \operatorname{dom} T$ we define $(\lambda T)(x) = \lambda \cdot T(x)$ and also, for $x \in \operatorname{dom}(T \cap S)$

$$(T + S)(x) = T(x) + S(x) = \{x_1^* + x_2^* : x_1^* \in T(x), x_2^* \in S(x)\},$$

while if $x \notin \operatorname{dom}(T \cap S)$, we set $(T + S)(x) = \emptyset$. Thus $\operatorname{dom} \lambda T = \operatorname{dom} T$ and $\operatorname{dom}(T + S) = \operatorname{dom} T \cap \operatorname{dom} S$. One can check that if T and S are monotone, then λT and $T + S$ are also monotone, and λT is maximal monotone whenever T is.

The next theorem shows that maximal monotone operators are not locally bounded at any point of the boundary of their domains.

Theorem 1.39 *Suppose that* Y *is a Banach space and* $T : Y \to 2^{Y^*}$ *is maximal monotone. If* $\operatorname{int}\operatorname{cl}\operatorname{dom} T \neq \emptyset$*, then for all* $z \in \operatorname{dom} T \backslash (\operatorname{int}\operatorname{cl}\operatorname{dom} T)$

(i) there exists a non-zero $z^* \in N_{\operatorname{dom} T}(z)$*;*

(ii) $T(z) + N_{\operatorname{dom} T}(z) \subset T(z)$*;*

(iii) T *is not locally bounded at* z*.*

Proof. A proof can be found in [35, Theorem 4.2.10]. ■

Note that Property (ii) above holds for all $z \in \operatorname{dom} T$, and does not need the assumption $\operatorname{int}\operatorname{cl}\operatorname{dom} T \neq \emptyset$.

There is a kind of converse of Theorem 1.38, due to Libor Veselý, that we now remind. This result is interesting because it does not assume anything about the nonemptiness of interiors.

Theorem 1.40 ***(Libor Veselý)*** *Suppose that* Y *is a Banach space and* $T : Y \to 2^{Y^*}$ *is maximal monotone. If* $y \in \operatorname{cl}\operatorname{dom} T$ *and* T *is locally bounded at* y*, then* $y \in \operatorname{dom} T$*. If in addition* $\operatorname{cl}\operatorname{dom} T$ *is convex, then* $y \in \operatorname{int}(\operatorname{dom} T)$*.*

Proof. See Phelps [92, Theorem 1.14] . ■

Proposition 1.41 *Let* Y *be a Banach space and* $T : Y \to Y^*$ *a single-valued monotone operator such that* $\operatorname{int}(\operatorname{co}\operatorname{dom} T) \neq \emptyset$*. If* T *is maximal, then* $\operatorname{dom} T$ *is open and* T *is continuous with respect to the norm topology in* Y *and the weak***-topology in* Y^* *at every point of* $\operatorname{dom} T$*.*

Proof. See [35, Theorem 4.6.4]. ■

We now mention a few results that concern the sum of monotone operators.

Theorem 1.42 *Let* Y *be a Banach space and let* $S, T : Y \to 2^{Y^*}$ *be monotone operators. Suppose that*

$$0 \in \operatorname{core}[\operatorname{co}\operatorname{dom} T - \operatorname{co}\operatorname{dom} S].$$

Then there exist $r, c > 0$ *such that, for each* $y \in \operatorname{dom} T \cap \operatorname{dom} S$*,* $t^* \in T(y)$ *and* $s^* \in S(y)$*,*

$$\max(||t^*||, ||s^*||) \leq c(r + ||y||)(r + ||t^* + s^*||).$$

Proof. A proof can be found in [113] or [25, Theorem 2.11]. ■

We recall that an operator T on a Banach space Y is said to be norm×weak*-closed (respectively, sequentially norm×weak*-closed) if $\operatorname{gr} T$ is closed (respectively, sequentially closed) in the norm×weak*-topology of $Y \times Y^*$. Borwein, Fitzpatrick and Girgensohn in [29] proved that, in general, $\operatorname{gr} T$ is only sequentially norm×weak*-closed, not norm×weak*-closed.

Proposition 1.43 *Let* Y *be any Banach space and let* $S, T : Y \to 2^{Y^*}$ *be maximal monotone operators. Suppose that*

$$0 \in \operatorname{core}[\operatorname{co}\operatorname{dom} T - \operatorname{co}\operatorname{dom} S].$$

For any $y \in \operatorname{dom} T \cap \operatorname{dom} S$*,* $T(y) + S(y)$ *is a weak***-closed subset of* Y^**.*

Proof. See [113]. ■

Proposition 1.44 *Suppose that Y is a reflexive Banach space and T is maximal monotone. Then the mapping $T + \mathcal{J}$ is surjective. i.e., $R(T + \mathcal{J}) = Y^*$.*

Proof. See [104, Theorem 10.7]. ■

Proposition 1.45 *Suppose that Y is a reflexive Banach space and T is monotone. If $R(T + \mathcal{J}) = Y^*$ and $\mathcal{J}$ and $\mathcal{J}^{-1}$ are both single-valued, then T is maximal monotone.*

Proof. See [104, Remark 10.8 and pages 38, 39]. ■

We observed in this section that if T and S are two monotone operators on X and $\lambda > 0$, then λT and $T + S$ are monotone, and λT is maximal monotone whenever T is. However, the sum of two maximal monotone operators is not maximal monotone in general. So the natural question regarding maximal monotone operators is, which conditions guarantee that the sum of two of them remains maximal monotone. These conditions concern the mutual position of their domains and are called constraint qualifications (CQ, from now on). Here we list some of these CQ (see also [57] and [118]):

(i) $(\operatorname{int} \operatorname{dom} T) \cap \operatorname{dom} S \neq \emptyset$ (The original one due to Rockafellar. See [100]);

(ii) $\operatorname{dom} S - \operatorname{dom} T$ is absorbing (due to Attouch, Riahi and Thera. See [9] and [104]);

(iii) $\operatorname{co} \operatorname{dom} S - \operatorname{co} \operatorname{dom} T$ is a neighborhood of 0 (due to Chu. See [41]);

(iv) $\operatorname{dom} S - \operatorname{dom} T$ is surrounding 0 (for the definition of surround point see [104]);

(v) $\operatorname{co} \operatorname{dom} S - \operatorname{co} \operatorname{dom} T$ is absorbing;

(vi) $\operatorname{dom} \chi_S - \operatorname{dom} \chi_T$ is absorbing (for the definition of χ_T see [104]).

Simons ([104]) proved that, in reflexive Banach spaces, all six (CQ) which are mentioned above are equivalent.

Theorem 1.46 *Let Y be a reflexive Banach space. Let T be maximal monotone and let f be closed and convex. Suppose that*

$$0 \in \operatorname{core}[\operatorname{co} \operatorname{dom} T - \operatorname{co} \operatorname{dom}(\partial f)].$$

Then

(i) $\partial f + T + \mathcal{J}$ is surjective.

(ii) $\partial f + T$ is maximal monotone.

(iii) ∂f is maximal monotone.

Proof. See [25, Theorem 4.2]. ■

An important consequence of preceding theorem is:

Corollary 1.47 *The sum of a maximal monotone operator T and a normal cone N_C on a reflexive Banach space, is maximal monotone whenever the transversality condition $0 \in \operatorname{core}\{C - \operatorname{co} \operatorname{dom} T\}$ holds.*

1.3.2 Fitzpatrick Function

The Fitzpatrick function [52], Krauss function [79, 80, 81] and the family of enlargements by Burachik, Svaiter [38], and Penot function [88] make a bridge between the results on convex functions and results on maximal monotone operators.
Let us start with the definition of Fitzpatrick function.

Definition 1.48 *Let Y be a Banach space and $T : Y \to 2^{Y^*}$ be a maximal monotone operator. The* Fitzpatrick function *associated with T is the function $\mathcal{F}_T : Y \times Y^* \to \mathbb{R} \cup \{+\infty\}$ defined by*

$$\mathcal{F}_T(x, x^*) = \sup_{(y,y^*)\in \operatorname{gr} T} (\langle x^*, y\rangle + \langle y^*, x\rangle - \langle y^*, y\rangle)$$

.

The Fitzpatrick function is norm×weak*- lsc and convex on $Y \times Y^*$. It can be easily verified that

$$\begin{aligned}\mathcal{F}_T(x, x^*) &= \sup_{(y,y^*)\in \operatorname{gr} T} \langle y^* - x^*, x - y\rangle + \langle x^*, x\rangle \\ &= \langle x^*, x\rangle - \inf_{(y,y^*)\in \operatorname{gr} T} \langle y^* - x^*, y - x\rangle .\end{aligned}$$

Theorem 1.49 *Let Y be a Banach space. For a maximal monotone operator $T : Y \to 2^{Y^*}$ one has*

$$\mathcal{F}_T(x, x^*) \geq \langle x^*, x\rangle. \tag{1.2}$$

with equality if and only if $x^* \in T(x)$. Actually, the equality $\mathcal{F}_T(x, x^*) = \langle x, x^*\rangle$ for all $x^* \in T(x)$, requires only monotonicity, not maximality.

Proof. See [52] or [25, Proposition 2.1]. ■

Let X be a LCS and T any monotone operator on X. A *representative function* for T is any function $\mathcal{H}_T : X \times X^* \to \mathbb{R} \cup \{+\infty\}$ such that

(i) $\mathcal{H}_T$ is lsc and convex;

(ii) $\mathcal{H}_T(x, x^*) \geq \langle x^*, x\rangle$, for all $(x, x^*) \in X \times X^*$;

(iii) $\mathcal{H}_T(x, x^*) = \langle x^*, x\rangle$, when $x^* \in T(x)$.

A representative is called *exact* if $\mathcal{H}_T(x, x^*) = \langle x^*, x\rangle$ exactly on the $\operatorname{gr} T$.

The *Penot function* [88] is defined on $Y \times Y^*$ by

$$\mathcal{P}_T(x, x^*) = \inf \left\{ \sum_{i=1}^{N} \lambda_i \langle x_i^*, x_i\rangle : \sum_{i=1}^{N} \lambda_i (x_i, x_i^*, 1) = (x, x^*, 1), x_i^* \in T(x_i), \lambda_i \geq 0 \right\}$$

One can easily check that $\mathcal{P}_T$ is convex and $\mathcal{P}_T(x, x^*) \geq \langle x^*, x\rangle$, for all x^* in $T(x)$. Moreover, it was shown in [88, 25] $\mathcal{P}_T^* = \mathcal{F}_T$.

We can combine the recent result with Theorem 1.20 and Proposition 1.22, and conclude that

$$(\mathcal{F}_T)^* = \operatorname{cl}\mathcal{P}_T = (\mathcal{P}_T)^{**}.$$

The theorem we present below can be found in [26] and [88].

Proposition 1.50 *Suppose that Y is a Banach space and T is a monotone operator on Y. Then*

(i) Penot's function $\mathcal{P}_T$ represents T;

(ii) if $\mathcal{H}_T$ represents T, then $\mathcal{H}_T \leq \operatorname{cl}\mathcal{P}_T$ pointwise;

(iii) if T is maximal monotone and $\mathcal{H}_T$ represents T, then $\mathcal{F}_T \leq \mathcal{H}_T \leq \operatorname{cl}\mathcal{P}_T$;

(iv) $\mathcal{F}_T(x, x^*) \geq \langle x^*, x\rangle$ if and only if (x, x^*) is monotonically related to $\operatorname{gr} T$;

(v) Assume that $\mathcal{F}_T$ represents T. Then $\mathcal{F}_T(x, x^*) = \langle x^*, x\rangle$ if and only if $\operatorname{cl}\mathcal{P}_T(x, x^*) = \langle x^*, x\rangle$.

We remark that $\mathcal{F}_T$ is not necessarily a representative function of T whenever T is not maximal monotone.

Next two theorems were shown by using the Fitzpatrick function, and generalize the celebrated Rockafellar sum theorems to general Banach spaces (with somewhat stronger assumptions). The following theorems are taken from [26] see also [114].

Theorem 1.51 ***(Maximality of sums, I)****. Let T and S be maximal monotone operators on a Banach space Y. Suppose also that either*

(i) $\operatorname{int}\operatorname{dom} T \cap \operatorname{int}\operatorname{dom} S \neq \emptyset$; or

(ii) $\operatorname{dom} T \cap \operatorname{int}\operatorname{dom} S \neq \emptyset$ while $\operatorname{dom} T \cap \operatorname{dom} S$ is closed and convex; or

(iii) both $\operatorname{dom} T, \operatorname{dom} S$ are closed and convex and $0 \in \operatorname{core}\operatorname{co}(\operatorname{dom} T - \operatorname{dom} S)$.

Then $T + S$ is maximal monotone.

Proof. See [26, Theorem 9]. ■

Theorem 1.52 ***(Maximality of sums, II)****. Let T and S be maximal monotone on a Banach space Y. Suppose also that* $\operatorname{core}\operatorname{co}\operatorname{dom} T \cap \operatorname{core}\operatorname{co}\operatorname{dom} S \neq \emptyset$*. Then $T + S$ is maximal monotone.*

Proof. See [26, Theorem 10]. ■

Chapter 2

Bifunctions

In this chapter, which is based on [5], we exhibit some correspondences between monotone operators and monotone bifunctions. Also, we establish new connections between maximal monotone operators and maximal monotone bifunctions. Most notably, we will prove that under weak assumptions, monotone bifunctions are locally bounded in the interior of the convex hull of their domain. As an immediate consequence, we get the corresponding property for monotone operators. Moreover, we show that in contrast to maximal monotone operators, monotone bifunctions (maximal or not maximal) can also be locally bounded at the boundary of their domain.

This chapter is organized as follows: In the next section, we define maximal monotonicity of bifunctions, and we present some preliminary definitions, properties and results. A part of our results is inspired by some analogous results from [64]. We will show in Section 2 that under very weak assumptions, local boundedness of monotone bifunctions is automatic at every point of $\operatorname{int} C$. In this way one can obtain an easy proof of the corresponding property of monotone operators. Moreover, in Section 3 we define and study cyclically monotone bifunctions. We prove that in any LCS a bifunction F is cyclically monotone, if and only if there exists a function $f : C \to R$ such that $F(x, y) \leq f(y) - f(x)$ for all $x, y \in C$. Especially, by assuming that F is maximal monotone and $\operatorname{int} C \neq \emptyset$, we get that f is convex on $\operatorname{int} C$ and uniquely defined up to a constant. In addition, we will show in Section 4 that monotone bifunctions are in some ways better behaved that the underlying monotone operators, since they can be locally bounded even at the boundary of their domain of definition. In contrast to this, it is known that maximal monotone operators T whose domain $\operatorname{dom} T$ has nonempty interior are never locally bounded at the boundary of $\operatorname{dom} T$. In fact, we will show that in $\mathbb{R}^n$ and for locally polyhedral domains C, an automatic local boundedness of bifunctions holds on the whole domain. We also show that each monotone operator is “inward locally bounded” at every point of the closure of its domain, a property which collapses to ordinary local boundedness at interior points of the domain. In Section 5, we collect some noteworthy counterexamples.

2.1 Monotone Bifunctions and Equilibrium Problems

In this section X is a TVS (unless explicitly stated otherwise) and C is a nonempty subset of X. By bifunction, in this chapter, we mean any function $F : C \times C \to \mathbb{R}$.

Definition 2.1 *A bifunction $F : C \times C \to \mathbb{R}$ is called* monotone *if*

$$F(x, y) + F(y, x) \leq 0 \quad \textit{for all } x, y \in C.$$

A direct consequence of the above definition is that $F(x, x) \leq 0$ for all $x \in C$. Also a bifunction $F : C \times C \to \mathbb{R}$ is called *strictly monotone* if

$$F(x, y) + F(y, x) < 0 \quad \text{for all } x, y \in C,\ x \neq y.$$

It should be noticed that in many papers, it is supposed that

$$F(x, x) = 0 \quad \text{for all } x \in C. \tag{2.1}$$

Monotone bifunctions were mainly studied in conjunction with the so-called equilibrium problem: Find $x_0 \in C$ such that

$$F(x_0, y) \geq 0 \quad \text{for all } y \in C.$$

In this case, such a point $x_0 \in C$ is called a solution of the equilibrium problem. The literature on equilibrium problems is quite extensive. Equilibrium problems were studied in many papers (see [23, 7, 8, 22, 64, 54, 71, 69, 75, 77, 78, 86] and the references therein), after Blum and Oettli showed in their highly influencing paper [23] that equilibrium problems include variational inequalities, fixed point problems, saddle point problems etc. In some of these papers [1, 8, 86] monotone bifunctions were related to monotone operators (see the next section for details) and maximal monotonicity of bifunctions was defined and studied. In [64] some results on maximal monotonicity of bifunctions were deduced assuming that the bifunction is locally bounded, i.e. its values are bounded from above for all x, y in a suitable neighborhood of each point of C or $\operatorname{int} C$.

The solution set of an equilibrium problem is the set $\mathrm{EP}(F)$ defined by

$$\mathrm{EP}(F) = \{z \in C : F(z, y) \geq 0 \quad \forall y \in C\}.$$

Assume that $F : C \times C \to \mathbb{R}$ is a bifunction. Following [1, 5, 23, 64], the operator $A^F : X \to 2^{X^*}$ is defined by

$$A^F(x) = \begin{cases} \{x^* \in X^* : \forall y \in C, F(x, y) \geq \langle x^*, y - x\rangle\} & \text{if } x \in C, \\ \emptyset & \text{if } x \notin C. \end{cases} \tag{2.2}$$

The following proposition will illustrate this concept further.

Proposition 2.2 *Let $F : C \times C \to \mathbb{R}$ be a monotone bifunction. Then the following statements are equivalent:*
(i) $z^* \in A^F(x)$;
(ii) $\langle z^*, x\rangle = \sup_{y\in C}(\langle z^*, y\rangle - F(x,y))$.

Proof. (i)$\Longrightarrow$(ii) Suppose that $z^* \in A^F(x)$. Then $F(x,y) \geq \langle z^*, y-x\rangle$ for all $y \in C$. Therefore

$$\langle z^*, x\rangle \geq \langle z^*, y\rangle - F(x,y) \quad \forall y \in C. \tag{2.3}$$

By taking the supremum from (2.3) we obtain

$$\langle z^*, x\rangle \geq \sup_{y\in C}(\langle z^*, y\rangle - F(x,y)). \tag{2.4}$$

Putting $y = x$ in (2.3) and taking in account that $F(x,x) \leq 0$, we deduce $F(x,x) = 0$. This together with (2.4) imply (ii).

(ii) $\Longrightarrow$ (i) Assume that (ii) holds. Then we have (2.3). This implies that $F(x,y) \geq \langle z^*, y-x\rangle$ for all $y \in C$. Hence $z^* \in A^F(x)$. ■

The following definition of maximality was used in [64] for reflexive Banach spaces. Now we redefine it for TVS.

Definition 2.3 *A monotone bifunction F is called* maximal monotone *if A^F is maximal monotone.*

The following remark presents some elementary properties of the multifunction A^F.

Remark 2.4 (i) If F is a monotone bifunction, then A^F is a monotone operator. Indeed, assume that $x, y \in C$ and $x^* \in A^F(x)$ and $y^* \in A^F(y)$. Then

$$F(x,y) \geq \langle x^*, y-x\rangle$$

and

$$F(y,x) \geq \langle y^*, x-y\rangle.$$

By adding the two inequalities we obtain

$$\langle y^* - x^*, y-x\rangle \geq -F(x,y) - F(y,x) \geq 0.$$

This means that A^F is monotone.

(ii) If F is monotone and $x^* \in A^F(x)$, then $F(x,x) = 0$. From monotonicity of F we get $F(x,x) \leq 0$. On the other hand $x^* \in A^F(x)$ which implies that

$$F(x,x) \geq \langle x^*, x-x\rangle = 0.$$

Thus $F(x,x) = 0$.

(iii) For each $x \in C$, $A^F(x)$ is convex. Let $x_1^*, x_2^* \in A^F(x)$ and $\lambda \in [0,1]$. Then for all $y \in C$, we have

$$\langle \lambda x_1^* + (1-\lambda)x_2^*, y-x\rangle = \lambda\langle x_1^*, y-x\rangle + (1-\lambda)\langle x_2^*, y-x\rangle \leq F(x,y).$$

This implies that $\lambda x_1^* + (1-\lambda) x_2^* \in A^F(x)$.

(iv) For each $x \in C$, $A^F(x)$ is weak*-closed. We will show that $X^* \backslash A^F(x)$ is weak*-open. Assume that $y^* \in X^* \backslash A^F(x)$. Then there exists $y_0 \in X$ with $\langle y^*, y_0 - x\rangle > F(x, y_0)$. Choose $t \in \mathbb{R}$ such that $\langle y^*, y_0 - x\rangle > t > F(x, y_0)$. Set $U = \{x^* \in X^* : \langle x^*, y_0 - x\rangle > t\}$. Then U is a nonempty neighborhood of y^* in weak*- topology, which does not meet $A^F(x)$. Therefore, $X^* \backslash A^F(x)$ is weak*-open.

(v) As it was remarked in [64], if we define an extension $\hat{F}$ of F on $C \times X$ by

$$\hat{F}(x, y) = \begin{cases} F(x, y) & \text{if } y \in C, \\ +\infty & \text{if } y \in X \backslash C, \end{cases}$$

then $A^F(x) = \partial \hat{F}(x, \cdot)(x)$ for all $x \in C$.

(vi) Suppose that $F_1, F_2 : C \times C \to \mathbb{R}$ are two bifunctions and t, s are two positive real numbers. Then $\left(tA^{F_1} + sA^{F_2}\right)(x) \subset A^{tF_1 + sF_2}(x)$ for each $x \in C$. If $x^* \in \left(tA^{F_1} + sA^{F_2}\right)(x)$, then $x^* = x_1^* + x_2^*$ where $x_1^* \in tA^{F_1}(x)$ and $x_2^* \in sA^{F_2}(x)$. Therefore,

$$tF_1(x, y) \geq \langle x_1^*, y - x\rangle \quad \forall y \in C$$

and

$$sF_2(x, y) \geq \langle x_2^*, y - x\rangle \quad \forall y \in C.$$

By adding the above inequalities, we obtain

$$tF_1(x, y) + sF_2(x, y) \geq \langle x^*, y - x\rangle \quad \forall y \in C.$$

Thus $x^* \in A^{tF_1 + sF_2}(x)$. We note that $tA^{F_1}(x) = A^{tF_1}(x)$.

(vii) One can easily check that if $F_1, F_2 : C \times C \to \mathbb{R}$ are two monotone bifunctions with $F_1 \leq F_2$, then $A^{F_1} \subset A^{F_2}$. In this case, maximality of F_1 implies the maximality of F_2. ♦

Definition 2.5 *[23] A monotone bifunction $F : C \times C \to \mathbb{R}$ is called* BO-maximal monotone *(where BO stands for Blum and Oettli), if for every $(x, x^*) \in C \times X^*$ the following implication holds:*

$$F(y, x) + \langle x^*, y - x\rangle \leq 0, \quad \forall y \in C \implies \langle x^*, y - x\rangle \leq F(x, y), \quad \forall y \in C. \quad (2.5)$$

In the last section of this chapter we provide an example (see Example 2.37), which shows that the maximality of F is different from BO-maximality even if it is defined on a closed convex set and $\operatorname{gr} A^F \neq \emptyset$. However, maximality of F implies BO-maximality. This fact is established in the following result.

Proposition 2.6 *If $F : C \times C \to \mathbb{R}$ is maximal monotone, then it is BO-maximal monotone.*

Proof. Assume that

$$F(y, x) + \langle x^*, y - x\rangle \leq 0, \quad \forall y \in C. \quad (2.6)$$

Then for every $y \in C$ and $y^* \in A^F(y)$,

$$\langle x^*, x-y\rangle \geq F(y,x) \geq \langle y^*, x-y\rangle .$$

Thus, $\langle x^* - y^*, x-y\rangle \geq 0$ holds for each $(y,y^*) \in \operatorname{gr} A^F$. Since A^F is maximal monotone, $x^* \in A^F(x)$. Consequently,

$$F(x,y) \geq \langle x^*, y-x\rangle , \quad \forall y \in C.$$

Hence, implication (2.5) holds. ■

The converse is true if X is a reflexive Banach space, C is convex, $F(x,\cdot)$ is lsc and convex for all $x \in C$, and property (2.1) holds (see [1, 8]). In the last chapter (see Theorem 4.16 and its discussion) we will generalize this result.

As we observed in Remark 2.4, to any bifunction F we attached the monotone operator A^F. Now, to each operator $T : X \to 2^{X^*}$ we will attach a corresponding bifunction. As in [5, 64], we define the bifunction $G_T : \operatorname{dom} T \times \operatorname{dom} T \to \mathbb{R} \cup \{+\infty\}$ by

$$G_T(x,y) = \sup_{x^* \in T(x)} \langle x^*, y-x\rangle.$$

In the following proposition we collect some useful properties of the bifunction G_T. Most of these properties are known in reflexive Banach spaces [64].

Proposition 2.7 *Suppose that X is LCS and $T : X \to 2^{X^*}$ is monotone. Then*
(i) G_T is real-valued and monotone;
(ii) $G_T(x,x) = 0$ for each $x \in \operatorname{dom} T$, i.e., G_T fulfils (2.1);
(iii) if T is maximal monotone, then G_T is maximal monotone and

$$A^{G_T} = T;$$

(iv) assume that T is monotone, has closed convex values, and $\operatorname{dom}(T) = X$. If G_T is maximal monotone, then T is maximal monotone;
(v) $G_T(x,\cdot)$ is lsc and convex for each $x \in \operatorname{dom} T$;
(vi) $G_T(x, \lambda y + (1-\lambda)x) = \lambda G_T(x,y)$ for all $x, y \in \operatorname{dom} T$ and each λ in $\mathbb{R}_+$;
(vii) $T^{-1}(0) \subset EP(G_T)$.

Proof. (i) Let T be a monotone operator. Then for $x, y \in \operatorname{dom} T$, $x^* \in T(x)$ and $y^* \in T(y)$ we have $\langle y^* - x^*, y-x\rangle \geq 0$. Thus $-\langle x^*, y-x\rangle \geq \langle y^*, x-y\rangle$ and so $\inf_{x^* \in T(x)}(-\langle x^*, y-x\rangle) \geq \sup_{y^* \in T(y)}\langle y^*, x-y\rangle$. Therefore

$$\sup_{y^* \in T(y)} \langle y^*, x-y\rangle + \sup_{x^* \in T(x)} \langle x^*, y-x\rangle \leq 0.$$

Hence, $G_T(x,y) + G_T(y,x) \leq 0$. This implies that $G_T(x,y) \in \mathbb{R}$ and G_T is a monotone bifunction.

(ii) It is obvious.

(iii) The proof of this part is based upon the original paper [64]. For any $x \in \operatorname{dom} T, x^* \in T(x)$ and every $y \in C$ from the definition of G_T we get $G_T(x,y) \geq \langle x^*, y-x \rangle$. This implies that $x^* \in A^{G_T}(x)$ and so $T(x) \subset A^{G_T}(x)$. By hypothesis, T is maximal monotone so $T = A^{G_T}$. Now it follows from Definition 2.3 that G_T is maximal.

(iv) The proof of this part is also very close to the proof of Proposition 2.4 in [64]; we include the proof for the sake of completeness. Since G_T is a maximal monotone bifunction, by definition, A^{G_T} is a maximal monotone operator. Now for every $x \in X$ and $z^* \in A^{G_T}(z)$ we have

$$G_T(x,y) = \sup_{x^* \in T(x)} \langle x^*, y-x \rangle \geq \langle z^*, y-x \rangle \quad \forall y \in X.$$

The separation theorem (see Chapter 1, Corollary 1.9) implies that $z^* \in T(x)$. Therefore $A^{G_T} \subset T$. Thus $A^{G_T} = T$ and so T is maximal because A^{G_T} is maximal.

(v) This is a direct consequence of Proposition 1.14 from Chapter 1.

(vi) We have

$$\begin{aligned} G_T(x, \lambda y + (1-\lambda)x) &= \sup_{x^* \in T(x)} \langle x^*, \lambda y + (1-\lambda)x - x \rangle \\ &= \sup_{x^* \in T(x)} \langle x^*, \lambda(y-x) \rangle = \lambda G_T(x,y). \end{aligned}$$

(vii) The proof is an immediate consequence of the definitions and so it is omitted. ■

We also note that for each $\lambda > 0$ we have $G_{\lambda T} = \lambda G_T$.

Given an arbitrary monotone bifunction $F : C \times C \to \mathbb{R}$, one can construct A^F and the monotone bifunction G_{A^F}. In this case for all y in C we have

$$G_{A^F}(x,y) = \sup_{x^* \in A^F(x)} \langle x^*, y-x \rangle \leq F(x,y). \tag{2.7}$$

Note that whenever F is maximal monotone then G_{A^F} is also maximal monotone and so $A^F = A^{G_{A^F}}$. However, Example 2.5 in [64] shows that correspondence $F \to A^F$ is not one-to-one. The next proposition shows that in a special case we have equality in (2.7).

Proposition 2.8 *Let $T : X \to 2^{X^*}$ be a monotone operator. Set $F = G_T$. Then $G_{A^F} = F$ on $\operatorname{dom} T \times \operatorname{dom} T$.*

Proof. Let $x, y \in C := \operatorname{dom} T$. For each $x^* \in A^{G_T}(x)$ one has $\langle x^*, y-x \rangle \leq G_T(x,y)$ by definition of A^{G_T}. Hence,

$$G_{A^{G_T}}(x,y) = \sup_{x^* \in A^{G_T}(x,y)} \langle x^*, y-x \rangle \leq G_T(x,y).$$

To show the reverse inequality, take $z^* \in T(x)$. Then for each $w \in C$,

$$\langle z^*, w-x \rangle \leq \sup_{x^* \in T(x)} \langle x^*, w-x \rangle = G_T(x,w).$$

This implies that $z^* \in A^{G_T}(x)$, i.e., A^{G_T} is an extension of T. Consequently,

$$G_T(x,y) = \sup_{x^* \in T(x)} \langle x^*, w - x \rangle \leq \sup_{x^* \in A^{G_T}(x,y)} \langle x^*, y - x \rangle = G_{A^{G_T}}(x,y).$$

Thus $G_{A^F} = F$ on $\operatorname{dom} T \times \operatorname{dom} T$. ■

In addition, as noted in [64], it is possible to have $G_T = G_S$ for two monotone operators T and S, while $T \neq S$. For instance, if T is maximal monotone and S is any operator different from T such that $\operatorname{cl} \operatorname{co} S = T$, then $G_S = G_T$ hence G_S is maximal monotone, while S is not.

Thus, to each monotone operator T corresponds a monotone bifunction G_T, and to each monotone bifunction F corresponds a monotone operator A^F. It is obvious that $T \subseteq A^{G_T}$ for each monotone operator T. In general equality does not hold; however part (iii) of Proposition 2.7 shows that if T is maximal monotone, then $T = A^{G_T}$ and so G_T is maximal monotone. More generally, one has:

Theorem 2.9 *Suppose that Y is a Banach space. Let $T : Y \to 2^{Y^*}$ be monotone with weak*-closed convex values, and such that* $\operatorname{cl} \operatorname{dom} T$ *is convex. For any $x \in \operatorname{dom} T$, set $K(x) = N_{\operatorname{dom} T}(x)$. If $T(x) + K(x) \subseteq T(x)$ for all $x \in \operatorname{dom} T$, then $A^{G_T} = T$.*

Proof. It is enough to prove that $A^{G_T}(x) \subseteq T(x)$ for all $x \in Y$. Let $x \in Y$ and $z^* \in A^{G_T}(x)$. Then

$$\sup_{x^* \in T(x)} \langle x^*, y - x \rangle \geq \langle z^*, y - x \rangle, \quad \forall y \in \operatorname{dom} T. \tag{2.8}$$

Assume that $z^* \notin T(x)$. Since $T(x)$ is weak*-closed and convex, by the separation theorem (see Chapter 1, Corollary 1.9) there exists $v \in Y$ such that

$$\sup_{x^* \in T(x)} \langle x^*, v \rangle < \langle z^*, v \rangle . \tag{2.9}$$

For every $y^* \in K(x)$ and every $x^* \in T(x)$ one has by assumption $x^* + ty^* \in T(x)$ for all $t \geq 0$. Hence (2.9) implies

$$\forall t \geq 0, \quad \langle x^*, v \rangle + t \langle y^*, v \rangle < \langle z^*, v \rangle . \tag{2.10}$$

It follows that $\langle y^*, v \rangle \leq 0$. Therefore v is in the polar cone of $K(x)$, which is equal to the tangent cone $T_{\operatorname{dom} T}(x)$ of $\operatorname{dom} T$ at x. Hence v can be written as a limit

$$v = \lim_{n \to \infty} \frac{y_n - x}{\lambda_n}$$

where $y_n \in \operatorname{dom} T$ and $\lambda_n \searrow 0$. It also follows from (2.10) that $\langle x^*, v \rangle < \langle z^*, v \rangle$. Thus for n sufficiently large,

$$\langle x^*, y_n - x \rangle < \langle z^*, y_n - x \rangle .$$

But this contradicts (2.8). Hence $z^* \in T(x)$. ■

We remark that in Banach spaces (see Chapter 1, Proposition 1.33 and Theorem 1.39), whenever T is maximal, its values are weak*-closed and convex and $T(x) + K(x) \subseteq T(x)$ for all $x \in \operatorname{dom} T$. If in addition Y is reflexive, then $\operatorname{cl} \operatorname{dom} T$ is convex so all assumptions of Theorem 2.9 hold. Another case where the assumptions obviously hold is provided by the following:

Corollary 2.10 *Let Y be a Banach space. Assume that $T : Y \to 2^{Y^*}$ is monotone with weak*-closed, convex values and such that* $\operatorname{dom} T = Y$. *Then* $A^{G_T} = T$.

Corollary 2.10 is true also in LCS. Next proposition extends it to LCS.

Proposition 2.11 *Let X be a LCS. Suppose that $T : X \to 2^{X^*}$ is monotone with weak*-closed, convex values and such that* $\operatorname{dom} T = X$. *Then* $A^{G_T} = T$.

Proof. Given $x \in X$ and $z^* \in A^{G_T}(x)$,

$$G_T(x, y) = \sup_{x^* \in T(x)} \langle x^*, y - x \rangle \geq \langle z^*, y - x \rangle \quad \forall y \in X.$$

By hypothesis $T(x)$ is weak*-closed and convex, so the separation theorem (see Chapter 1, Corollary 1.9) together with preceding inequality imply that $A^{G_T}(x) \subset T(x)$. This enables us to obtain the desired equality. ■

Corollary 2.12 *Let $T : X \to X^*$ be a single-valued monotone operator with* $\operatorname{dom} T = X$. *Then* $A^{G_T} = T$.

Given a monotone operator T, one may define another monotone bifunction $\hat{G}_T$ by the following procedure which is taken from [80] and is reproduced here for the convenience of the reader. First, define $G_T : \operatorname{dom} T \times \operatorname{co} \operatorname{dom} T \to \mathbb{R} \cup \{+\infty\}$ as usual:

$$G_T(x, y) = \sup_{x^* \in T(x)} \langle x^*, y - x \rangle, \quad x \in \operatorname{dom} T, y \in \operatorname{co} \operatorname{dom} T.$$

Then define $\hat{G}_T : \operatorname{co} \operatorname{dom} T \times \operatorname{co} \operatorname{dom} T \to \mathbb{R} \cup \{+\infty\}$ as follows

$$\hat{G}_T(x, y) = \sup\{\sum_{i=1}^{k} \alpha_i G_T(x_i, y) : x = \sum_{i=1}^{k} \alpha_i x_i, x_i \in \operatorname{dom} T, \sum_{i=1}^{k} \alpha_i = 1, \alpha_i \geq 0\}.$$

This is the concave hull of the function $G_T(\cdot, y)$ (see formula (1.1) in Chapter 1). Note that $\hat{G}_T$ is well-defined, its values cannot be $-\infty$, and $\hat{G}_T(x, \cdot)$ is lsc and convex as supremum of lsc and convex functions.

Proposition 2.13 *$\hat{G}_T$ is real-valued, monotone, and such that $G_T(x, y) \leq \hat{G}_T(x, y)$ for all* $(x, y) \in \operatorname{dom} T \times \operatorname{co} \operatorname{dom} T$.

Proof. The inequality $G_T(x,y) \le \hat{G}_T(x,y)$ for $(x,y) \in \operatorname{dom} T \times \operatorname{co} \operatorname{dom} T$ is obvious from the definition of $\hat{G}_T$. Since for $(x,y) \in \operatorname{dom} T \times \operatorname{dom} T$ one has $G_T(x,y) \le -G_T(y,x)$ and $-G_T(y,\cdot)$ is concave, it follows that

$$\forall (x,y) \in \operatorname{co} \operatorname{dom} T \times \operatorname{dom} T, \quad \hat{G}_T(x,y) \le -G_T(y,x). \tag{2.11}$$

Now take the convex envelope with respect to y of both sides of (2.11). $\hat{G}_T(x,y)$ remains unchanged since $\hat{G}_T(x,\cdot)$ is convex, and $-G_T(y,x)$ becomes $-\hat{G}_T(y,x)$. It follows that

$$\hat{G}_T(y,x) + \hat{G}_T(x,y) \le 0, \quad \forall (x,y) \in \operatorname{co} \operatorname{dom} T \times \operatorname{co} \operatorname{dom} T. \tag{2.12}$$

Thus, $\hat{G}_T$ is monotone. Also, it follows from (2.12) that $\hat{G}_T$ is real-valued since $\hat{G}_T$ does not take the value $-\infty$. ∎

Note that $\hat{G}_T(x,x) \le 0$ for all $x \in \operatorname{co} \operatorname{dom} T$, while for $x \in \operatorname{dom} T$ one has $\hat{G}_T(x,x) = 0$ since $\hat{G}_T(x,x) \ge G_T(x,x)$. It is not true in general that $\hat{G}_T(x,x) = 0$ for all $x \in \operatorname{co} \operatorname{dom} T$.

2.2 Local Boundedness of Monotone Bifunctions

The aim of the present section is to study local boundedness of monotone bifunctions in relation with the corresponding property of monotone operators in Banach spaces. We will show that under very weak assumptions, local boundedness of monotone bifunctions is automatic at every point of $\operatorname{int} C$. In this way one can obtain an easy proof of the corresponding property of monotone operators.

Throughout this section, unless otherwise stated, X is a Banach space. We start off with reproducing the following definition from [64].

Definition 2.14 *A bifunction F is called locally bounded at $x_0 \in X$ if there exist $\epsilon > 0$ and $k \in \mathbb{R}$ such that $F(x,y) \le k$ for all x and y in $C \cap B(x_0,\epsilon)$. We call F locally bounded on a set $K \subseteq X$ if it is locally bounded at every $x \in K$.*

Local boundedness of operators is defined in Chapter 1, Definition 1.37.

Remark 2.15 (i) If a bifunction (not necessarily monotone) $F : C \times C \to \mathbb{R}$ is locally bounded at $x_0 \in \operatorname{int} C$, then A^F is locally bounded at x_0. Indeed, assume that $\varepsilon > 0$ and $k \in \mathbb{R}$ are such that $B(x_0,\varepsilon) \subseteq C$ and $F(x,y) \le k$ for all $x,y \in B(x_0,\varepsilon)$. Then for every $x \in B(x_0,\frac{\varepsilon}{2})$, $x^* \in A^F(x)$ and $v \in B(0,1)$, one has $x + \frac{\varepsilon}{2}v \in B(x_0,\varepsilon)$ and

$$k \ge F(x, x + \frac{\varepsilon}{2}v) \ge \frac{\varepsilon}{2}\langle x^*, v\rangle .$$

Thus $\|x^*\| \le \frac{2k}{\varepsilon}$ and A^F is locally bounded at x_0. The converse is not true in general (see Example 2.39 in Section5 of this chapter and the subsequent discussion).

(ii) Likewise, given an operator T, if G_T is locally bounded at $x_0 \in \operatorname{int}\operatorname{dom} T$, then T is locally bounded at x_0. Indeed, A^{G_T} is locally bounded at x_0 by the above argument, so T is also locally bounded since $T \subseteq A^{G_T}$. ♦

Local boundedness of bifunctions is a useful property. We reproduce here two of the results in [64].

Proposition 2.16 *Assume that X is reflexive, C is convex, and F is maximal monotone, locally bounded on* $\operatorname{cl} C$, *and such that $F(x,x) = 0$ for all $x \in C$. Then $C \subseteq \operatorname{cl}\operatorname{dom}(A^F)$.*

Proposition 2.17 *Let F be maximal monotone, locally bounded on* $\operatorname{int} C$ *and such that $F(x,x) = 0$ for all $x \in C$. If $C \subseteq \operatorname{cl}\operatorname{dom}(A^F)$, then*

$$\operatorname{int} C = \operatorname{int}\operatorname{dom}(A^F).$$

Note that in [64] all results are stated for reflexive spaces, but in fact the proof of Proposition 2.17 does not use reflexivity.

We will show that, under mild assumptions, any monotone bifunction is locally bounded in the interior of its domain. We will need the following lemma, which generalizes to quasi-convex functions a well-known property of convex functions.

Lemma 2.18 *Let X be a Banach space and $f : X \to \mathbb{R} \cup \{+\infty\}$ be lsc and quasi-convex. If $x_0 \in \operatorname{int}\operatorname{dom} f$, then f is bounded from above in a neighborhood of x_0.*

Proof. Let $\varepsilon > 0$ be such that $\overline{B}(x_0, \varepsilon) \subseteq \operatorname{dom} f$. Set $S_n = \{x \in \overline{B}(x_0, \varepsilon) : f(x) \leq n\}$. Then S_n are convex and closed and $\bigcup_{n\in\mathbb{N}} S_n = \overline{B}(x_0, \varepsilon)$. By Baire's theorem, there exists $n \in \mathbb{N}$ such that $\operatorname{int} S_n \neq \emptyset$. Take any $x_1 \in \operatorname{int} S_n$ and any $x_2 \neq x_0$ such that $x_2 \in \overline{B}(x_0, \varepsilon)$ and $x_0 \in \operatorname{co}\{x_1, x_2\}$. Choose $n_1 > \max\{n, f(x_2)\}$. Then $x_1 \in \operatorname{int} S_{n_1}$, $x_2 \in S_{n_1}$ hence $x_0 \in \operatorname{int} S_{n_1}$ so f is bounded by n_1 at a neighborhood of x_0. ∎

Note that, if in the above lemma f is lsc and convex, then the result is obvious since f is continuous at every interior point of $\operatorname{dom} f$.

Theorem 2.19 *Let X be a Banach space, $C \subseteq X$ a set, and $F : C \times C \to \mathbb{R}$ a monotone bifunction such that for every $x \in C$, $F(x, \cdot)$ is lsc and quasi-convex. Assume that for some $x_0 \in \operatorname{int} C$ there exists a neighborhood $B(x_0, \varepsilon) \subseteq C$ such that for each $x \in B(x_0, \varepsilon)$, $F(x, \cdot)$ is bounded from below[1] on $B(x_0, \varepsilon)$. Then F is locally bounded at x_0.*

Proof. Let $\varepsilon > 0$ be as in the assumption and define $g : B(x_0, \varepsilon) \to \mathbb{R} \cup \{+\infty\}$ by

$$g(y) = \sup\{F(x, y) : x \in B(x_0, \varepsilon)\}.$$

[1]This bound may depend on x.

We show that g is real-valued. Given $y \in B(x_0, \varepsilon)$, for each $x \in B(x_0, \varepsilon)$,

$$F(x, y) \leq -F(y, x).$$

By assumption, there exists M_y such that $F(y, x) \geq M_y$ for all $x \in B(x_0, \varepsilon)$. Hence $g(y) \leq -M_y < \infty$, i.e., g is real-valued.

Now g is lsc and quasi-convex, and $x_0 \in \operatorname{int} \operatorname{dom} g$. By Lemma 2.18, we can find $\varepsilon_1 < \varepsilon$ and $M \in \mathbb{R}$ such that $g(y) \leq M$ for all $y \in B(x_0, \varepsilon_1)$. Then $F(x, y) \leq M$ for all $x, y \in B(x_0, \varepsilon_1)$ so F is locally bounded at x_0. ■

Corollary 2.20 *Let X be reflexive and $F : C \times C \to \mathbb{R}$ be monotone and such that for every $x \in C$, $F(x, \cdot)$ is lsc and quasi-convex. Then F is locally bounded on* $\operatorname{int} C$.

Proof. Given $x_0 \in \operatorname{int} C$ choose $\varepsilon > 0$ such that $\overline{B}(x_0, \varepsilon) \subseteq C$. Since X is reflexive Banach space, $\overline{B}(x_0, \varepsilon)$ is weakly compact, hence for each $y \in C$, $F(y, \cdot)$ has a minimum on $\overline{B}(x_0, \varepsilon)$. Consequently, all assumptions of Theorem 2.19 are satisfied. ■

When $F(x, \cdot)$ is lsc and convex, reflexivity of X is not necessary:

Corollary 2.21 *Let $F : C \times C \to \mathbb{R}$ be monotone and such that for every $x \in C$, $F(x, \cdot)$ is lsc and convex. Then F is locally bounded on* $\operatorname{int} C$.

Proof. Let $x_0 \in \operatorname{int} C$. Choose $\varepsilon > 0$ be such that $B(x_0, \varepsilon) \subseteq C$. For every $x \in B(x_0, \varepsilon)$, the subdifferential of $\partial F(x, \cdot)$ is nonempty at x. For every x^* in $\partial F(x, \cdot)(x)$ and $y \in B(x_0, \varepsilon)$ one has

$$F(x, y) - F(x, x) \geq \langle x^*, y - x \rangle \geq - \|x^*\| \|x - y\| \geq -2\varepsilon \|x^*\| .$$

Thus $F(x, \cdot)$ is bounded from below on $B(x_0, \varepsilon)$. According to the Theorem 2.19, F is locally bounded at x_0. ■

If $T : X \to 2^{X^*}$ is monotone, then G_T is monotone while $G_T(x, \cdot)$ is lsc and convex. According to the Corollary 2.21 and Remark 2.15, we immediately obtain:

Corollary 2.22 *Let X be a Banach space and $T : X \to 2^{X^*}$ be monotone. Then T is locally bounded at every point of* $\operatorname{int} \operatorname{dom} T$.

We see that the well-known local boundedness of monotone operators can be shown very easily through Corollary 2.21 on local boundedness of bifunctions. In fact, whenever property (2.1) holds, one can also easily show the converse, i.e., provide a proof of Corollary 2.21 assuming that Corollary 2.22 is known:

Proposition 2.23 *Assume that F is monotone, satisfies (2.1) and $F(x, \cdot)$ is lsc and convex for each $x \in C$. Then F is locally bounded on* $\operatorname{int} C$.

Proof. Under our assumptions, $A^F(x)$ is actually the subdifferential $\partial F(x,\cdot)(x)$ of the lsc and convex function $F(x,\cdot)$ at x. It is known that this is nonempty for all $x \in \operatorname{int} C$. Hence, the monotone operator A^F is locally bounded on $\operatorname{int} C$.

For each $x_0 \in \operatorname{int} C$ choose $\varepsilon > 0$ and $k \in \mathbb{R}$ such that $B(x_0, \varepsilon) \subseteq C$ and $||y^*|| \leq k$ for every $y^* \in A^F(y)$, $y \in B(x_0,\varepsilon)$. Then for each $x, y \in B(x_0,\varepsilon)$ and $y^* \in A^F(y)$,

$$F(x,y) \leq -F(y,x) \leq -\langle y^*, x-y\rangle \leq \|y^*\| \|x-y\| \leq 2\varepsilon k.$$

Thus F is locally bounded on $\operatorname{int} C$. ■

In fact, with the same proof as in the above proposition, we obtain the slightly more general result, which is a kind of converse of Proposition 2.17:

Proposition 2.24 *Assume F is a monotone bifunction and* $\operatorname{int} C = \operatorname{int} \operatorname{dom} A^F$. *Then F is locally bounded on* $\operatorname{int} C$.

Corollary 2.25 *Suppose that $F : X \times X \to \mathbb{R}$ is monotone and* $\operatorname{dom} A^F = X$. *Then F is locally bounded on X.*

One can also obtain a well-known generalization of Corollary 2.22 by using bifunctions.

Lemma 2.26 *Suppose that X is a Banach space and $T : X \to 2^{X^*}$ is monotone. Then*

(i) $T \subseteq A^{G_T} \subseteq A^{\hat{G}_T}$;

(ii) $T = A^{G_T} = A^{\hat{G}_T}$, if T is maximal monotone.

Proof. (i) $T \subseteq A^{G_T}$ is obvious. Since $G_T(x,y) \leq \hat{G}_T(x,y)$ for all (x,y) in $C \times \operatorname{co} C$, we deduce that $A^{G_T} \subseteq A^{\hat{G}_T}$.

(ii) Obvious consequence of (i). ■

Proposition 2.27 *Suppose that X is a Banach space and $T : X \to 2^{X^*}$ is monotone and* $\operatorname{int}(\operatorname{co} \operatorname{dom} T) \neq \emptyset$. *Then T is locally bounded on* $\operatorname{int}(\operatorname{co} \operatorname{dom} T)$.

Proof. We know that $\hat{G}_T$ is monotone and $\hat{G}_T(x,\cdot)$ is lsc and convex for all $x \in \operatorname{co} \operatorname{dom} T$. Thus by Corollary 2.21, $\hat{G}_T$ is locally bounded on $\operatorname{int}(\operatorname{co} \operatorname{dom} T)$. It follows from Remark 2.15 that $A^{\hat{G}_T}$ is locally bounded on $\operatorname{int}(\operatorname{co} \operatorname{dom} T)$. Now Lemma 2.26 implies that T is locally bounded on $\operatorname{int}(\operatorname{co} \operatorname{dom} T)$. ■

2.3 Cyclically Monotone Bifunctions

In this section we will derive some properties of cyclically monotone bifunctions. Indeed, we generalize some results of [64] to Hausdorff LCS.

Definition 2.28 *Suppose that X is a vector space and C is a nonempty subset of X. A bifunction $F : C \times C \to \mathbb{R}$ is called* cyclically monotone *if for any cycle $x_1, x_2, ..., x_{n+1} = x_1$ in C*

$$F(x_1, x_2) + F(x_2, x_3) + \cdots + F(x_n, x_{n+1}) \leq 0.$$

The following proposition provides a necessary and sufficient condition for a bifunction to be cyclically monotone. We follow Hadjisavvas and Khatibzadeh's proof for the cyclically monotone bifunctions in reflexive Banach spaces [64], which we include for the sake of completeness.

Proposition 2.29 *Suppose that X is a vector space, C is a nonempty subset of X and $F : C \times C \to \mathbb{R}$ is a bifunction. Then F is cyclically monotone if and only if there exists a function $f : C \to \mathbb{R}$ such that*

$$F(x, y) \leq f(y) - f(x) \quad \forall x, y \in C. \tag{2.13}$$

Proof. Assume that there exists a function $f : C \to \mathbb{R}$ such that (2.13) holds. Then for every cycle $x_1, x_2, ..., x_{n+1} = x_1$ in C we have

$$F(x_1, x_2) + F(x_2, x_3) + \cdots + F(x_n, x_{n+1}) \leq \sum_{i=1}^{n} (f(x_{i+1}) - f(x_i)) = 0.$$

This means that F is cyclically monotone.

Conversely, let F be a cyclically monotone bifunction. Choose any $x_0 \in C$ and define f on C by

$$f(x) = \sup \{F(x_0, x_1) + F(x_1, x_2) + \cdots + F(x_n, x)\} \tag{2.14}$$

where the supremum is taken over all families $x_1, x_2, ..., x_n$ in C and $n \in \mathbb{N}$. Since F is cyclically monotone,

$$F(x_0, x_1) + F(x_1, x_2) + \cdots + F(x_n, x) + F(x, x_0) \leq 0.$$

This implies that $F(x_0, x_1) + F(x_1, x_2) + \cdots + F(x_n, x) \leq -F(x, x_0)$. Now by taking the supremum again over $x_1, x_2, ..., x_n \in C$ we get $f(x) \leq -F(x, x_0)$. Thus f is real-valued and also for any $x, y \in C$ and $x_1, x_2, ..., x_n \in C$

$$F(x_0, x_1) + F(x_1, x_2) + \cdots + F(x_n, x) + F(x, y) \leq f(y).$$

Taking the supremum over all families $x_1, x_2, ..., x_n$ in C, the preceding inequality yields

$$f(x) + F(x, y) \leq f(y).$$

This means that inequality (2.13) holds. ■

Whenever F is also maximal monotone, more can be said on f.

Proposition 2.30 *Suppose that X is a Hausdorff LCS and $\operatorname{int} C \neq \emptyset$ and $F : C \times C \to \mathbb{R}$ is maximal monotone, cyclically monotone and satisfies (2.1). Then:*

(i) The sets $\operatorname{cl} C$ and $\operatorname{int} C$ are convex, and equalities $\operatorname{cl} C = \operatorname{cl}\operatorname{dom} A^F$ and $\operatorname{int} C = \operatorname{int}\operatorname{dom}(A^F)$ hold; the function f in relation (2.13) is uniquely defined up to a constant on $\operatorname{int} C$, and is lsc and convex on $\operatorname{int} C$.

(ii) If in addition $F(x,\cdot)$ is lsc for every $x \in C$, then f is uniquely defined up to a constant, and lsc and convex on C.

Proof. The proof we present here is borrowed from [64] and it is a simplification of the original proof. Although the proof in [64] is for reflexive Banach spaces, it works for LCS.

(i) Maximal monotonicity of A^F is a direct consequence of Definition 2.3. For any cycle $x_1, x_2, ..., x_{n+1} = x_1$ in X and each $x_i^* \in A^F(x_i)$ for $i = 1, ..., n$, we have

$$F(x_i, x_{i+1}) \geq \langle x_i^*, x_{i+1} - x_i \rangle .$$

By adding the above inequalities for $i = 1, ..., n$, we obtain

$$F(x_1, x_2) + F(x_2, x_3) + \cdots + F(x_n, x_{n+1}) \geq \sum_{i=1}^{n} \langle x_i^*, x_{i+1} - x_i \rangle .$$

By assumption F is cyclically monotone, hence the left hand right of above inequality is less than or equal to zero. Thus from the preceding inequality we get

$$\sum_{i=1}^{n} \langle x_i^*, x_{i+1} - x_i \rangle \leq 0.$$

This means that A^F is cyclically monotone. Now, by Proposition 1.36 from Chapter 1 for any $(x_0, x_0^*) \in \operatorname{gr} A^F$ the function defined as

$$\Phi(x) = \sup_{\substack{(x_i, x_i^*) \in \operatorname{gr} A^F \\ n \in \mathbb{N}, i=1,...,n}} \left(\langle x_n^*, x - x_n \rangle + \sum_{i=0}^{n-1} \langle x_i^*, x_{i+1} - x_i \rangle \right)$$

is proper, lsc and convex, $\Phi(x_0) = 0$ and $A^F(x) \subset \partial\Phi(x)$. From the maximality of A^F we conclude that $A^F = \partial\Phi$. For each x in C we have

$$\begin{aligned} \Phi(x) &= \sup_{\substack{(x_i, x_i^*) \in \operatorname{gr} A^F \\ n \in \mathbb{N}, i=1,...,n}} \left(\langle x_n^*, x - x_n \rangle + \sum_{i=0}^{n-1} \langle x_i^*, x_{i+1} - x_i \rangle \right) \\ &\leq \sup_{\substack{(x_i, x_i^*) \in \operatorname{dom} F \\ n \in \mathbb{N}, i=1,...,n}} \left(F(x_n, x) + \sum_{i=0}^{n-1} F(x_i, x_{i+1}) \right) \leq -F(x, x_0), \end{aligned}$$

since $F(x_1, x_2) + F(x_2, x_3) + \cdots + F(x_n, x) + F(x, x_0) \leq 0$ by cyclic monotonicity. Hence Φ is real-valued on C so that $C \subset \operatorname{dom}\Phi$. It follows that

$$\operatorname{cl} C \subset \operatorname{cl}\operatorname{dom}\Phi = \operatorname{cl}\operatorname{dom}(\partial\Phi) = \operatorname{cl}\operatorname{dom} A^F \subset \operatorname{cl} C,$$

$$\operatorname{int} C \subset \operatorname{int} \operatorname{dom} \Phi = \operatorname{int} \operatorname{dom} (\partial \Phi) = \operatorname{int} \operatorname{dom} A^F \subset \operatorname{int} C.$$

From the above relations we conclude that $\operatorname{cl} C = \operatorname{cl} \operatorname{dom} A^F = \operatorname{cl} \operatorname{dom} \Phi$ and $\operatorname{int} C = \operatorname{int} \operatorname{dom} \Phi = \operatorname{int} \operatorname{dom} A^F$. Since Φ is a lsc and convex, $\operatorname{dom} \Phi$ is convex, so $\operatorname{cl} C = \operatorname{cl} \operatorname{dom} \Phi$ and $\operatorname{int} C = \operatorname{int} \operatorname{dom} \Phi$ are convex. Now let $f : C \to \mathbb{R}$ be any function such that (2.13) holds. Then for every $(x, x^*) \in \operatorname{gr} A^F$ and every $y \in C$, we have

$$f(y) - f(x) \geq \langle x^*, y - x \rangle .$$

This means that $\partial \Phi \subset \partial f$ and by maximal monotonicity of $\partial \Phi$, $\partial \Phi = \partial f$.

For each $x, y \in \operatorname{int} C$ and $t \in]0, 1[$ with $z := (1-t)x + ty \in \operatorname{int} C$, select an element $z^* \in A^F(z)$. Then we have

$$f(x) - f(z) \geq \langle z^*, x - z \rangle ,$$

$$f(y) - f(z) \geq \langle z^*, y - z \rangle .$$

Multiplying the first inequality with t and the second one with $(1-t)$, then adding them, we obtain

$$f((1-t)x + ty) \leq (1-t) f(x) + t f(y) . \tag{2.15}$$

Which means that f is convex on $\operatorname{int} C$. Also, f is lsc on $\operatorname{int} C$ since $\partial f \neq \emptyset$ there. Since $\partial \Phi = \partial f$ the functions Φ and f differ by a constant on $\operatorname{int} C$.

(ii) Assume that F is lsc and let f be a function satisfying (2.13). Then for each $y \in C$, we have

$$\liminf_{y \to x} (f(y) - f(x)) \geq \liminf_{y \to x} F(x, y) \geq F(x, x) = 0$$

thus f is lsc. From part (i) of the proof we know that $\operatorname{int} C$ and $\operatorname{cl} C$ are convex. Adding a constant if necessary, we may assume that $f = \Phi$ on $\operatorname{int} C$. For any $x \in C$, choose $y \in \operatorname{int} C$ and a sequence $x_n = (1-t_n)x + t_n y, n \in \mathbb{N}$ with $t_n > 0$ and $t_n \to 0$. Since $C \subset \operatorname{dom} \Phi$ and $\operatorname{int} C = \operatorname{int} \operatorname{dom} \Phi$ we have $x_n \in \operatorname{int} C = \operatorname{int} \operatorname{dom} \Phi$. Applying (2.15) which is valid whenever $(1-t)x + ty \in \operatorname{int} C$ and lower semi-continuity of f we get

$$\liminf_{n \to \infty} f(x_n) \leq \liminf_{n \to \infty} ((1-t_n) f(x) + t_n f(y)) = f(x) \leq \liminf_{n \to \infty} f(x_n) .$$

Therefore $f(x) = \liminf_{n \to \infty} f(x_n)$. Applying the same argument for Φ, we conclude that $\Phi(x) = \liminf_{n \to \infty} \Phi(x_n)$. Consequently,

$$f(x) = \liminf_{n \to \infty} f(x_n) = \liminf_{n \to \infty} \Phi(x_n) = \Phi(x).$$

Thus $f = \Phi$ on C and this implies that f is convex. ■

Example 5.3 in [64] shows that a convex function such that (2.13) holds may not exist, if $F(x, \cdot)$ is not lsc. In addition, Example 5.4 in [64] shows that cyclic monotonicity of A^F does not imply cyclic monotonicity of F, even if F is monotone, C is a convex subset of $\mathbb{R}$ and A^F is a subdifferential of a proper, lsc and convex function.

Proposition 2.24 induces the following result, which does not assume lower semi-continuity or quasi-convexity.

Proposition 2.31 *Suppose that X is a Banach space,* $\operatorname{int} C \neq \emptyset$ *and* $F : C \times C \to \mathbb{R}$ *is maximal monotone, cyclically monotone and satisfies* $F(x,x) = 0$ *for all* $x \in C$. *Then* F *is locally bounded on* $\operatorname{int} C$.

Proof. Since F is maximal monotone and cyclically monotone, by part (i) of Proposition 2.30 we have

$$\operatorname{int} C = \operatorname{int} \operatorname{dom} A^F.$$

Now, Proposition 2.24 implies that F is locally bounded on $\operatorname{int} C$. ■

2.4 Local Boundedness at Arbitrary Points

In Proposition 2.16 one asks for the bifunction to be maximal monotone and locally bounded on $\operatorname{cl} C$. This assumption seems to be in contradiction with the theory of maximal monotone operators. In fact, if $T : X \to 2^{X^*}$ is a maximal monotone operator and $\operatorname{int} \operatorname{dom} T \neq \emptyset$, then T is never locally bounded on elements of the boundary of $\operatorname{dom} T$; see Theorem 1.39. However, this does not imply that the maximal monotone bifunction G_T is also unbounded at x_0. In fact, in $\mathbb{R}^n$ we have a result of local boundedness at arbitrary points and in particular at boundary points, for more general bifunctions.

Let us denote by $||x||_\infty$ the sup norm of $x = (x_1, x_2, ..., x_n) \in \mathbb{R}^n$,

$$||x||_\infty = \max\{|x_1|, |x_2|, ..., |x_n|\},$$

and by $\overline{B}_\infty(x, \varepsilon)$ the closed $\varepsilon-$ball around x with respect to $\|\cdot\|_\infty$. A set which is the convex hull of finitely many points is called a *polytope*. We call a subset C of $\mathbb{R}^n$ *locally polyhedral at* $x_0 \in C$ if there exists $\varepsilon > 0$ such that $\overline{B}_\infty(x, \varepsilon) \cap C$ is a polytope.

In the following proposition we do not assume that F is monotone.

Proposition 2.32 *Let* $C \subset \mathbb{R}^n$ *be locally polyhedral at* $x_0 \in C$ *and* $F : C \times C \to \mathbb{R}$ *be a bifunction. If* $F(x, \cdot)$ *is quasi-convex for each* $x \in C$, *and* $F(\cdot, y)$ *is upper semi-continuous (usc) for all* $y \in C$, *then* F *is locally bounded at* x_0.

Proof. Choose $\varepsilon > 0$ such that $\overline{B}_\infty(x_0, \varepsilon) \cap C$ is a polytope. Then there exist $x_1, x_2, ..., x_k$ such that

$$\overline{B}_\infty(x_0, \varepsilon) \cap C = \operatorname{co}\{x_1, x_2, ..., x_k\}.$$

Since $F(x, \cdot)$ is quasi-convex, for all x and y in $\overline{B}_\infty(x_0, \varepsilon) \cap C$ we have

$$F(x, y) \leq \max\{F(x, x_1), F(x, x_2), ..., F(x, x_k)\}.$$

On the other hand $F(\cdot, x_i)$ is usc and $\overline{B}_\infty(x_0, \varepsilon) \cap C$ is a compact set, thus $F(\cdot, x_i)$ attains its maximum on $\overline{B}_\infty(x_0, \varepsilon) \cap C$; that is, there exists M_i such that

$$F(x, x_i) \leq M_i \text{ for } i = 1, 2, ..., k \text{ and } x \in \overline{B}_\infty(x_0, \varepsilon) \cap C.$$

Set $M = \max\{M_1, M_2, ..., M_k\}$. Then

$$F(x, y) \leq M \text{ for all } x, y \in \overline{B}_\infty(x_0, \varepsilon) \cap C.$$

This means that F is locally bounded at x_0. ∎

Proposition 2.33 *Let $C \subset \mathbb{R}^n$ be locally polyhedral at x_0 and $F : C \times C \to \mathbb{R}$ be a monotone bifunction. If $F(x, .)$ is quasi-convex and lsc for all $x \in C$, then F is locally bounded at x_0.*

Proof. Choose $\varepsilon > 0$ such that $\overline{B}_\infty(x_0, \varepsilon) \cap C$ is a polytope. Since $F(x, \cdot)$ is quasi-convex, as the proof of the previous proposition there exist $x_1, x_2, ..., x_k \in \overline{B}_\infty(x_0, \varepsilon) \cap C$ such that for all $x, y \in \overline{B}_\infty(x_0, \varepsilon) \cap C$ we have

$$F(x, y) \leq \max\{F(x, x_1), F(x, x_2), ..., F(x, x_k)\}. \tag{2.16}$$

Since $F(x, y)$ is monotone,

$$F(x, x_i) \leq -F(x_i, x) \quad \text{for } i = 1, 2, ..., k. \tag{2.17}$$

For each i, $-F(x_i, \cdot)$ is usc. Therefore, $-F(x_i, \cdot)$ has a maximum M_i on $\overline{B}_\infty(x_0, \varepsilon) \cap C$. Set $M = \max\{M_1, M_2, ..., M_k\}$. Then (2.16) and (2.17) entail

$$F(x, y) \leq M \quad \text{for all } x, y \in \overline{B}_\infty(x_0, \varepsilon) \cap C,$$

i.e., F is locally bounded at x_0. ∎

Thus, if C is a polyhedral set and F satisfies the assumptions of Proposition 2.32 or 2.33, then it is locally bounded on C, not only on $\operatorname{int} C$. However, the following example shows that this property may fail if C is not locally polyhedral.

Example 2.34 Set $C = \{(\alpha, \beta) \in \mathbb{R}^2 : \alpha \geq \beta^4\}$. Define the function f on $\mathbb{R}^2$ by

$$f(\alpha, \beta) = \begin{cases} \frac{\beta^2}{2\alpha} & \text{if } \alpha \geq \beta^4, \alpha > 0, \\ 0 & \text{if } \alpha = \beta = 0, \\ +\infty & \text{otherwise.} \end{cases}$$

This function is lsc and convex (it is the restriction to C of the function in [98, page 83]).

Now define the bifunction $F : C \times C \to \mathbb{R}$ by $F(x, y) = f(y) - f(x)$, $x, y \in C$. This bifunction F has very nice properties: it is cyclically monotone, $F(x, \cdot)$ is lsc and convex, $F(\cdot, y)$ is concave and usc, it is defined on a closed convex set thus it is maximal monotone (see Proposition 3.1 in [64]). Nevertheless, it is not locally bounded at 0. Indeed, consider the sequences $x_n = (0, 0)$ and $y_n = (\frac{1}{n^4}, \frac{1}{n})$. Then $F(x_n, y_n) \to +\infty$, hence every neighborhood of 0 contains pairs x, y with $F(x, y)$ as large as we wish. ▲

Since monotone bifunctions can be locally bounded at the boundary of their domain, it is interesting to investigate an analogous property for monotone operators.
Given a subset $C \subseteq X$, let us denote by $\operatorname{inw} C(x_0) := \bigcup_{\lambda>0} \frac{1}{\lambda}(\operatorname{int} C - x_0)$ the set of inward directions of C at x_0. Note that if $v \in \operatorname{inw} C(x_0)$ then v is also an inward direction at all x sufficiently close to x_0. Indeed, it is sufficient to take $x \in B(x_0, \varepsilon)$ where $\varepsilon > 0$ is such that $B(x_0 + \lambda v, \varepsilon) \subseteq C$.

Definition 2.35 *An operator $T : X \to 2^{X^*}$ is called inward locally bounded at $x_0 \in \operatorname{cl} \operatorname{dom} T$ if for each $v \in \operatorname{inw} C(x_0)$ there exist $k > 0$ and $\varepsilon > 0$ such that for all $x \in B(x_0, \varepsilon) \cap C$ and $x^* \in T(x)$, one has $\langle x^*, v \rangle \leq k$.*

We remark that if T is inward locally bounded at an interior point x_0 of $\operatorname{dom} T$, then by the uniform boundedness principle (see Chapter 1, Theorem 1.6) it is locally bounded at x_0, since $\operatorname{inw} \operatorname{dom} T(x_0) = X$.

Proposition 2.36 *A monotone operator T is inward locally bounded at every point of* $\operatorname{cl} \operatorname{dom} T$.

Proof. Let $x_0 \in \operatorname{cl} \operatorname{dom} T$ and $v \in \operatorname{inw} \operatorname{dom} T(x_0)$ be given. Choose $\lambda > 0$ such that $x_0 + \lambda v \in \operatorname{int} \operatorname{dom} T$. Since T is locally bounded at $x_0 + \lambda v$, there exist $\varepsilon > 0$ and $k > 0$ such that $B(x_0 + \lambda v, \varepsilon) \subseteq \operatorname{dom} T$ and $\|y^*\| \leq k$ for all $y^* \in T(y)$, $y \in B(x_0 + \lambda v, \varepsilon)$. For every $x \in B(x_0, \varepsilon) \cap \operatorname{dom} T$, one has $x + \lambda v \in B(x_0 + \lambda v, \varepsilon)$. Thus for every $x^* \in T(x)$ and $y^* \in T(x + \lambda v)$,

$$\langle x^*, v \rangle = \frac{1}{\lambda} \langle x^*, x + \lambda v - x \rangle \leq \frac{1}{\lambda} \langle y^*, x + \lambda v - x \rangle \leq k \|v\| .$$

Thus T is inward locally bounded at x_0. ■

Comparing this last result with Propositions 2.32 and 2.33, we should remark that these propositions imply a somewhat stronger local boundedness than inward local boundedness. Indeed, if T is monotone and $\operatorname{dom} T$ is locally polyhedral, then by Proposition 2.33 the bifunction G_T is locally bounded everywhere; thus, $\langle x^*, y - x \rangle$ is bounded from above for all $x^* \in T(x)$ where x, y are near a point x_0 of the boundary, even if $y - x$ is "outward" rather than inward. This is because whenever $y - x$ is outward, its norm is small, so that $\langle x^*, y - x \rangle$ is bounded even if the norm of x^* is large.

2.5 Counterexamples

We indicate that when $F : C \times C \to \mathbb{R}$ is a bifunction such that

$$F(x, y) = -F(y, x) \quad \forall x, y \in C$$

then, as one can easily check, the implication (2.5) holds and so F is BO-maximal monotone.

The first example of this section shows that it is possible for a bifunction to be BO-maximal monotone, without being maximal monotone, even if it is defined on a closed convex set and $\operatorname{gr} A^F \neq \emptyset$. In our knowledge, the only example of a bifunction F published so far [64] which is BO-maximal monotone but not maximal monotone, is in some sense trivial since $\operatorname{gr} A^F = \emptyset$.

Example 2.37 Define the bifunction $F : [0,1] \times [0,1] \to \mathbb{R}$ by

$$F(x,y) = \begin{cases} 0 & \text{if} \quad x, y \in]0,1[, \\ -\sqrt{y} & \text{if} \quad x = 0, y \in [0,1], \\ \sqrt{x} & \text{if} \quad y = 0, x \in [0,1], \\ \sqrt{1-y} & \text{if} \quad x = 1, y \in [0,1], \\ -\sqrt{1-x} & \text{if} \quad y = 1, x \in [0,1]. \end{cases}$$

Note that $F(x,y)$ is well-defined if both x, y are endpoints of $[0,1]$. In addition F is monotone and is BO-maximal, since $F(x,y) = -F(y,x)$ for all $x, y \in [0,1]$. Next we will show that F is not maximal monotone. So, we calculate the A^F. Let $x \in [0,1]$ and $x^* \in A^F(x)$. We consider the following cases:

Case 1: Assume $x \in]0,1[$, by taking $y = 1$ we have $F(x,1) \geq x^*(1-x)$ and so $-\sqrt{1-x} \geq x^*(1-x)$. From here we obtain $x^* \leq -\frac{1}{\sqrt{1-x}} < 0$. However for $y \in]0,1[$, one can easily get $x^* = 0$, a clear contradiction. Therefore, $A^F(x) = \emptyset$ for $x \in]0,1[$.

Case 2: If $x = 0$, then $F(0,y) \geq x^* y$. From this we obtain $x^* \leq -\frac{1}{\sqrt{y}}$ for all $y \in]0,1]$ which is impossible. Therefore, $A^F(0) = \emptyset$.

Case 3: If $x = 1$, then $F(1,y) = \sqrt{1-y} \geq x^*(y-1)$ for all $y \in]0,1]$. Thus $x^* \geq -\frac{1}{\sqrt{1-y}}$ for all $y \in]0,1[$ and so $x^* \geq -1$. Now if we take $y = 0$, we get $F(1,0) = 1 \geq x^*(-1)$. Thus $x^* \leq -1$. Therefore, $A^F(1) = \{-1\}$. Consequently,

$$A^F(x) = \begin{cases} \emptyset & \text{if} \quad x \in [0,1[, \\ -1 & \text{if} \quad x = 1, \end{cases} \quad \text{and } \operatorname{gr} A^F = \{(1,-1)\}.$$

Obviously A^F and so F is not maximal monotone. ▲

When the assumptions of lower semi-continuity or quasi-convexity do not hold, then local boundedness may fail (see Theorem 2.19, related corollaries and Proposition 2.33) as shown by the following examples.

Example 2.38 Let x^* be a noncontinuous linear functional on X, that is x^* in $X' \backslash X^*$ and set $F(x,y) = \langle x^*, y - x \rangle$. Then F is a monotone (cyclically monotone) bifunction, which is affine (convex) but obviously is not locally bounded at any $x \in X$. ▲

Example 2.39 Define F on $\mathbb{R} \times \mathbb{R}$ by $F(x,y) = \frac{1}{|y|} - \frac{1}{|x|}$ for $x \neq 0$ and $y \neq 0$, and $F(x,0) = x = -F(0,x)$, $x \in \mathbb{R}$. Then F is monotone, $F(x,\cdot)$ is lsc for every $x \in \mathbb{R}$, but F is not locally bounded at 0. In addition, this bifunction is a counterexample to the converse of Proposition 2.6 and of Remark 2.15: one can readily show that F is BO-maximal monotone and that $\operatorname{dom} A^F = \{0\}$, with $A^F(0) = \{-1\}$. It follows that F is not maximal monotone and also A^F is locally bounded at $\{0\}$ while F is not. ▲

In contrast to the previous example, if a monotone operator T is locally bounded at $x_0 \in X$, then G_T is not only locally bounded but also locally bounded by an arbitrarily small positive number at x_0. Indeed, if $\|x^*\| \le k$ for all $x \in B(x_0, \varepsilon)$, $x^* \in T(x)$, then for all $x, y \in \operatorname{dom} T \cap B(x_0, \varepsilon)$ and $y^* \in T(y)$ we find

$$G_T(x, y) \le -G_T(y, x) \le -\langle y^*, x - y\rangle \le 2\varepsilon k.$$

The next simple example shows that G_T is not necessary locally bounded on the closure of the domain of T.

Example 2.40 Set $C = [0, \frac{\pi}{2})$ and define $T : C \to \mathbb{R}$ by

$$T(x) = \begin{cases} \tan^2 x & \text{if } \; x \in]0, \frac{\pi}{2}[, \\ (-\infty, 0] & \text{if } \; x = 0. \end{cases}$$

Then T is a monotone operator. Now consider the sequences $x_n = \frac{\pi}{2} - \frac{2}{n}$ and $y_n = \frac{\pi}{2} - \frac{1}{n}$ in C. Then

$$G_T(x_n, y_n) = \frac{\tan^2\left(\frac{\pi}{2} - \frac{2}{n}\right)}{n} \to +\infty$$

when $n \to \infty$. Thus G_T is not locally bounded at $\frac{\pi}{2} \in \operatorname{cl} C$. ▲

Example 2.41 Define

$$T(x_1, x_2) = \begin{cases} \left(-\frac{1}{2}\left(\frac{x_2}{x_1}\right)^2, \frac{x_2}{x_1}\right) & \text{if } \; x_1 \ge x_2^4 \text{ and } x_1 > 0, \\ (-\infty, 0] \times \{0\} & \text{if } \; x_1 = x_2 = 0. \end{cases}$$

Then T is monotone on C where

$$C := \left\{(u, v) \in \mathbb{R}^2 | u \ge v^4\right\}.$$

Set $x = (x_1, x_2)$ and $y = (y_1, y_2)$, then

$$G_T(x, y) = \begin{cases} \frac{x_2 y_2}{x_1} - \frac{x_2^2}{2x_1}\left(1 + \frac{y_1}{x_1}\right) & \text{if } \; x \in C - \{(0,0)\} \text{ and } y \in C, \\ \frac{y_2^2}{2y_1} & \text{if } \; x_1 = 0 \text{ and } y_1 \ne 0, \\ 0 & \text{if } \; y \in C \text{ and } x = (0,0). \end{cases}$$

G_T is defined on C but it is not locally bounded on C, because if $u_n = \left(\frac{1}{n^4}, \frac{1}{n^2}\right)$ and $v_n = \left(\frac{1}{n^4}, \frac{1}{n}\right)$, then

$$G_T(u_n, v_n) = n - 1.$$

Note that $u_n \to (0,0)$ and $v_n \to (0,0)$ but $G_T(u_n, v_n) \to +\infty$. ▲

Chapter 3

σ-Monotone Bifunctions and Operators

In recent years, operators which have some kind of generalized monotonicity property have received a lot of attention (see for example [63] and the references therein). Many papers considering generalized monotonicity were devoted to the investigation of its relation to generalized convexity; others studied the existence of solutions of generalized monotone variational inequalities and, in some cases, derived algorithms for finding such solutions.

Monotone operators are known to have many very interesting properties. For instance, we have seen that a monotone operator T defined on a Banach space is locally bounded in the interior of its domain. Furthermore, if T is maximal monotone and $\mathcal{J}$ is the duality map, then $T + \lambda\mathcal{J}$ is surjective for every $\lambda > 0$. So the question naturally arises: are these properties shared by other operators which satisfy a more relaxed kind of monotonicity?

In a recent paper, Iusem, Kassay and Sosa [71] introduced the class of the so-called pre-monotone operators. This class includes monotone operators, but contains many more: for example, if T is monotone and R is globally bounded, then $T + R$ is pre-monotone. In fact, it includes ε-monotone operators which are related to the very useful ε-subdifferentials [73, 87]. In [71] it is shown that, in a finite dimensional space, pre-monotone operators retain the two above mentioned properties.

The present chapter is based on the original paper [6]. We will show that most results of [71] remain valid in infinite dimensional Banach spaces. We also prove that some other properties of monotone operators remain valid in a much more general context. More precisely, in Section 1 we will introduce the class of σ-monotone and maximal σ-monotone operators and we will study their properties. In Section 2, we will introduce the class of pre-monotone bifunctions which are related to the notion of pre-monotone operator. We will show that such bifunctions are locally bounded in the interior of their domain and we will deduce local boundedness of pre-monotone operators. We will also state and prove a generalization of the Libor Veselý theorem. In Section 3 we will prove that under some (CQ) conditions, the values of the sum of two maximal σ-monotone operators are weak*-closed. In Section 4 we will confine our attention to finite dimensions and prove the existence of solutions for an equilibrium problem in a (generally unbounded) closed convex subset of an Euclidean space. This result does not involve any kind of monotonicity. We will conclude this chapter by comparing some types of generalized monotone operators.

3.1 σ-Monotone Operators

Most definitions and many of the results of the section are essentially due to [71], the main difference being that in [71] one considers pre-monotone operators in $\mathbb{R}^n$, without specifying a given σ. In this section after some preliminary definitions and results, we show that every maximal σ-monotone operator is convex-valued and weak*-closed valued. In addition, if σ is usc, then the operator is sequentially norm$\times$weak*-closed. Moreover, we provide an example which shows that upper semicontinuity of σ is a necessary condition. In very special case, $X = \mathbb{R}$, we establish that if T is pre-monotone and closed, then σ_T is continuous.

Definition 3.1 *(i) Given an operator $T : X \to 2^{X^*}$ and a map $\sigma : \operatorname{dom} T \to \mathbb{R}_+$, T is said to be* σ-monotone *if for every $x, y \in \operatorname{dom} T$, $x^* \in T(x)$ and $y^* \in T(y)$,*

$$\langle x^* - y^*,\ x - y\rangle \geq -\min\{\sigma(x), \sigma(y)\}\|x - y\|. \tag{3.1}$$

(ii) An operator T is called pre-monotone *if it is σ-monotone for some $\sigma : \operatorname{dom} T \to \mathbb{R}_+$.*

(iii) A σ-monotone operator T is called maximal σ-monotone*, if for every operator T' which is σ'-monotone with $\operatorname{gr} T \subseteq \operatorname{gr} T'$ and σ' an extension of σ, one has $T = T'$.*

The notion of pre-monotone operators for the finite-dimensional case is introduced in [71]. The same paper also contains examples of maximal σ-monotone operators (see for instance [71, Proposition 5.8]).

Remark 3.2 (i) It should be noticed that $T : X \to 2^{X^*}$ is σ-monotone if and only if

$$\forall x, y \in \operatorname{dom} T,\ x^* \in T(x),\ y^* \in T(y), \quad \langle x^* - y^*,\ x - y\rangle \geq -\sigma(y)\|x - y\|. \tag{3.2}$$

(ii) If $\sigma(y) = 2\varepsilon \geq 0$ for each $y \in \operatorname{dom} T$, then T is called *ε-monotone* [87]. Therefore, every monotone and each ε-monotone operator is pre-monotone.

(iii) Definition 3.1 does not allow negative values for σ, since this can only happen in very special cases. For instance, if T satisfies (3.1) and its domain contains any line segment $[x_0, y_0] := \{(1 - t)x_0 + ty_0 : t \in [0, 1]\}$, then the set of points $x \in [x_0, y_0]$ where $\sigma(x) < 0$ is at most countable. Indeed, if this is not the case, then there exists $\varepsilon > 0$ such that $\sigma(x) < -\varepsilon$ for infinitely many $x \in [x_0, y_0]$. Choose $x_0^* \in T(x_0)$, $y_0^* \in T(y_0)$. Given $n \in \mathbb{N}$, choose $x_k = x_0 + t_k(y_0 - x_0)$, $k = 1, \dots n-1$, such that $0 < t_1 < \cdots < t_{n-1} < 1$ and $\sigma(x_k) < -\varepsilon$. Then choose $x_k^* \in T(x_k)$. Set $x_n = y_0$ and $x_n^* = y_0^*$. Relation (3.2) gives for all $k = 0, 1, \dots n-1$:

$$\left\langle x_{k+1}^* - x_k^*, x_{k+1} - x_k \right\rangle \geq \varepsilon \left\| x_{k+1} - x_k \right\| \Rightarrow$$
$$\left\langle x_{k+1}^* - x_k^*, y_0 - x_0 \right\rangle \geq \varepsilon \left\| y_0 - x_0 \right\|.$$

Adding these inequalities for $k = 0, 1, \dots n-1$ yields

$$\langle y_0^* - x_0^*, y_0 - x_0 \rangle \geq n\varepsilon \left\| y_0 - x_0 \right\|.$$

This should hold for each $n \in \mathbb{N}$, which is impossible.

(iv) The notion of pre-monotonicity is not suited to linear operators, since every σ-monotone linear operator $T : X \to X^*$ is in fact monotone. Indeed, in this case putting $y = 0$ in (3.2) we find

$$\forall x \in X, \quad \langle Tx, x \rangle \geq -\sigma(0) \|x\|. \tag{3.3}$$

By putting nx instead of x in (3.3) and then using $T(nx) = nT(x)$ one gets easily

$$\forall x \in X, \quad \langle Tx, x \rangle \geq -\frac{\sigma(0)}{n} \|x\|. \tag{3.4}$$

By taking the limit in inequality (3.4) as $n \to \infty$ we get $\langle Tx, x \rangle \geq 0$. Replacing x by $x - y$ we conclude that

$$\forall x, y \in X, \quad \langle Tx - Ty, x - y \rangle \geq 0.$$

This means that T is monotone.

(v) Every globally bounded operator is pre-monotone. Assume that T is globally bounded. Then there exists $M > 0$ such that $\|x^*\| \leq M$ for all (x, x^*) in $\operatorname{gr} T$. Now by setting $\sigma(y) = 2M$ for all $y \in \operatorname{dom} T$, we infer that T is pre-monotone. Note that if T and S are σ_1-monotone and σ_2-monotone respectively, such that $\operatorname{dom} T \cap \operatorname{dom} S \neq \emptyset$, then by taking $\sigma = \sigma_1 + \sigma_2$ on the $\operatorname{dom} T \cap \operatorname{dom} S$ one can get $T + S$ is σ-monotone.

(vi) From (ii) and (v) we deduce that if S is monotone and R is globally bounded then $T = R + S$ is pre-monotone.

(vii) A σ-monotone operator is maximal σ-monotone if and only if, for every operator T' which is σ'-monotone with $\operatorname{gr} T \subseteq \operatorname{gr} T'$ and $\sigma'(x) \leq \sigma(x)$ for all $x \in \operatorname{dom} T$, one has $T = T'$. ⧫

The following proposition is an easy consequence of Zorn's lemma, as for monotone operator.

Proposition 3.3 *Every σ-monotone operator has a maximal σ-monotone extension.*

Note: As it was pointed out in [71, page 817] maximal pre-monotonicity must refer to a given σ. As in [71], if we define a maximal pre-monotone operator as a pre-monotone one whose graph is not properly contained in the graph of another pre-monotone operator, then with this notion, no operator would be maximal premonotone. For instance, assume that $T : X \to 2^{X^*}$ is any pre-monotone operator which satisfies (3.1) for a given σ. Now define $T_n : X \to 2^{X^*}$ by $T_n(x) = T(x) + \overline{B}(0, n)$ for $n = 1, 2, ...$ where $\overline{B}(0, n)$ is the closure of $B(0, n)$. Then by part (vi) of Remark 3.2, T_n is pre-monotone with $\sigma_n = \sigma + 2n$ and $\operatorname{gr} T_n \subset \operatorname{gr} T_{n+1}$ for $n \in \mathbb{N}$. Thus $\operatorname{gr} T_n$ is an increasing chain and $\cup_{n=1}^{\infty} \operatorname{gr} T_n = X \times X^*$ and the operator with this graph is not pre-monotone. Thus, with this notion of maximal pre-monotonicity there would be no maximal pre-monotone operators.

Definition 3.4 *Let A be a subset of X. Given a mapping $\sigma : A \to \mathbb{R}_+$, two pairs $(x, x^*), (y, y^*) \in A \times X^*$ are σ-monotonically related if*

$$\langle x^* - y^*, x - y\rangle \geq -\min\{\sigma(x), \sigma(y)\}\|x - y\|.$$

The proof of the following proposition is obvious.

Proposition 3.5 *The σ-monotone operator $T : X \to 2^{X^*}$ is maximal σ-monotone if and only if, for every point $(x_0, x_0^*) \in X \times X^*$ and every extension σ' of σ to $\operatorname{dom} T \cup \{x_0\}$ such that (x_0, x_0^*) is σ'-monotonically related to all pairs $(y, y^*) \in \operatorname{gr} T$, we have $(x_0, x_0^*) \in \operatorname{gr} T$.*

Given an operator $T : X \to 2^{X^*}$, we define the function $\sigma_T : \operatorname{dom} T \to \mathbb{R}_+ \cup \{+\infty\}$ by

$$\sigma_T(y) = \inf\{a \in \mathbb{R}_+ : \langle x^* - y^*, x - y\rangle \geq -a\|x - y\|, \forall (x, x^*) \in \operatorname{gr} T, y^* \in T(y)\}.$$

Note that if the operator T is pre-monotone, then

$$\sigma_T = \inf\{\sigma : T \text{ is } \sigma\text{-monotone}\}$$

and thus σ_T is finite, and T is σ_T-monotone. Also in this case, it is obvious that

$$\sigma_T(y) = \max\left\{\sup\{\frac{\langle x^* - y^*, y - x\rangle}{\|y - x\|} : x \in X\backslash\{y\}, x^* \in T(x), y^* \in T(y)\}, 0\right\} \tag{3.5}$$

(see also [71]). The following result is due to [71] but we include a proof for the convenience of the reader.

Proposition 3.6 *Let an operator T be given.*

(i) σ_T is finite and T is σ_T-monotone, if and only if T is σ-monotone for some σ.

(ii) σ_T is finite and T is maximal σ_T-monotone, if and only if T is maximal σ-monotone for some σ.

Proof. We have only to prove that whenever T is maximal σ-monotone for some σ, then it is maximal σ_T-monotone. Assume that $S : X \to 2^{X^*}$ is σ'-monotone with $\operatorname{gr} T \subseteq \operatorname{gr} S$ and σ' an extension of σ_T. Since $\sigma' = \sigma_T \leq \sigma$ on $\operatorname{dom} T$, by Remark 3.2 part (vii) we get that $S = T$. Hence, T is maximal σ_T-monotone. This completes the proof. ■

Proposition 3.7 *Every maximal σ-monotone operator T is convex-valued and weak*-closed valued. Moreover, if σ is defined and usc at some point x_0 in $\operatorname{cl}\operatorname{dom} T$, then T is sequentially norm$\times$weak*-closed at x_0.*

Proof. Let $T : X \to 2^{X^*}$ be a maximal σ-monotone operator and $(x, x_1^*), (x, x_2^*)$ in $\operatorname{gr} T$, $\lambda \in [0, 1]$. Then for each $(y, y^*) \in \operatorname{gr} T$,

$$\begin{aligned}\langle \lambda x_1^* + (1-\lambda)x_2^* - y^*, x - y\rangle &= \lambda\langle x_1^* - y^*, x - y\rangle + (1-\lambda)\langle x_2^* - y^*, x - y\rangle \\ &\geq -\lambda \min\{\sigma(x), \sigma(y)\}\|x - y\| \\ &\quad - (1-\lambda)\min\{\sigma(x), \sigma(y)\}\|x - y\| \\ &= -\min\{\sigma(x), \sigma(y)\}\|x - y\|.\end{aligned}$$

That is, $(x, \lambda x_1^* + (1-\lambda)x_2^*)$ is σ-monotonically related with all $(y, y^*) \in \operatorname{gr} T$. Now, it follows from Proposition 3.5 that $(x, \lambda x_1^* + (1-\lambda)x_2^*) \in \operatorname{gr} T$ which implies that $T(x)$ is convex.

Assume that x^* is in the weak*-closure of $T(x)$. Then there exists a sequence x_n^* in $T(x)$ such that $x_n^* \overset{w^*}{\to} x^*$. For each $(y, y^*) \in \operatorname{gr} T$ we have

$$\langle x_n^* - y^*, x - y\rangle \geq -\min\{\sigma(x), \sigma(y)\}\|x - y\|.$$

Passing to limit when n goes to ∞, the preceding inequality implies that

$$\langle x^* - y^*, x - y\rangle \geq -\min\{\sigma(x), \sigma(y)\}\|x - y\|.$$

Thus $(x, x^*) \in \operatorname{gr} T$. This means that $T(x)$ is weak*-closed.

We now show sequential closedness: suppose that (x_n, x_n^*) is a sequence in $\operatorname{gr} T$ such that $x_n \to x_0$ and $x_n^* \overset{w^*}{\to} x_0^*$. It follows from the σ-monotonicity of T that for each $(y, y^*) \in \operatorname{gr} T$ we have

$$\langle x_n^* - y^*, x_n - y\rangle \geq -\min\{\sigma(x_n), \sigma(y)\}\|x_n - y\|.$$

By taking limits in the above inequality and using the upper semicontinuity of σ at x_0 we get

$$\langle x_0^* - y^*, x_0 - y\rangle \geq -\min\{\sigma(x_0), \sigma(y)\}\|x_0 - y\|$$

which implies that (x_0, x_0^*) is σ-monotonically related with all $(y, y^*) \in \operatorname{gr} T$. By using Proposition 3.5 we deduce that $(x_0, x_0^*) \in \operatorname{gr} T$. ■

We note that, as for monotone operators, in general $\operatorname{gr} T$ is only sequentially norm$\times$weak*-closed, not norm$\times$weak*-closed [29]. However, we will see in the next section that maximal σ-monotone operators are actually usc in the interior of their domains.

The assumption of upper semicontinuity of σ cannot be omitted from Proposition 3.7, as the following example shows. This is also an example of a pre-monotone operator which is not ε-monotone. Note that for $T : \mathbb{R} \to \mathbb{R}$ we have

$$\sigma_T(y) = \max\left\{\sup_{x \le y}\{T(x) - T(y)\}, \sup_{x \ge y}\{T(y) - T(x)\}\right\}. \tag{3.6}$$

Example 3.8 We define the functions $\varphi, \sigma : \mathbb{R} \to \mathbb{R}$ by

$$\varphi(x) = \begin{cases} x\sin^2 x & \text{if} \quad x \ge 0, \\ 0 & \text{if} \quad x < 0, \end{cases}$$

and

$$\sigma(x) = \max\{\varphi(x), \max_{z \le x}\varphi(z) - \varphi(x)\}.$$

We show that φ is σ-monotone, i.e., for all $x, y \in \mathbb{R}$ the following inequality holds:

$$(\varphi(x) - \varphi(y))(x - y) \ge -\min\{\sigma(x), \sigma(y)\}|x - y|.$$

We may assume without loss of generality that $x \le y$, so we have to prove that $\varphi(x) - \varphi(y) \le \min\{\sigma(x), \sigma(y)\}$. Indeed,

$$\varphi(x) - \varphi(y) \le \varphi(x) \le \sigma(x)$$

and

$$\varphi(x) - \varphi(y) \le \max_{z \le y}\varphi(z) - \varphi(y) \le \sigma(y)$$

so φ is σ-monotone. Note that φ is not ε-monotone since

$$(\varphi(x) - \varphi(y))\operatorname{sgn}(x - y)$$

is not bounded from below (take $y = 2k\pi + \pi/2$, $x = 2k\pi + \pi$ for large $k \in \mathbb{N}$).

We now change φ and σ at one point: define $T, \sigma_1 : \mathbb{R} \to \mathbb{R}$ by

$$T(x) = \begin{cases} \varphi(x) & \text{if} \quad x \ne \frac{\pi}{2}, \\ \frac{\pi}{4} & \text{if} \quad x = \frac{\pi}{2}, \end{cases} \qquad \text{and} \quad \sigma_1(x) = \begin{cases} \sigma(x) & \text{if} \quad x \ne \frac{\pi}{2}, \\ \frac{\pi}{4} & \text{if} \quad x = \frac{\pi}{2}. \end{cases}$$

One can readily show that T is σ_1-monotone.

Now let $\widetilde{T}$ be a maximal σ_1-monotone extension of T. Its graph is not closed; indeed $(\pi/2, \pi/2)$ belongs to the closure of $\operatorname{gr}\widetilde{T}$. However, it does not belong to $\operatorname{gr}\widetilde{T}$ since it is not σ_1-monotonically related to $(\pi, 0) \in \operatorname{gr}\widetilde{T}$: since

$$\sigma_1(\pi) = \max_{z \le \pi}\varphi(z) \ge \varphi(\pi/2) = \pi/2,$$

one has

$$(\frac{\pi}{2} - 0)\operatorname{sgn}(\frac{\pi}{2} - \pi) = -\frac{\pi}{2} < -\frac{\pi}{4} = -\min\{\sigma_1(\frac{\pi}{2}), \sigma_1(\pi)\}. \qquad \blacktriangle$$

Note that in Proposition 3.7 we observed that if T is σ-monotone and σ is usc, then $\operatorname{gr} T$ is sequentially norm×weak*-closed. Moreover, in Example 3.8 it is shown that upper semicontinuity of σ cannot be omitted from the statement of Proposition 3.7. Now in the following (in case $X = \mathbb{R}$) we show that closedness of the graph of a σ-monotone and single-valued operator T implies the continuity of σ_T in case T is single-valued.

Lemma 3.9 *Assume that $T : \mathbb{R} \to \mathbb{R}$ is σ-monotone. Then T is locally bounded. Moreover, if $\operatorname{gr} T$ is closed, then T is continuous.*

Proof. First we show that T is locally bounded on $\mathbb{R}$. Assume that $a < b$. Note that

$$\sigma_T(y) = \max\left\{\sup_{x\le y}\{T(x)-T(y)\}, \sup_{x\ge y}\{T(y)-T(x)\}\right\}.$$

Thus $\sigma_T(b) \ge \sup_{x\le b}(T(x)-T(b))$ and so $T(x) \le \sigma_T(b) + T(b)$ for all $x \le b$. i.e., T is bounded above on $(-\infty, b]$. Likewise, $\sigma_T(a) \ge \sup_{a\le x}(T(a)-T(x))$. Therefore, $T(x) \ge T(a) - \sigma_T(a)$, that is T is bounded below on $[a, +\infty)$. Hence T is bounded on every interval $[a, b]$.

Now assume that $\operatorname{gr} T$ is closed but it is not continuous. Then there exists a sequence $\{x_n\}$ in $\mathbb{R}$ converging to some x, such that $\{T(x_n)\}$ does not converge to $T(x)$. Thus there exists $\varepsilon > 0$ such that $|T(x_n) - T(x)| \ge \varepsilon$ for infinitely many $n \in \mathbb{N}$. Since T is locally bounded, there would be a subsequence (which we denote again by $\{T(x_n)\}$ for simplicity) converging to a point $a \in \mathbb{R}$ such that $|a - T(x)| \ge \varepsilon$. This means that $(x_n, T(x_n)) \to (x, a) \ne (x, T(x))$, thus contradicting the fact that T is closed. ∎

Proposition 3.10 *Suppose that $T : \mathbb{R} \to \mathbb{R}$ is σ-monotone and $\operatorname{gr} T$ is closed. Then σ_T is continuous.*

Proof. For the continuity of σ_T it is enough to show that $\sup_{x\le y}\{T(x)-T(y)\}$ and $\sup_{x\ge y}\{T(y)-T(x)\}$ are continuous as functions of y. First we show that $\sup_{x\le y}\{T(x)-T(y)\}$ is continuous. By the above lemma T is continuous. So it is enough we prove that $f(y) = \sup_{x\le y} T(x)$ is continuous. The continuity of T implies that T is locally uniformly continuous. Let $y_0 \in \mathbb{R}$. For a given $\varepsilon > 0$ there exists $\delta > 0$ such that

$$|T(x) - T(y_0)| < \frac{\varepsilon}{2} \quad \forall x \in [y_0 - \delta, y_0 + \delta]. \tag{3.7}$$

Set $A = \left[y_0 - \frac{\delta}{2}, y_0 + \frac{\delta}{2}\right]$ and take $y \in A$. It follows from (3.7) that

$$\left|\sup_{x\in A, x\le y} T(x) - \sup_{x\in A, x\le y_0} T(x)\right| < \varepsilon. \tag{3.8}$$

Note that

$$f(y) = \sup_{x\le y} T(x) = \max\left\{\sup_{x<y_0-\frac{\delta}{2}} T(x), \sup_{y_0-\frac{\delta}{2}\le x\le y} T(x)\right\}$$

and

$$f(y_0) = \sup_{x \le y_0} T(x) = \max \left\{ \sup_{x < y_0 - \frac{\delta}{2}} T(x), \sup_{y_0 - \frac{\delta}{2} \le x \le y_0} T(x) \right\}$$

For simplicity in calculations, set

$$a = \sup_{x < y_0 - \frac{\delta}{2}} T(x), \ b = \sup_{y_0 - \frac{\delta}{2} \le x \le y} T(x) \quad \text{and} \quad c = \sup_{y_0 - \frac{\delta}{2} \le x \le y_0} T(x).$$

Therefore $f(y) = \max\{a, b\}$ and $f(y_0) = \max\{a, c\}$. Using (3.8) we infer that $|b - c| < \varepsilon$, i.e.,

$$-\varepsilon + c < b < \varepsilon + c$$

which implies

$$\max\{c - \varepsilon, a\} < \max\{b, a\} < \max\{\varepsilon + c, a\}. \tag{3.9}$$

On the other hand,

$$-\varepsilon + \max\{c, a\} = \max\{c - \varepsilon, a - \varepsilon\} \le \max\{c - \varepsilon, a\} \tag{3.10}$$

and

$$\max\{\varepsilon + c, a\} \le \max\{\varepsilon + c, a + \varepsilon\} = \max\{c, a\} + \varepsilon. \tag{3.11}$$

Combining (3.9), (3.10) and (3.11) we obtain

$$-\varepsilon + \max\{c, a\} < \max\{b, a\} < \max\{c, a\} + \varepsilon,$$

so $|f(y) - f(y_0)| < \varepsilon$. This means that f is continuous. In a similar manner one can get $\sup_{x \ge y} \{T(y) - T(x)\}$ is continuous. ■

So the question naturally arises: Can we extend the above result to more general spaces? For instance, given a pre-monotone operator T with norm×weak*-closed graph, is σ_T usc?

3.2 Local Boundedness and Related Properties

In this section we will point out the connection between locally boundedness of σ-monotone bifunctions and σ-monotone operators. Also, we will prove a generalization of the Libor Veselý theorem: if T is maximal pre-monotone, $\operatorname{cl}\operatorname{dom} T$ is convex and T is locally bounded at $x_0 \in \operatorname{cl}\operatorname{dom} T$, then x_0 is an interior point of $\operatorname{dom} T$. Moreover, we will see some properties of σ-monotone operators can be more easily investigated through the use of σ-monotone bifunctions that we now introduce. Let X be a Banach space, C a nonempty subset of X and $\sigma : C \to \mathbb{R}_+$ be a map. A bifunction $F : C \times C \to \mathbb{R}$ will be called *σ-monotone* if

$$\forall x, y \in C, \quad F(x, y) + F(y, x) \le \min\{\sigma(x), \sigma(y)\}\|x - y\|. \tag{3.12}$$

Equivalently, F is σ-monotone if

$$\forall x, y \in C, \quad F(x, y) + F(y, x) \le \sigma(y)\|x - y\|. \tag{3.13}$$

This notion is a generalization of the notion of monotone bifunction introduced in Chapter 2, where σ is identically zero.

Given any bifunction $F : C \times C \to \mathbb{R}$, we define as in Chapter 2 the operator $A^F : X \to 2^{X^*}$ by

$$A^F(x) = \begin{cases} \{x^* \in X^* : \forall y \in C, F(x,y) \geq \langle x^*, y-x\rangle\} & \text{if } x \in C, \\ \emptyset & \text{if } x \notin C. \end{cases}$$

Note that in case $F(x,x) = 0$ for all $x \in C$, one has $A^F(x) = \partial F(x,\cdot)(x)$ (the subdifferential of the function $F(x,\cdot)$ at x).

Proposition 3.11 *For a σ-monotone bifunction F, A^F is σ-monotone.*

Proof. Let $x^* \in A^F(x)$ and $y^* \in A^F(y)$. By the definition of A^F,

$$F(x,y) \geq \langle x^*, y-x\rangle$$

and

$$F(y,x) \geq \langle y^*, x-y\rangle.$$

From these inequalities we obtain

$$\langle x^* - y^*, x-y\rangle \geq -F(x,y) - F(y,x) \geq -\min\{\sigma(x), \sigma(y)\}\|x-y\|.$$

This means that A^F is σ-monotone. ■

Definition 3.12 *A σ-monotone bifunction F is called* maximal σ-monotone *if A^F is maximal σ-monotone.*

For a given operator $T : X \to 2^{X^*}$, as in Chapter 2 we define $G_T : \operatorname{dom} T \times \operatorname{dom} T \to \mathbb{R} \cup \{+\infty\}$ by $G_T(x,y) = \sup_{x^* \in T(x)} \langle x^*, y-x\rangle$. For each $x \in \operatorname{dom} T$, $G_T(x,\cdot)$ is lsc and convex, and $G_T(x,x) = 0$. The following result shows that G_T is actually real-valued whenever T is σ-monotone, and establishes some relations between σ-monotonicity of G_T and T.

Proposition 3.13 *Let T be an operator. Then the following statements are true.*

(i) If T is σ-monotone, then G_T is a real-valued, σ-monotone bifunction.

(ii) If T is maximal σ-monotone, then G_T is a maximal σ-monotone bifunction and $A^{G_T} = T$.

(iii) Suppose that T is a σ-monotone operator with weak-closed convex values and $\operatorname{dom} T = X$. If G_T is maximal σ-monotone, then T is maximal σ-monotone.*

Proof. (i) Let $T : X \to 2^{X^*}$ be σ-monotone. Given $x, y \in \operatorname{dom} T$, for every $x^* \in T(x)$ and $y^* \in T(y)$, we have

$$\langle x^* - y^*, x-y\rangle \geq -\sigma(y)\|x-y\|.$$

Thus

$$\langle y^*, x - y\rangle + \langle x^*, y - x\rangle \le \sigma(y)\|x - y\|.$$

This implies that

$$\sup_{y^*\in T(y)} \langle y^*, x - y\rangle + \sup_{x^*\in T(x)} \langle x^*, y - x\rangle \le \sigma(y)\|x - y\|.$$

Form here we conclude that

$$\forall x, y \in \operatorname{dom} T, \quad G_T\,(x, y) + G_T\,(y, x) \le \sigma(y)\|x - y\|.$$

Consequently, $G_T\,(x, y) \in \mathbb{R}$ for all $x, y \in \operatorname{dom} T$ and G_T is a σ-monotone bifunction.

(ii) Let $(x, z^*) \in \operatorname{gr} T$. For every $y \in C$ we have

$$G_T\,(x, y) = \sup_{x^*\in T(x)} \langle x^*, y - x\rangle \ge \langle z^*, y - x\rangle.$$

This means that $z^* \in A^{G_T}\,(x)$; i.e., $T(x) \subseteq A^{G_T}(x)$. It follows from Proposition 3.11 and part (i) that A^{G_T} is σ-monotone. Since T is maximal σ-monotone, we conclude that $T = A^{G_T}$.

(iii) Since G_T is maximal σ-monotone by assumption, A^{G_T} is maximal σ-monotone. Let $x \in X$ and $z^* \in A^{G_T}\,(x)$. Then

$$G_T\,(x, y) = \sup_{x^*\in T(x)} \langle x^*, y - x\rangle \ge \langle z^*, y - x\rangle.$$

Now, the separation theorem [see Chapter 1, Corollary 1.9] implies that $z^* \in T\,(x)$. Thus, $\operatorname{gr} A^{G_T} \subseteq \operatorname{gr} T$. This implies that $T = A^{G_T}$ and T is maximal σ-monotone. ■

Remark 3.14 Given a maximal σ-monotone bifunction F, according to Proposition 3.13, we can construct A^F and the σ-monotone bifunction $G := G_{A^F}$. One has $G(x, y) \le F(x, y)$ for all $x, y \in \operatorname{dom} A^F$. It follows from Proposition 3.13 that $A^F = A^G$. However (as it was noted in Chapter 2), Example 2.5 of [64] implies that the correspondence $F \mapsto A^F$ is not one to one, even for the monotone case $\sigma \equiv 0$. ♦

We now generalize the definition of locally bounded bifunctions (see Definition 2.14 from Chapter 2).

Definition 3.15 *A bifunction $F : C \times C \to \mathbb{R}$ is called:*

(i) Locally bounded *at $(x_0, y_0) \in X \times X$ if there exist an open neighborhood V of x_0, an open neighborhood W of y_0 and $M \in \mathbb{R}$ such that $F(x, y) \le M$ for all $(x, y) \in (V \times W) \cap (C \times C)$.*

(ii) Locally bounded *on $K \times L \subseteq X \times X$, if it is locally bounded at each $(x, y) \in K \times L$.*

(iii) Locally bounded *at $x_0 \in X$ if it is locally bounded at (x_0, x_0), i.e., there exist an open neighborhood V of x_0 and $M \in \mathbb{R}$ such that $F(x, y) \le M$ for all $x, y \in V \cap C$.*

(iv) Locally bounded on $K \subseteq X$, if it is locally bounded at each $x \in K$.

If a bifunction (not necessarily σ-monotone) $F : C \times C \to \mathbb{R}$ is locally bounded at $x_0 \in \operatorname{int} C$, then A^F is locally bounded at x_0 [Chapter 2, Remark 2.15]. Consequently, if T is an operator such that G_T is locally bounded at $x_0 \in \operatorname{int} \operatorname{dom} T$, then T is locally bounded at x_0 since $T(x) \subseteq A^{G_T}(x)$ for all $x \in X$. As in Chapter 2, this will be the main instrument for showing local boundedness of operators.

We will show that σ-monotone bifunctions are locally bounded in the interior of their domain, under mild assumptions. In case $X = \mathbb{R}^n$ we can give a constructive proof.

Proposition 3.16 *Let $X = \mathbb{R}^n$ and $C \subseteq \mathbb{R}^n$. Assume that $F : C \times C \to \mathbb{R}$ is σ-monotone and $F(x, \cdot)$ is lsc and quasi-convex for every $x \in C$. Then F is locally bounded at every point of* $\operatorname{int} C \times \operatorname{int} C$.

Proof. Let $(x_0, y_0) \in \operatorname{int} C \times \operatorname{int} C$. Since the space is finite-dimensional, we can find $z_1, z_2, \ldots, z_m \in C$ such that $V := \operatorname{co}\{z_1, z_2, \ldots, z_m\} \subseteq C$ is a neighborhood of y_0. Let $U \subseteq C$ be a compact neighborhood of x_0 in C. Set $M_k = \min_{x \in U} F(z_k, x)$; the minimum exists since $F(z_k, \cdot)$ is lsc. For every $x \in U$, $y \in V$ we find, using quasi-convexity of $F(x, \cdot)$ and σ-monotonicity of F:

$$\begin{aligned} F(x, y) &\leq \max_{1 \leq k \leq m} F(x, z_k) \\ &\leq \max_{1 \leq k \leq m} \{\sigma(z_k) \|x - z_k\| - F(z_k, x)\} \\ &\leq \max_{1 \leq k \leq m} \sigma(z_k) \sup_{z \in U, w \in V} \|z - w\| + \max_{1 \leq k \leq m} (-M_k). \end{aligned}$$

Since U and V are both bounded, $\sup_{z \in U, w \in V} \|z - w\|$ is finite. This completes the proof. ■

For the general case of a Banach space X, we will apply Lemma 2.18 from Chapter 2.

Theorem 3.17 *Suppose X is a Banach space, C is a subset of X and $F : C \times C \to \mathbb{R}$ is a σ-monotone bifunction such that for every $x \in C$, $F(x, \cdot)$ is lsc and quasi-convex. Further, suppose that for some $x_0 \in C$ and $y_0 \in \operatorname{int} C$ there exists $\varepsilon > 0$ such that $B(y_0, \varepsilon) \subseteq C$ and for each $y \in B(y_0, \varepsilon)$, $F(y, \cdot)$ is bounded from below on $B(x_0, \varepsilon) \cap C$ (note that this bound may depend on y). Then F is locally bounded at (x_0, y_0).*

Proof. Let $\varepsilon > 0$ be as in the assumption. Define $g : B(y_0, \varepsilon) \to \mathbb{R} \cup \{+\infty\}$ by

$$g(y) := \sup\{F(x, y) : x \in B(x_0, \varepsilon) \cap C\}.$$

For every $y \in B(y_0, \varepsilon)$ and $x \in B(x_0, \varepsilon) \cap C$, σ-monotonicity of F implies

$$F(x, y) \leq \min\{\sigma(x), \sigma(y)\}\|x - y\| - F(y, x) \leq \sigma(y)(\varepsilon + \|y - x_0\|) - M_y$$

where M_y is a lower bound of $F(y, \cdot)$ on $B(x_0, \varepsilon) \cap C$. Therefore, g is real-valued. On the other hand, g is lsc and quasi-convex and also $y_0 \in \operatorname{int} \operatorname{dom} g$. By Lemma

2.18 from Chapter 2, there exists $\delta < \varepsilon$ and $M \in \mathbb{R}$ such that $g(y) \le M$ for all $y \in B(y_0, \delta)$. Then by the definition of g we get $F(x, y) \le M$ for all $y \in B(y_0, \delta)$ and $x \in B(x_0, \delta) \cap C$; i.e., F is locally bounded at (x_0, y_0). ■

The condition "$F(y, \cdot)$ is bounded from below on $B(x_0, \varepsilon) \cap C$" can be easily removed by imposing some usual assumptions on the bifunction F or the space X, as shown in the following two results.

Corollary 3.18 *Suppose X is a reflexive Banach space, C is a subset of X and $F : C \times C \to \mathbb{R}$ is a σ-monotone bifunction such that for every $x \in C$, $F(x, \cdot)$ is lsc and quasi-convex. Then F is locally bounded at every point of $\operatorname{int} C \times \operatorname{int} C$. If in addition C is weakly closed, then F is locally bounded on $C \times \operatorname{int} C$.*

Proof. Let $x_0 \in \operatorname{int} C$. Choose $\varepsilon > 0$ such that $\overline{B}(x_0, \varepsilon) \subseteq C$. By assumption $F(x, \cdot)$ is lsc and quasi-convex, so it is weakly lsc. For every $y \in C$, $F(y, \cdot)$ attains its minimum on the weakly compact set $\overline{B}(x_0, \varepsilon)$ and so $F(y, \cdot)$ is bounded from below on $B(x_0, \varepsilon)$. Therefore, all conditions of Theorem 3.17 are satisfied. Thus F is locally bounded at every point of $\operatorname{int} C \times \operatorname{int} C$.

If in addition C is weakly closed, then for any $x_0 \in C$ and $\varepsilon > 0$, $\overline{B}(x_0, \varepsilon) \cap C$ is weakly compact and we can repeat the previous argument. ■

Corollary 3.19 *Suppose X is a Banach space, C is a subset of X and $F : C \times C \to \mathbb{R}$ is a σ-monotone bifunction such that for every $x \in C$, $F(x, \cdot)$ is lsc and convex. Then F is locally bounded at any point of $C \times \operatorname{int} C$.*

Proof. Let $x_0 \in C$ and $y_0 \in \operatorname{int} C$. Choose $\varepsilon > 0$ such that $B(y_0, \varepsilon) \subseteq C$. For every $y \in B(y_0, \varepsilon)$, the subdifferential of $\partial F(y, \cdot)$ is nonempty at y. Choose $y^* \in \partial F(y, \cdot)(y)$. Then for every $x \in B(x_0, \varepsilon) \cap C$ one has

$$F(y, x) - F(y, y) \ge \langle y^*, x - y\rangle \ge -\|y^*\| \, \|x - y\| \ge -\|y^*\| \, (\varepsilon + \|x_0 - y\|).$$

Thus $F(y, \cdot)$ is bounded from below on $B(x_0, \varepsilon) \cap C$. By Theorem 3.17, F is locally bounded at (x_0, y_0). ■

We immediately obtain a generalization of Proposition 3.5 in [71] to general Banach spaces:

Corollary 3.20 *Suppose that X is a Banach space and $T : X \to 2^{X^*}$ is a pre-monotone operator. Then T is locally bounded at every point of $\operatorname{int} \operatorname{dom} T$.*

Proof. Apply Corollary 3.19 to G_T. ■

Corollary 3.21 (Rockafellar) *Every set-valued monotone operator T from X to X^* is locally bounded on $\operatorname{int} \operatorname{dom} T$.*

For maximal σ-monotone operators, there is a kind of converse to Corollary 3.20, generalizing the Libor Veselý theorem [see Chapter 1 Theorem 1.40]. We first show:

Lemma 3.22 *If T is maximal σ-monotone, then for all $x \in \operatorname{dom} T$ one has $T(x) + N_{\operatorname{dom} T}(x) \subseteq T(x)$.*

Proof. Take $w^* \in N_{\operatorname{dom} T}(z)$ and define

$$T_1(x) = \begin{cases} T(x) & \text{if } x \neq z, \\ T(x) + \mathbb{R}_+ w^* & \text{if } x = z. \end{cases}$$

Then $T(x) \subseteq T_1(x)$ for all $x \in \operatorname{dom} T$. For $z^* \in T(z)$, $y^* \in T(y)$ and $\lambda > 0$,

$$\begin{aligned} \langle z^* + \lambda w^* - y^*, z - y\rangle &= \langle z^* - y^*, z - y\rangle + \lambda \langle w^*, z - y\rangle \\ &\geq -\min\{\sigma(z), \sigma(y)\} \|z - y\|. \end{aligned}$$

Thus T_1 is σ-monotone. By the maximality of T we get $T = T_1$, which completes the proof. ■

Theorem 3.23 *Suppose that T is maximal σ-monotone, σ is defined and usc on* $\operatorname{cl} \operatorname{dom} T$. *Let $x_0 \in \operatorname{cl} \operatorname{dom} T$. If T is locally bounded at x_0, then $x_0 \in \operatorname{dom} T$. If in addition* $\operatorname{cl} \operatorname{dom} T$ *is convex, then $x_0 \in \operatorname{int} \operatorname{dom} T$.*

Proof. Since T is locally bounded at x_0, there exists a neighborhood U of x_0 such that $T(U)$ is bounded. Choose a sequence $\{x_n\} \subseteq (\operatorname{dom} T) \cap U$ such that $x_n \to x_0$ and choose $x_n^* \in T(x_n)$. It follows from Alaoglou's theorem [see Chapter 1, Theorem 1.11] that there exist a subnet $\{(x_\alpha, x_\alpha^*)\}$ of $\{(x_n, x_n^*)\}$ and $x_0^* \in X^*$ such that $x_\alpha^* \xrightarrow{w^*} x_0^*$. Therefore for all $(y, y^*) \in \operatorname{gr} T$, by upper semicontinuity of σ,

$$\begin{aligned} \langle x_0^* - y^*, x_0 - y\rangle &= \lim_\alpha \langle x_\alpha^* - y^*, x_\alpha - y\rangle \\ &\geq -\limsup_\alpha \min\{\sigma(x_\alpha), \sigma(y)\} \|x_\alpha - y\| \\ &\geq -\min\{\sigma(x_0), \sigma(y)\} \|x_0 - y\|. \end{aligned}$$

Thus (x_0, x_0^*) is σ-monotonically related with all $(y, y^*) \in \operatorname{gr} T$. So $x_0^* \in T(x_0)$ and $x_0 \in \operatorname{dom} T$.

Now let $\operatorname{cl} \operatorname{dom} T$ be convex. We will show that $U \subseteq \operatorname{int} \operatorname{cl} \operatorname{dom} T$. Indeed, if not, then U contains a boundary point of $\operatorname{cl} \operatorname{dom} T$. By the Bishop-Phelps theorem (see [91, Chapter 3]) it will also contain a support point of $\operatorname{cl} \operatorname{dom} T$, i.e., there exist $z \in U \cap \operatorname{cl} \operatorname{dom} T$ and $0 \neq w^* \in X^*$ such that

$$\langle w^*, z\rangle = \sup\{\langle w^*, y\rangle : y \in \operatorname{cl} \operatorname{dom} T\}.$$

We know that T is locally bounded at z, hence $z \in \operatorname{dom} T$. On the other hand, $w^* \in N_{\operatorname{dom} T}(z)$, thus the cone $N_{\operatorname{dom} T}(z)$ is not equal to $\{0\}$. Then Lemma 3.22 shows that $T(z)$ cannot be bounded, a contradiction.

Thus $U \subseteq \operatorname{int} \operatorname{cl} \operatorname{dom} T$. Since T is locally bounded on U, we obtain $U \subseteq \operatorname{dom} T$, hence $x_0 \in \operatorname{int} \operatorname{dom} T$. ■

We now deduce some properties related to local boundedness.

Proposition 3.24 *Suppose $T : X \to 2^{X^*}$ is maximal σ-monotone and σ is usc. Then*

(i) The operator T is usc in $\operatorname{int}\operatorname{dom} T$ *from the norm topology in X to the weak*-topology in X^*;*

(ii) If X is finite-dimensional, then for every $y \in \operatorname{int}\operatorname{dom} T$, $\sigma_T(y)$ is given by the following formula:

$$\sigma_T(y) = \sup\left\{ \frac{\langle x^* - y^*, y - x\rangle}{\|y - x\|} : x \neq y, (x, x^*) \in \operatorname{gr} T, y^* \in T(y) \right\}. \tag{3.14}$$

Proof. Fix $y \in \operatorname{int}\operatorname{dom} T$. To show upper semicontinuity at y, it is sufficient to show that for any net $\{(y_\alpha, y_\alpha^*)\}$ in $\operatorname{gr} T$ such that $y_\alpha \to y$ in X, there exists a weak*-cluster point of $\{y_\alpha^*\}$ in $T(y)$. Since T is locally bounded at y we may assume that both $\{y_\alpha\}$ and $\{y_\alpha^*\}$ are bounded and, by selecting a subnet if necessary, $y_\alpha^* \xrightarrow{w^*} y^*$. Since $\{y_\alpha^*\}$ is bounded, we have

$$\langle y_\alpha^*, y_\alpha\rangle \to \langle y^*, y\rangle .$$

As in the proof of Proposition 3.7 we deduce that $y^* \in T(y)$.

To show part (ii), choose any sequence $\{x_n\}_{n\in\mathbb{N}} \subseteq \operatorname{dom} T$ converging to y with $y \neq x_n$, and let $x_n^* \in T(x_n)$. Then the sequence $\{x_n^*\}$ is bounded. By selecting a subsequence if necessary, we may again assume that $x_n^* \to z^* \in T(y)$. Since

$$\sup\left\{ \frac{\langle x^* - y^*, y - x\rangle}{\|y - x\|} : x \neq y, (x, x^*) \in \operatorname{gr} T, y^* \in T(y) \right\} \geq \frac{\langle x_n^* - z^*, y - x_n\rangle}{\|y - x_n\|}$$
$$\geq -\|x_n^* - z^*\| \to 0,$$

relation (3.14) follows from relation (3.5). ■

Next we show that under appropriate conditions, a σ-monotone bifunction is not only locally bounded, but also bounded by a small number in a neighborhood of any interior point. This is a consequence of the following more general result.

Proposition 3.25 *Suppose that $F : C \times C \to \mathbb{R}$ is a σ-monotone bifunction such that $F(x, x) = 0$ for all $x \in C$. Assume that $F(x, \cdot)$ is lsc and convex for each $x \in C$ and σ is usc. If $x_0 \in \operatorname{int} C$, then there exist an open neighborhood V of x_0 and $K \in \mathbb{R}$ such that $F(y, x) \leq K\|x - y\|$ for all $x \in V$ and $y \in C$.*

Proof. From $F(x, x) = 0$ for all $x \in C$, we infer that $A^F(x) = \partial F(x, \cdot)(x)$. Since $F(x, \cdot)$ is lsc and convex, the subdifferential of $F(x, \cdot)$ at each $x \in \operatorname{int} C$ is nonempty-valued. Thus $\operatorname{int} C \subseteq \operatorname{dom} A^F$, so the σ-monotone operator A^F is locally bounded at x_0. Therefore, there exist an open neighborhood $V_1 \subseteq C$ of x_0 and $K_1 \in \mathbb{R}$ such that $\|x^*\| \leq K_1$ for all $x^* \in A^F(x)$, $x \in V_1$. Since σ is usc at x_0, it is bounded from above by a number K_2 on a neighborhood V_2 of x_0. Then for each $y \in C$ and $x \in V := V_1 \cap V_2$, if we choose $x^* \in A^F(x)$ we get

$$\begin{aligned} F(y, x) &\leq -F(x, y) + \sigma(x)\|y - x\| \\ &\leq -\langle x^*, y - x\rangle + K_2\|y - x\| \leq (K_1 + K_2)\|y - x\| \end{aligned}$$

and the proof of the proposition is complete. ■

3.3 Pre-monotonicity and Related Results

In this section we generalized Theorems 1.42 and 1.43 form Chapter 1 to σ-monotone operators. In addition, we extend Theorem 2.29 from Chapter 2. Moreover, we introduce the notion of σ-convexity and we investigate some further results.

Let the operators $T, S : X \to 2^{X^*}$ and a map $\sigma : \operatorname{dom} T \cup \operatorname{dom} S \to \mathbb{R}_+$ be given. Then T (respectively S) is σ-monotone with respect to this σ if for every $x, y \in \operatorname{dom} T$ (respectively $x, y \in \operatorname{dom} S$), $x^* \in T(x)$ and $y \in T(y)$ (respectively $x^* \in S(x)$ and $y^* \in S(y)$) relation (3.1) is satisfied. Roughly speaking, when we consider two operators and say that they are σ-monotone with respect to the same function σ, we tacitly assume that σ defined on the union of their domains.

Next theorem is an extension of Theorem 1.42 from Chapter 1 to σ-monotone operators. The idea of the proof was first used for monotone operators by A. Verona and M. E. Verona [113] and then by J. M. Borwein [25].

Theorem 3.26 *Let X be a Banach space and let S and $T : X \to 2^{X^*}$ be σ-monotone operators. Suppose that*

$$0 \in \operatorname{core}[\operatorname{co} \operatorname{dom} T - \operatorname{co} \operatorname{dom} S]. \tag{CQ}$$

Then there exist $r, c > 0$ such that, for any $x \in \operatorname{dom} T \cap \operatorname{dom} S$, $t^* \in T(x)$ and $s^* \in S(x)$,

$$\max(||t^*||, ||s^*||) \leq c(r + ||x||)(2r + ||t^* + s^*||).$$

Proof. Consider the function

$$\rho_T(x) = \sup\left\{\frac{\langle z^*, x - z\rangle}{1 + ||z||} : (z, z^*) \in \operatorname{gr} T\right\}.$$

ρ_T is lsc and convex as supremum of affine functions. If $x \in \operatorname{dom} T$, $x^* \in T(x)$ then for all $z \in \operatorname{dom} T$ and $z^* \in T(z)$ we have

$$\begin{aligned}
\frac{\langle z^*, x - z\rangle}{1 + ||z||} &= \frac{\langle z^* - x^*, x - z\rangle}{1 + ||z||} + \frac{\langle x^*, x - z\rangle}{1 + ||z||} \\
&\leq \frac{\min\{\sigma(x), \sigma(z)\}}{1 + ||z||}||x - z|| + ||x^*||\frac{||x - z||}{1 + ||z||} \\
&\leq (||x^*|| + \min\{\sigma(x), \sigma(z)\})\left(\frac{||x||}{1 + ||z||} + \frac{||z||}{1 + ||z||}\right) \\
&< (||x^*|| + \sigma(x))(||x|| + 1)
\end{aligned}$$

which shows that $\rho_T(x) < +\infty$, that is $\operatorname{dom} T \subset \operatorname{dom} \rho_T$. Since ρ_T is convex we conclude that $\operatorname{co} \operatorname{dom} T \subset \operatorname{dom} \rho_T$. Likewise, we get $\operatorname{co} \operatorname{dom} S \subset \operatorname{dom} \rho_S$. Thus

$$\operatorname{co} \operatorname{dom} T - \operatorname{co} \operatorname{dom} S \subset \operatorname{dom} \rho_T - \operatorname{dom} \rho_S \tag{3.15}$$

The assumption and (3.15) imply that $0 \in \operatorname{core}(\operatorname{dom}\rho_T - \operatorname{dom}\rho_s)$. Therefore

$$\begin{aligned} X &= \cup_{n=1}^{\infty} n\,(\operatorname{dom}\rho_T - \operatorname{dom}\rho_S) \\ &= \cup_{n=1}^{\infty} \cup_{i=1}^{\infty} n(\{x : \rho_T(x) \le i, ||x|| \le i\} - \{x : \rho_S(x) \le i, ||x|| \le i\}). \end{aligned}$$

By the way $\{x : \rho_T(x) \le i, ||x|| \le i\}$ and $\{x : \rho_s(x) \le i, ||x|| \le i\}$ are closed, convex and compact so $(\{x : \rho_T(x) \le i, ||x|| \le i\} - \{x : \rho_s(x) \le i, ||x|| \le i\})$ is closed and convex. By the Baire category theorem [see Chapter 1 Corollary 1.3] there exists $j \in \mathbb{N}$ such that

$$\operatorname{int}(\{x : \rho_T(x) \le j, ||x|| \le j\} - \{x : \rho_s(x) \le j, ||x|| \le j\}) \ne \emptyset.$$

Set $S_i = (\{x : \rho_T(x) \le n, ||x|| \le i\} - \{x : \rho_s(x) \le i, ||x|| \le i\})$. Pick up any $x_1 \in S_j$ and $x_2 \in X$ such that $0 \in \operatorname{co}\{x_1, x_2\}$. Choose

$$r > \max\{j, \rho_T(x_2), \rho_s(x_2)\}$$

then $0 \in \operatorname{int} S_r$. Thus there exist $\varepsilon > 0$ such that

$$B(0,\varepsilon) \subset (\{x : \rho_T(x) \le r, ||x|| \le r\} - \{x : \rho_S(x) \le r, ||x|| \le r\}). \tag{3.16}$$

Let now $z \in B(0,\varepsilon)$, $x \in \operatorname{dom} T \cap \operatorname{dom} S$, $t^* \in T(x)$ and $s^* \in S(x)$. Then $z = a - b$ where $\rho_T(a) \le r$, $||a|| \le r$, $\rho_S(b) \le r$, $||b|| \le r$. We have

$$\begin{aligned} \langle t^*, z\rangle &= \langle t^*, a - x\rangle + \langle s^*, b - x\rangle + \langle t^* + s^*, x - b\rangle \\ &\le \rho_T(a)(1 + ||x||) + \rho_S(b)(1 + ||x||) + ||t^* + s^*||(||x|| + r) \\ &\le (r + ||x||)(2r + ||t^* + s^*||). \end{aligned}$$

From here it follows that

$$||t^*|| \le \frac{(r + ||x||)(2r + ||t^* + s^*||)}{\varepsilon}. \tag{3.17}$$

Likewise

$$||s^*|| \le \frac{(r + ||x||)(2r + ||t^* + s^*||)}{\varepsilon}. \tag{3.18}$$

Set $c = \frac{1}{\varepsilon}$, now (3.17) and (3.18) imply that desired assertion. ■

In the following we recall the Krein-Smulian theorem.

Theorem 3.27 ***(Krein–Šmulian)*** *Let X be a Banach space. A convex set in X^* is weak*-closed if and only if its intersection with $B(0,\varepsilon)$ is weak*-closed for every $\varepsilon > 0$.*

We recall that a set $A \subset X^*$ is *bounded weak*-closed* if every bounded and weak*-convergent net in A has its limit in A. The Krein–Šmulian theorem obviously implies the following.

Corollary 3.28 *A convex set in X^* is weak*-closed if and only if it is bounded weak*-closed.*

The following result extends Theorem 1.43 from Chapter 1 to σ-monotone operators. Our proof is very close to the proof of A. Verona and M. E. Verona in [113].

Proposition 3.29 *Let X be any Banach space and let $S, T : X \to 2^{X^*}$ be maximal σ-monotone operators. Suppose that*

$$0 \in \operatorname{core}[\operatorname{co}\operatorname{dom} T - \operatorname{co}\operatorname{dom} S].$$

For any $x \in \operatorname{dom} T \cap \operatorname{dom} S, T(x) + S(x)$ is a weak-closed subset of X^*.*

Proof. Since T and $S : X \to 2^{X^*}$ are maximal σ-monotone by Proposition 3.7 we infer that $T(z)$ and $S(z)$ are convex. Therefore $T(z) + S(z)$ is also convex. By Corollary 3.28 it is suffices to prove that $T(z) + S(z)$ is bounded weak*-closed, that is, every bounded weak*-convergent net in $T(z) + S(z)$ has a limit in $T(z) + S(z)$.

Let $\{t_i^*\} \subset T(z)$ and $\{s_i^*\} \subset S(z)$ be nets such that $\{t_i^* + s_i^*\}$ is bounded and weak*-convergent to z^*. By Theorem 3.26,

$$\max(||t_i^*||, ||s_i^*||) \leq c(r + ||x||)(2r + ||t_i^* + s_i^*||).$$

Thus the nets $\{t_i^*\}$ and $\{s_i^*\}$ are bounded. So they are relatively weak*-compact. By replacing them with subnets we may assume that weak*-limit $t_i^* = t$ and weak*-limit $s_i^* = s$. Since T and S are maximal σ-monotone, Proposition 3.7, $T(z)$ and $S(z)$ are weak*-closed. Therefore $t^* \in T(z)$ and $s^* \in S(z)$. Then $z^* = t^* + s^* \in T(z) + S(z)$. ■

Assume that F and $G : C \times C \to \mathbb{R}$ are two σ-monotone bifunctions and $\alpha > 0$. Then the bifunctions αF and $F + G$ are defined from $C \times C$ to $\mathbb{R}$ by $(\alpha F)(x, y) = \alpha \cdot (F(x, y))$ and $(F + G)(x, y) = F(x, y) + G(x, y)$.

Proposition 3.30 *Suppose that F and $G : C \times C \to \mathbb{R}$ are two σ-monotone bifunctions. Then $F + G$ is 2σ-monotone bifunction and αF is $\alpha\sigma$-monotone bifunction. Moreover,*

$$A^F(x) + A^G(x) \subset A^{F+G}(x) \quad \forall x \in X. \tag{3.19}$$

Proof. We have

$$F(x, y) + F(y, x) \leq \min\{\sigma(x), \sigma(y)\} ||y - x|| \tag{3.20}$$

and

$$G(x, y) + G(y, x) \leq \min\{\sigma(x), \sigma(y)\} ||y - x||. \tag{3.21}$$

Adding the inequalities (3.20) and (3.21) we deduce

$$(F + G)(x, y) + (F + G)(y, x) \leq 2\min\{\sigma(x), \sigma(y)\} ||y - x||,$$

that is, $F + G$ is 2σ-monotone. Also it follows from (3.20) that αF is $\alpha\sigma$-monotone.

Now suppose that $x \in X$. If $x \in X \backslash C$, then the inclusion is trivial. So, let $x \in C$ and $x^* \in \left(A^F + A^G\right)(x)$. Hence, there exist $x_1^* \in A^F(x)$ and $x_2^* \in A^G(x)$ with $x^* = x_1^* + x_2^*$. Thus

$$F(x,y) \geq \langle x_1^*, y - x\rangle \quad \forall y \in C, \tag{3.22}$$

and

$$G(x,y) \geq \langle x_2^*, y - x\rangle \quad \forall y \in C. \tag{3.23}$$

From (3.22) and (3.23) we obtain

$$\begin{aligned}(F+G)(x,y) &\geq \langle x_1^* + x_2^*, y - x\rangle \\ &= \langle x^*, y - x\rangle \quad \forall y \in C,\end{aligned}$$

i.e., $A^F(x) + A^G(x) \subset A^{F+G}(x)$ for all $x \in C$. ■

Note: One can easily verify that if F is maximal σ-monotone, then αF is also maximal $\alpha\sigma$-monotone. However, $F + G$ is not necessarily maximal 2σ-monotone when F and G are maximal σ-monotone.

The following example shows that the inclusion in (3.19) can be proper.

Example 3.31 Define $F, G : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ by $F(x,y) = (y-x)^3$ and $G(x,y) = -F(x,y)$. Then F and G are monotone bifunctions and for each $x \in \mathbb{R}$ we have $A^F(x) = A^G(x) = \emptyset$ and $A^{F+G}(x) = \{0\}$. ▲

Definition 3.32 *A bifunction $F : C \times C \to \mathbb{R}$ is called* cyclically σ-monotone *if for any cycle $x_1, x_2, ..., x_{n+1} = x_1$ in C,*

$$\sum_{i=1}^{n} F(x_i, x_{i+1}) \leq \frac{1}{2}\sum_{i=1}^{n} \min\{\sigma(x_i), \sigma(x_{i+1})\} ||x_i - x_{i+1}||.$$

It is easy to check that every cyclically σ-monotone bifunction is a σ-monotone bifunction. Moreover assume that F is σ-monotone (cyclically σ-monotone); if we define

$$F_1(x,y) = F(x,y) - \frac{1}{2}\min\{\sigma(x), \sigma(y)\} ||x - y||$$

then F_1 is monotone (cyclically monotone) and vice versa.

The following proposition will enable us to obtain the extension of Proposition 2.29 from Chapter 2.

Proposition 3.33 *A bifunction $F : C \times C \to \mathbb{R}$ is cyclically σ-monotone if and only if there exists a function $f : C \to \mathbb{R}$ such that*

$$\forall x, y \in C, \quad F(x,y) - \frac{1}{2}\min\{\sigma(x), \sigma(y)\} ||x - y|| \leq f(y) - f(x). \tag{3.24}$$

Proof. Set

$$F_1(x,y) = F(x,y) - \frac{1}{2}\min\{\sigma(x),\sigma(y)\}||x-y||.$$

Then F is cyclically σ-monotone if and only if F_1 is cyclically monotone. Thus by Proposition 2.29, F_1 is cyclically monotone if and only if there exists a function $f : C \to \mathbb{R}$ such that

$$F_1(x,y) \leq f(y) - f(x) \quad \forall x,y \in C. \tag{3.25}$$

Therefore the inequality (3.25) holds if and only if (3.24) holds. ■

Now we are going to introduce the notion of σ-convexity. First we recall from [73] that a function $f : X \to \mathbb{R} \cup \{+\infty\}$ is ε-convex if it satisfies the following inequality for every $a, b \in X$, and $\lambda \in]0,1[$:

$$f(\lambda a + (1-\lambda)b) \leq \lambda f(a) + (1-\lambda) f(b) + \lambda(1-\lambda)\varepsilon||a-b||$$

Definition 3.34 *A function $f : X \to \mathbb{R} \cup \{+\infty\}$ is called* σ-convex *if*

$$f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda) f(y) + \lambda(1-\lambda)\min\{\sigma(x),\sigma(y)\}||x-y|| \tag{3.26}$$

for all $x, y \in X$, and $\lambda \in]0,1[$.

For a proper function $f : X \to \mathbb{R} \cup \{+\infty\}$ the Clarke-Rockafellar generalized directional derivative at x in a direction $z \in X$ is defined by

$$f^{\uparrow}(x,z) = \sup_{\delta>0} \limsup_{(y,\alpha)\xrightarrow{f} x, \lambda\searrow 0} \inf_{u\in B(z,\delta)} \frac{f(y+\lambda u)-\alpha}{\lambda}$$

where $(y,\alpha) \xrightarrow{f} x$ means that $y \to x, \alpha \to f(x)$ and $\alpha \geq f(y)$. If f is lsc at x, the above definition coincides with

$$f^{\uparrow}(x,z) = \sup_{\delta>0} \limsup_{y\xrightarrow{f} x, \lambda\searrow 0} \inf_{u\in B(z,\delta)} \frac{f(y+\lambda u)-f(y)}{\lambda}.$$

Here, $y \xrightarrow{f} x$ means that $y \to x$ and $f(y) \to f(x)$. The Clarke-Rockafellar subdifferential of f at $x \in \operatorname{dom} f$ is defined by

$$\partial^{CR} f(x) = \left\{x^* \in X^* : \langle x^*, z\rangle \leq f^{\uparrow}(x,z) \quad \forall z \in X\right\}.$$

Whenever f is locally Lipschitz we have $f^{\uparrow}(x,z) = f^{o}(x,z)$, where $f^{o}(x,z)$ is the Clarke directional derivative at x in a direction $z \in X$ which is defined by

$$f^{o}(x,z) = \limsup_{y\to x, \lambda\searrow 0} \frac{f(y+\lambda z)-f(y)}{\lambda}.$$

Moreover, the Clarke's subdifferential of f at $x \in \operatorname{dom} f$ is defined by

$$\partial^{C} f(x) = \{x^* \in X^* : \langle x^*, z\rangle \leq f^{o}(x,z) \quad \forall z \in X\}.$$

Lemma 3.35 *Assume that $f : X \to \mathbb{R}\cup\{+\infty\}$ is lsc and σ-convex. If σ is usc, then*

$$\partial^{CR} f(x) \subseteq \left\{ \begin{array}{c} x^* \in X^* : \langle x^*, z\rangle \le f(x+z) - f(x) \\ + \min\{\sigma(x), \sigma(z+x)\} ||z|| \quad \forall z \in X \end{array} \right\}.$$

Proof. For each $y, u \in X$, $\lambda \in]0,1[$ by σ-convexity of f we obtain

$$f(y+\lambda u) \le \lambda f(y+u) + (1-\lambda) f(y) + \lambda(1-\lambda)\min\{\sigma(y+u), \sigma(y)\} ||u||,$$

that is,

$$\frac{f(y+\lambda u) - f(y)}{\lambda} \le f(y+u) - f(y) + (1-\lambda)\min\{\sigma(y+u), \sigma(y)\} ||u||.$$

Let us fix z and x. Take $u = z+x-y$ in the above inequality, so for an arbitrary $\delta > 0$ we have

$$\begin{aligned} &\limsup_{y \xrightarrow{f} x, \lambda \searrow 0} \inf_{u \in B(z,\delta)} \frac{f(y+\lambda u) - f(y)}{\lambda} \\ &\le \limsup_{y \xrightarrow{f} x, \lambda \searrow 0} \frac{f(y+\lambda(z+x-y)) - f(y)}{\lambda} \\ &\le \limsup_{y \xrightarrow{f} x, \lambda \searrow 0} \left[f(x+z) - f(y) + (1-\lambda)\min\{\sigma(x+z), \sigma(y)\} ||z+x-y|| \right] \\ &\le f(x+z) - f(x) + \min\{\sigma(x), \sigma(z+x)\} ||z||. \end{aligned}$$

Since $\delta > 0$ was arbitrary we get

$$f^{\uparrow}(x,z) \le f(x+z) - f(x) + \min\{\sigma(x), \sigma(z+x)\} ||z||.$$

We are done. ■

The idea and a proof of the above lemma and the following proposition is in essence contained in [87], where only ε-convexity and ε-monotonicity were considered.

Proposition 3.36 *Let $f : X \to \mathbb{R}\cup\{+\infty\}$ be lsc and σ-convex. If σ is usc, then $\partial^{CR} f$ is 2σ-monotone.*

Proof. Assume that $x, y \in X$, $x^* \in \partial^{CR} f(x)$ and $y^* \in \partial^{CR} f(y)$. It follows from Lemma 3.35 that

$$\langle x^*, y-x\rangle \le f(y) - f(x) + \min\{\sigma(x), \sigma(y)\} ||y-x||$$

and

$$\langle y^*, x-y\rangle \le f(x) - f(y) + \min\{\sigma(x), \sigma(y)\} ||y-x||.$$

By adding these equalities we get $\partial^{CR} f$ is 2σ-monotone. ■

3.4 Equilibrium Problem and Pre-monotonicity

For a closed convex subset C of X and $w \in X$, the *projection* of w onto C is the set $P_C(w) = \{z \in C : \|w - z\| \leq \|w - x\|, \forall x \in C\}$. If $z \in P_C(w)$ then

$$\exists w^* \in \mathcal{J}(w - z) : \forall y \in C, \quad \langle w^*, y - z \rangle \leq 0. \tag{3.27}$$

When C is closed and convex and X is reflexive, $P_C(w)$ is nonempty. It should be noted that if $X = \mathbb{R}^n$ and we consider the Euclidean norm on $\mathbb{R}^n$, then the duality map is the identity map and $P_C(w)$ is unique provided that C is closed and convex.

An operator $T : X \to 2^{X^*}$ is called *coercive* if

$$\lim_{\|x\| \to \infty} \frac{\inf_{x^* \in T(x)} \langle x^*, x \rangle}{\|x\|} = \infty.$$

We introduce a weaker notion than coercivity: an operator T will be called *quasi coercive* if $\lim_{\|x\| \to \infty} \inf_{x^* \in T(x)} \|x^*\| = \infty$, and

$$\liminf_{\|x\| \to \infty} \frac{\inf_{x^* \in T(x)} \langle x^*, x \rangle}{\|x\|} > -\infty.$$

Clearly, each coercive operator is quasi coercive. The operator $T : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $T(a, b) = (-b, a)$ is quasi coercive without being coercive.

Given a subset C of X and a bifunction $F : C \times C \to \mathbb{R}$, the equilibrium problem [8, 23] corresponding to F and C is the following:

$$\text{find } x_0 \in X \text{ such that } F(x_0, x) \geq 0, \text{ for all } x \in C. \tag{3.28}$$

If $C \subseteq X$ is convex, a function $g : C \to \mathbb{R}$ is called *semi-strictly quasiconvex* [63] if for all $x, y \in C$ the following implication holds:

$$g(x) < g(y) \Rightarrow \quad \forall \lambda \in]0, 1[, \quad g(\lambda x + (1 - \lambda)y) < g(y).$$

If g is semi-strictly quasiconvex and lsc, then it is quasiconvex. A lsc function g is semi-strictly quasiconvex if and only if for all $x, y \in C$ and $\lambda \in]0, 1[$,

$$g(\lambda x + (1 - \lambda)y) \geq g(x) \Rightarrow g(\lambda x + (1 - \lambda)y) \leq g(y).$$

Such functions were called *pseudoconvex* in [71]. We recall the following theorem, due to Ky Fan.

Theorem 3.37 *[50] Let C be a compact convex set in a Hausdorff TVS. If $F : C \times C \to \mathbb{R}$ is such that for every $x \in C$, $F(x, \cdot)$ is quasiconvex and for every $y \in C$, $F(\cdot, y)$ is usc, then there exists $x_0 \in C$ such that*

$$\forall y \in C, \quad F(x_0, y) \geq \inf_{x \in C} F(x, x).$$

From now on $X = \mathbb{R}^n$.

The following proposition will permit application of Ky Fan's theorem to G_T.

Proposition 3.38 *Let $T : X \to 2^{X^*}$ be such that* $\operatorname{int}\operatorname{dom} T \neq \emptyset$. *If T is locally bounded on* $\operatorname{int}\operatorname{dom} T$ *and* $\operatorname{gr} T$ *is closed, then T is usc on* $\operatorname{int}\operatorname{dom} T$ *and also for each $y \in \operatorname{dom} T$, $G_T(\cdot, y)$ is usc on* $\operatorname{int}\operatorname{dom} T$.

Proof. The first part of the proposition is standard, see for instance the proof of Proposition 3.24(i). For the second part, we note that for each $y \in \operatorname{dom} T$ the function $(x, x^*) \to \langle x^*, y - x\rangle$ is continuous, so by the well-known "Berge's Maximum Theorem" (see for instance Proposition I.3.3., page 83 in [67]), $G_T(\cdot, y)$ is usc. ■

We now present a result for equilibrium problems. Given $C \subseteq X$ and $r > 0$, set $C_r = \{x \in C : \|x\| \leq r\}$.

Proposition 3.39 *Suppose that $C \subseteq X$ is closed and convex. Let $F : C \times C \to \mathbb{R}$ be a bifunction satisfying*

(i) $F(x, x) = 0$ for all $x \in C$;

(ii) $F(\cdot, y)$ is usc for all $y \in C$;

(iii) $F(x, \cdot)$ is semi-strictly quasiconvex and lsc for all $x \in C$;

(iv) there exists $r > 0$ such that for each $x \in C \backslash C_r$, there exists $y \in C$ with $\|y\| < \|x\|$ and $F(x, y) \leq 0$.

Then the equilibrium problem (3.28) has a solution.

Proof. Choose $r > 0$ so that condition (iv) holds, and set $s = r + 1$. Then C_s is compact. According to Theorem 3.37, there exists $x_0 \in C_s$ such that $F(x_0, x) \geq 0$ for all $x \in C_s$. Then there exists $y \in C$ with $\|y\| < s$ such that $F(x_0, y) \leq 0$; indeed, if $\|x_0\| = s$ we can apply condition (iv), whereas if $\|x_0\| < s$ then we can set $y = x_0$ and use condition (i). In both cases we actually have $F(x_0, y) = 0$ by the definition of x_0.

Now for every $x \in C$ we can find $\lambda \in]0, 1[$ such that $x_\lambda := \lambda y + (1-\lambda)x \in C_s$. Hence $F(x_0, x_\lambda) \geq 0$. If we assume that $F(x_0, x) < 0$ then $F(x_0, y) > F(x_0, x)$ would imply by semi-strict quasiconvexity that $F(x_0, y) > F(x_0, x_\lambda) \geq 0$, a contradiction. Thus $F(x_0, x) \geq 0$ for all $x \in C$. ■

The following surjectivity theorem extends Theorem 4.1 in [71] to quasi-coercive operators.

Theorem 3.40 *Assume that $T : X \to 2^{X^*}$ is locally bounded, convex-valued,* $\operatorname{gr} T$ *is closed, and that* $\operatorname{dom} T = X$. *If T is quasi-coercive, then T is surjective.*

Proof. Given $z^* \in X^*$, define $F : X \times X \to \mathbb{R}$ by

$$F(x, y) = G_T(x, y) - \langle z^*, y - x\rangle$$

for all $x, y \in X$. By using Proposition 3.38 it is easy to check that F satisfies (i)-(iii) of Proposition 3.39. We now check the validity of condition (iv).

Since T is quasi-coercive, we can find $r_1 > 1$ and $k \in \mathbb{R}$ such that

$$\forall \|x\| \geq r_1, \forall x^* \in T(x) : \frac{\langle x^*, x\rangle}{\|x\|} > k, \tag{3.29}$$

and then $r > r_1$ such that

$$\forall \|x\| \geq r : \inf_{x^* \in T(x)} \|x^*\| > \max\{0, 3\|z^*\| - 2k\}. \tag{3.30}$$

For each x such that $\|x\| \geq r$, choose $w_x^* \in P_{T(x)}(0)$. We now apply property (3.27) of the duality map to the set $T(x)$ in the space X^*. According to (3.27), since $w = 0$, $\mathcal{J} = I$ we have

$$\forall x^* \in T(x), \quad \langle x^* - w_x^*, w_x^*\rangle \geq 0. \tag{3.31}$$

By relation (3.30), $w_x^* \neq 0$. Set

$$y_x = x\left(1 - \frac{1}{\|x\|}\right) - \frac{w_x^*}{2\|w_x^*\|}.$$

Since $\|x\| \geq r > 1$, y_x satisfies

$$\|y_x\| \leq \|x\| - 1 + \frac{1}{2} < \|x\|.$$

Note that

$$y_x - x = -\frac{x}{\|x\|} - \frac{w_x^*}{2\|w_x^*\|}. \tag{3.32}$$

Using successively relations (3.32), (3.29), (3.31), $\langle w_x^*, w_x^*\rangle = \|w_x^*\|^2$, relation (3.30) and $w_x^* \in P_{T(x)}(0)$, we deduce

$$\begin{aligned} F(x, y_x) &= \sup_{x^* \in T(x)} \langle x^*, y_x - x\rangle - \langle z^*, y_x - x\rangle \\ &\leq -\inf_{x^* \in T(x)} \frac{1}{\|x\|}\langle x^*, x\rangle - \frac{1}{2\|w_x^*\|}\inf_{x^* \in T(x)}\langle x^*, w_x^*\rangle - \langle z^*, y_x - x\rangle \\ &\leq -k - \frac{1}{2\|w_x^*\|}\langle w_x^*, w_x^*\rangle + \frac{1}{\|x\|}\langle z^*, x\rangle + \frac{1}{2\|w_x^*\|}\langle z^*, w_x^*\rangle \\ &\leq -k - \frac{1}{2}\|w_x^*\| + \|z^*\| + \frac{1}{2}\|z^*\| \\ &\leq 0. \end{aligned}$$

Therefore, condition (iv) in Proposition 3.39 also holds. Hence there exists $z_0 \in X$ such that for all $y \in X$ we have

$$F(z_0, y) = \sup_{y^* \in T(z_0)} \langle y^* - z^*, y - z_0\rangle \geq 0.$$

Since $T(z_0)$ is closed and bounded, we get $T(z_0)$ is compact. On the other hand $\langle \cdot - z^*, y - z_0\rangle$ is continuous on $T(z_0)$, so the supremum is attained at some element φ_y. It follows that for all $v \in X$ there exists an element $v^* = \varphi_{v+z_0}$ in $T(z_0)$ so that $\langle v^* - z^*, v \rangle \geq 0$. This means that z^* cannot be separated from the closed convex set $T(z_0)$, so $z^* \in T(z_0)$ and T is surjective. ■

Theorem 3.40 has many applications. As an example, in Theorem 4.2 of [71] we can replace the identity operator by a more general operator S.

Theorem 3.41 *Assume that $T : X \to 2^{X^*}$ is pre-monotone, convex-valued and $\operatorname{gr} T$ is closed, and $S : X \to 2^{X^*}$ is such that $\operatorname{gr} S$ closed, convex-valued, locally bounded and coercive. If $\operatorname{dom} T = \operatorname{dom} S = X$, then $T + S$ is surjective.*

Proof. It is clear that $T + S$ is convex valued. Also, by Corollary 3.20 the operator T is locally bounded, so $T + S$ is locally bounded. We show that $\operatorname{gr}(T + S)$ is closed. Indeed, let $\{(x_i, z_i^*)\}$ be a sequence in $\operatorname{gr}(T + S)$ such that $x_i \to x$ and $z_i^* \to z^*$. Let $z_i = x_i^* + y_i^*$ with $x_i^* \in T(x_i)$ and $y_i^* \in S(x_i)$. Take a neighborhood U of x such that $S(U)$ is bounded. There exists some $N \in \mathbb{N}$ such that for $i > N$, $x_i \in U$; then $\{y_i^*\}$ is bounded. Thus, by taking a subsequence if necessary we can assume that $\{y_i^*\}$ converges to some $y^* \in X^*$. Then x_i^* converges to $x^* := z^* - y^*$. By closedness of $\operatorname{gr} T$ and $\operatorname{gr} S$, $x^* \in T(x)$ and $y^* \in S(x)$, i.e., $z^* \in (T + S)(x)$ and $\operatorname{gr}(T + S)$ is closed.

Finally we show that $T + S$ is coercive. Choose any $x_0^* \in T(0)$. We estimate

$$\begin{aligned}
\inf_{z^* \in (T+S)(x)} \langle z^*, x \rangle &\geq \inf_{x^* \in T(x)} \langle x^*, x \rangle + \inf_{y^* \in S(x)} \langle y^*, x \rangle \\
&\geq \inf_{x^* \in T(x)} \langle x^* - x_0^*, x - 0 \rangle + \langle x_0^*, x \rangle + \inf_{y^* \in S(x)} \langle y^*, x \rangle \\
&\geq -\sigma(0) \|x\| - \|x_0^*\| \|x\| + \inf_{y^* \in S(x)} \langle y^*, x \rangle .
\end{aligned}$$

Since S is coercive, we infer that $T + S$ is also coercive. By Theorem 3.40, $T + S$ is surjective. ■

The preceding theorem, together with Proposition 3.7 imply the following:

Corollary 3.42 *Let T be maximal σ-monotone with an usc σ. If $\operatorname{dom} T = X$, then $T + \lambda I$ is surjective for each $\lambda > 0$.*

Proof. According to Proposition 3.7, T is convex-valued. It is also locally bounded and usc by Proposition 3.24(i), hence it is closed. Since I is defined everywhere, it is maximal monotone and in particular $\operatorname{gr} T$ is closed, convex-valued and locally bounded. Also, I is obviously coercive. Now Theorem 3.41 yields the result. ■

3.5 Comparison with other Notions

In this section, we will compare some types of generalized monotone operators. In the next definition $S = \{x \in X : \|x\| = 1\}$, is the unit sphere of Banach space

X, and $x \to_e x_0$ means that x converges to x_0 in direction e, i.e., $x \to x_0$ and $\frac{x-x_0}{\|x-x_0\|} \to e$. Also, define

$$U(x_0, e, \delta) = \left\{ x \in X : x \neq x_0, \|x - x_0\| < \delta, \left\| \frac{\|x - x_0\|}{x - x_0} - e \right\| < \delta \right\}.$$

Definition 3.43 *Suppose that $T : X \to 2^{X^*}$ is an operator. We recall that T is*

(i) ε-monotone *in the sense of Luc-Ngai-Thera if for a given $\varepsilon > 0$ and for every $x, y \in \operatorname{dom} T$, $x^* \in T(x)$ and $y^* \in T(y)$*

$$\langle y^* - x^*, y - x \rangle \geq -2\varepsilon ||y - x||.$$

(ii) Submonotone *at $x_0 \in X$ in the sense of Aussel-Daniilidis-Thibault if for every $\varepsilon > 0$ there exists $\delta > 0$ such that*

$$\langle y^* - x^*, y - x \rangle \geq -\varepsilon ||y - x||.$$

for all $x, y \in B(x_0, \delta)$, all $x^* \in T(x)$ and $y^* \in T(y)$.

(iii) Submonotone *in the sense of Georgiev at $x_0 \in X$ if*

$$\liminf_{\substack{x_0 \neq x \to_e x_0 \\ y \in T(x),\, y_0 \in T(x_0)}} \frac{\langle x - x_0, y - y_0 \rangle}{\|x - x_0\|} \geq 0.$$

Equivalently, T is submonotone at x_0 if and only if

$$\forall e \in S, \quad \forall \varepsilon > 0 \quad \exists \delta > 0 : \frac{\langle x - x_0, y - y_0 \rangle}{\|x - x_0\|} > -\varepsilon$$
$$\forall x \in U(x_0, e, \delta), \quad \forall y \in T(x), \quad \forall y_0 \in T(x_0).$$

Example 3.44 Define $T : \mathbb{R} \to \mathbb{R}$ by

$$T(x) = \begin{cases} -1 & \text{if} \quad x \geq 0, \\ 0 & \text{if} \quad x < 0. \end{cases}$$

It is clear that T is pre-monotone with the constant (continuous) map $\sigma(y) \equiv 1$. Thus it is ε-monotone in the sense of Luc-Ngai-Thera. But it is not submonotone in the sense of Aussel-Daniilidis-Thibault. ▲

Remark 3.45 Assume that $\Phi : \mathbb{R}^n \to \mathbb{R}^n$ is a continuous map. Then Φ is submonotone in the sense of Aussel-Daniilidis-Thibault. Indeed, for a given $\varepsilon > 0$ and $x_0 \in \mathbb{R}^n$ there exists by assumption $\delta > 0$ such that for $x_1, x_2 \in B(x_0, \delta/2)$

$$||\Phi(x_1) - \Phi(x_2)|| < \varepsilon.$$

Therefore

$$\begin{aligned}\langle \Phi(x_2) - \Phi(x_1), x_1 - x_2 \rangle &\leq ||\Phi(x_1) - \Phi(x_2)||\ ||x_1 - x_2|| \\ &< \varepsilon ||x_1 - x_2||.\end{aligned}$$

This implies that

$$\langle \Phi(x_2) - \Phi(x_1), x_2 - x_1 \rangle \geq -\varepsilon ||x_1 - x_2||.$$

Thus Φ is submonotone. ♦

Note that since the function φ which is represented in Example 3.8 is continuous, the above remark implies that φ is submonotone in the sense of Aussel-Daniilidis-Thibault and we know that it is not ε-monotone in the sense of Luc-Ngai-Thera.

We now represent an example, that is submonotone in the sense of Georgiev but it is not pre-monotone.

Example 3.46 Define the function $T : \mathbb{R} \to \mathbb{R}$ by

$$T(x) = \begin{cases} \frac{1}{x} & \text{if} \quad x > 0, \\ 0 & \text{if} \quad x \leq 0. \end{cases}$$

Then T is submonotone in the sense of Georgiev [55, Example 1.3]. But it is not pre-monotone. To show this, suppose that there exists $\sigma : \mathbb{R} \to \mathbb{R}_+$ such that T is pre-monotone. Take $y = 1$ and $0 < x < 1$. Then we have

$$\left(1 - \frac{1}{x}\right)(1 - x) \geq -\sigma(1)(1 - x).$$

This implies that $x \geq \frac{1}{1+\sigma(1)}$. If we choose $x = \frac{1}{2+\sigma(1)}$, then we have

$$\frac{1}{2 + \sigma(1)} \geq \frac{1}{1 + \sigma(1)} \Longrightarrow 2 \leq 1.$$

This is a contradiction. ▲

Chapter 4

Fitzpatrick Transform

Most of the results of the Sections 2 and 3 in the present chapter are based on [4]. In this chapter we will introduce the notion of normal bifunction. Also, we will present a new definition of monotone bifunctions, which is a slight generalization of the original definition given by Blum and Oettli, but which is better suited for relating monotone bifunctions to monotone operators. In this new definition, we will introduce the Fitzpatrick transform of a BO-maximal monotone bifunction so as to correspond exactly to the Fitzpatrick function of a maximal monotone operator in case the bifunction is constructed starting from the operator. Whenever the monotone bifunction is lower semicontinuous and convex with respect to its second variable, the Fitzpatrick transform permits to obtain results on its maximal monotonicity.

We now outline the contents of this chapter. After describing our motivation in the first section, in the second section we will define normal bifunctions and their monotonicity and then we will portray their properties. A central result of this section is that an operator with weak*-closed convex values is maximal monotone if and only if the corresponding bifunction is BO-maximal monotone.

In Section 3 we will introduce the notion of Fitzpatrick transformation and we will derive some consequences of this notion. Indeed, we will prove that at each point $(x, x^*) \in X \times X^*$, the Fitzpatrick transform of a BO-maximal monotone is greater than or equal to $\langle x^*, x \rangle$; and equality holds if and only if (x, x^*) belong to the graph of corresponding operator, an analogous property of the Fitzpatrick function of a maximal monotone operator. Moreover, in Proposition 4.12 we will find a link between the Fitzpatrick transform and the Fitzpatrick function. In addition, we will define the upper Fitzpatrick transform and will see that in conjunction with the Fitzpatrick transform, it is very useful in our analysis. In particular we will prove that the maximality of A^F and BO-maximality of F are equivalent whenever the space is reflexive, and F is lsc and convex with respect to its second variable. The other theme of Section 3 is characterizing the BO-maximality through some equivalent statements in Theorem 4.19. In Section 4, we make use of the notion of pair and partial convolutions. We will find an upper bound for the Fitzpatrick transform of a sum of two monotone operators, and then will deduce an inequality for the Fitzpatrick transform of a monotone bifunction which is subadditive with respect to its second variable. The final result of Section 4 will extend the Fitzpatrick inequality of operators to Fitzpatrick transforms. In Section 5 we will consider some existence theorems. The proof of these results are based on ideas of Blum and Oettli in [23]. In fact, we will generalize their theorems to BO-maximality. In Section 6 we will collect various examples concerning the Fitzpatrick transform of bifunctions. In Section 7 we will introduce the notion of n-cyclically monotone and BO-n-cyclically maximal monotone bifunctions. Also, we will bring forward their relation to n-cyclically monotone operators. Afterwards, we will prove a theorem for BO-n-cyclically maximal monotone bifunctions which is similar to the corresponding theorem of Fitzpatrick functions. Subsequently, we will generalize some results from Section 3 to cyclically monotone bifunction. Finally, we will include some examples in the last section of this chapter.

4.1 Motivation

Given a nonempty subset of a Banach space X, the term "monotone bifunction" on C is often used (as we did in the previous chapters) for functions $F : C \times C \to \mathbb{R}$ such that

$$F(x, y) + F(y, x) \leq 0, \quad \forall x, y \in C.$$

Starting from the paper by Blum and Oettli [23], monotone bifunctions were studied mainly in view of their application to equilibrium problems. Here, we will focus our interest on their relation to monotone operators. Let us recall from previous chapters the basic definitions, in order to understand the need for some changes to them. Given a multivalued monotone operator $T : X \to 2^{X^*}$ with domain $\operatorname{dom} T = \{x \in X : T(x) \neq \emptyset\}$, the bifunction G_T defined on $\operatorname{dom} T \times \operatorname{dom} T$ by

$$G_T(x, y) = \sup_{x^* \in T(x)} \langle x^*, y - x \rangle \tag{4.1}$$

is real-valued and monotone (see Chapter 2). On the other hand, given any monotone bifunction F, the operator defined by

$$A^F(x) = \{x^* \in X^* : \langle x^*, y - x \rangle \leq F(x, y), \quad \forall y \in C\}$$

whenever $x \in C$ while $A^F(x) = \emptyset$ for $x \notin C$, is monotone.

A monotone bifunction $F : C \times C \to \mathbb{R}$ is called BO-maximal monotone [23] if for all $x \in C$ and $x^* \in X$, the following implication holds:

$$F(y, x) + \langle x^*, y - x \rangle \leq 0, \quad \forall y \in C \Longrightarrow \langle x^*, y - x \rangle \leq F(x, y), \quad \forall y \in C.$$

The bifunction F is called maximal monotone if the operator A^F is maximal monotone. In Chapter 2 we observed that every maximal monotone bifunction is BO-maximal monotone; the converse is not true in general, but it holds under some additional assumptions: For instance, if F is BO-maximal monotone, C is closed and convex, and $F(x,\cdot)$ is lsc and convex for every $x \in C$ and $F(x,x) = 0$ for each $x \in C$, then F is maximal monotone [1, 30].

A very powerful tool in the study of maximal monotone operators is the notion of Fitzpatrick function [52], (see also Chapter 1). Given a monotone operator T with graph $\operatorname{gr} T = \{(x,x^*) \in X \times X^* : x^* \in T(x)\}$, its Fitzpatrick function $\mathcal{F}_T$ can be written as

$$\mathcal{F}_T(x,x^*) = \sup_{(y,y^*)\in \operatorname{gr} T} \left(\langle x^*, y\rangle + \langle y^*, x-y\rangle\right).$$

A lsc and convex function φ on $X \times X^*$ is called a representative function of a monotone operator T if $\varphi(x,x^*) \geq \langle x^*, x\rangle$ for all $(x,x^*) \in X \times X^*$, and $\varphi(x,x^*) = \langle x^*, x\rangle$ for all $(x,x^*) \in \operatorname{gr} T$. It is known that the Fitzpatrick function of a maximal monotone operator T is a representative function of T. It has been shown recently that some important results on maximal monotone operator theory may be obtained by using methods of convex analysis on representative functions; see for instance [13, 15, 16, 25, 26, 38, 83, 89] etc.

If we compare the definitions of $\mathcal{F}_T$ and G_T we obtain

$$\mathcal{F}_T(x,x^*) = \sup_{y\in \operatorname{dom} T} \left(\langle x^*, y\rangle + G_T(y,x)\right).$$

Note that $\mathcal{F}_T$ is defined for all $x \in X$ (although y needs only to be in $\operatorname{dom} T$), and that in fact formula (4.1) can be used to define G_T on all $X \times X$. Obviously, $G_T(x,y) = -\infty$ for $x \notin \operatorname{dom} T$. This motivates the definition of a Fitzpatrick transform for every monotone bifunction, but we need to have bifunctions defined on $X \times X$. In fact, such kind of functions were introduced in [30] for bifunctions $F : C \times C \to \mathbb{R}$, where it was shown that one can recover some nice results and find new ones by using tools of convex analysis. In the present chapter we will introduce the so-called "normal bifunctions" defined on $X \times X$ and taking on values from $\overline{\mathbb{R}}$; we will see that the new formulation includes the previous one and gives simpler, more appealing formulas. Note that in [8], one considers bifunctions $F : X \times X \to \overline{\mathbb{R}}$ and defines monotonicity with respect to a subset C by $F(x,y) \leq -F(y,x)$, x, y in C. However, all other definitions and all results in [8] actually concern the restriction of F on $C \times C$, where F is real.

4.2 BO-Monotone Bifunctions

In what follows, X will be a LCS unless otherwise stated.

Definition 4.1 *A function $F : X \times X \to \overline{\mathbb{R}}$ is called* normal bifunction *if there exists a nonempty subset C of X such that*

$$F(x,y) = -\infty \quad \textit{iff} \quad x \notin C.$$

C will be called the domain of F. In what follows, it will be denoted by $\operatorname{dom} F$.

Note that in this definition we do not impose the assumption $F(x,x) = 0$ for all $x \in \operatorname{dom} F$.
According to Definition 4.1, $F : X \times X \to \overline{\mathbb{R}}$ is a normal bifunction if and only if we have that

$$\begin{aligned} &\{x \in X : \exists y \in X \text{ such that } F(x,y) > -\infty\} \\ &= \{x \in X : \forall y \in X, \quad F(x,y) > -\infty\} \neq \emptyset. \end{aligned}$$

In this case C coincides with the sets from above.

Definition 4.2 *A normal bifunction* $F : X \times X \to \overline{\mathbb{R}}$ *is called* monotone *if*

$$F(x,y) \leq -F(y,x), \quad \forall x,y \in X. \tag{4.2}$$

Remark 4.3 Let $F : X \times X \to \overline{\mathbb{R}}$ be a monotone bifunction. If x and y are both in $\operatorname{dom} F$, then $F(y,x) > -\infty$ and so $-F(y,x) < +\infty$, thus

$$-\infty < F(x,y) < +\infty.$$

In a similar manner we get $F(y,x) \in \mathbb{R}$. Hence we see that a normal bifunction is monotone if and only if

$$F(x,y) + F(y,x) \leq 0, \quad \forall x,y \in \operatorname{dom} F,$$

see also [8]. Therefore for all x in $\operatorname{dom} F$, we have $F(x,x) \leq 0$. ♦

For any operator $T : X \to 2^{X^*}$ one can define a normal bifunction G_T with domain $\operatorname{dom} G_T = \operatorname{dom} T$ by the formula

$$G_T(x,y) = \sup_{x^* \in T(x)} \langle x^*, y - x \rangle, \quad \forall x,y \in X.$$

Then $G_T(x,x) = 0$ for all x in $\operatorname{dom} T$. Moreover $G_T(x,\cdot)$ is lsc and convex for all x in $\operatorname{dom} T$.

Let $F : X \times X \to \overline{\mathbb{R}}$ be a normal bifunction. Define the operator A^F by

$$A^F(x) = \{x^* : \langle x^*, y - x \rangle \leq F(x,y), \quad \forall y \in X\}. \tag{4.3}$$

One can easily check that $\operatorname{dom} A^F \subseteq \operatorname{dom} F$; also, whenever F is monotone, A^F is also monotone and one has $F(x,x) = 0$ for all $x \in \operatorname{dom} A^F$.

Remark 4.4 So far, papers in the literature consider a bifunction to be defined on $C \times C$, where C is a subset of X, and defined A^F by requiring (4.3) to hold for $x,y \in C$. This is a particular case of what we are considering here. Indeed, for any $F : C \times C \to \mathbb{R}$ one can define a normal bifunction $\widetilde{F} : X \times X \to \overline{\mathbb{R}}$ which extends F on the whole space, by setting

$$\widetilde{F}(x,y) = \begin{cases} F(x,y) & \text{if } x \in C \text{ and } y \in C, \\ +\infty & \text{if } x \in C \text{ and } y \notin C, \\ -\infty & \text{if } x \notin C. \end{cases}$$

Then $A^{\tilde{F}}$ satisfies

$$A^{\tilde{F}}(x) = \begin{cases} \{x^* : \langle x^*, y-x \rangle \le F(x,y), \quad \forall y \in C\} & \text{if } x \in C, \\ \emptyset & \text{if } x \notin C, \end{cases}$$

i.e., it is the operator A^F considered in previous papers and Chapter 2. ♦

In the same spirit, we redefine the notion of BO-maximality.

Definition 4.5 *(i) A monotone bifunction F is called* BO-maximal monotone *if for all $(x, x^*) \in X \times X^*$,*

$$F(y,x) + \langle x^*, y-x \rangle \le 0, \quad \forall y \in X \Longrightarrow \langle x^*, y-x \rangle \le F(x,y), \quad \forall y \in X. \tag{4.4}$$

(ii) A monotone bifunction F is called maximal *if A^F is maximal monotone.*

Note that F is BO-maximal monotone if and only if

$$F(y,x) + \langle x^*, \ y-x \rangle \le 0, \forall y \in \operatorname{dom} F \Longrightarrow \langle x^*, y-x \rangle \le F(x,y), \forall y \in X. \tag{4.5}$$

Remark 4.6 The right-hand side of (4.4) says that $x^* \in \operatorname{dom} A^F$; thus, if F is BO-maximal monotone and $F(y,x) + \langle x^*, \ y-x \rangle \le 0$ holds for some $x \in X$ and for every $y \in \operatorname{dom} F$, then $x \in \operatorname{dom} F$ and $F(x,x) = 0$. ♦

In view of Remark 4.4, the definition of BO-maximal monotonicity considered in previous papers and Chapter 2 where F is defined on $C \times C$ and the right-hand side of (4.5) is required to hold only for $y \in C$, is again a particular case of the definition considered here.

Given an operator T, we denote by $\operatorname{cl}\operatorname{co} T$ the operator whose value at each $x \in X$ is weak*-$\operatorname{cl}\operatorname{co}(T(x))$. Then $G_T = G_{\operatorname{cl}\operatorname{co} T}$.

Theorem 4.7 *Let $T : X \to 2^{X^*}$ be an operator. Then $\operatorname{cl}\operatorname{co} T$ is maximal monotone if and only if G_T is BO-maximal monotone.*

Proof. Let $\operatorname{cl}\operatorname{co} T$ be maximal monotone. Since $G_T = G_{\operatorname{cl}\operatorname{co} T}$ we may suppose without loss of generality that T is maximal monotone. Now assume that T is maximal monotone and for some $(x, x^*) \in X \times X^*$,

$$G_T(y,x) + \langle x^*, y-x \rangle \le 0, \quad \forall y \in X.$$

Then

$$\sup_{y^* \in T(y)} \langle y^*, x-y \rangle + \langle x^*, y-x \rangle \le 0, \quad \forall y \in \operatorname{dom} T.$$

Thus for all $(y, y^*) \in \operatorname{gr} T$,

$$\langle y^* - x^*, y-x \rangle \ge 0.$$

On the other hand T is maximal monotone, therefore $(x, x^*) \in \operatorname{gr} T$. This implies that for every $y \in X$,

$$\langle x^*, y - x\rangle \leq \sup_{z^* \in T(x)} \langle z^*, y - x\rangle = G_T(x, y).$$

Thus G_T is BO-maximal.

Conversely, suppose that $G_T = G_{\operatorname{cl}\operatorname{co} T}$ is BO-maximal monotone. Then, for all $x, y \in \operatorname{dom} T, x^* \in \operatorname{cl}\operatorname{co} T(x)$ and $y^* \in \operatorname{cl}\operatorname{co} T(y)$,

$$\langle y^* - x^*, y - x\rangle \geq -(G_{\operatorname{cl}\operatorname{co}}(x, y) + G_{\operatorname{cl}\operatorname{co} T}(y, x)) \geq 0.$$

It follows that $\operatorname{cl}\operatorname{co} T$ is monotone. To show that it is maximal monotone, let $(x, x^*) \in X \times X^*$ be such that $\langle y^* - x^*, y - x\rangle \geq 0$ for all $(y, y^*) \in \operatorname{gr}\operatorname{cl}\operatorname{co} T$. Then $\langle x^*, y - x\rangle + \langle y^*, x - y\rangle \leq 0$ for all $(y, y^*) \in \operatorname{gr}\operatorname{cl}\operatorname{co} T$. By taking the supremum over $y^* \in T(y)$ we get $G_T(y, x) + \langle x^*, y - x\rangle \leq 0$ for all $y \in X$. Since $G_T = G_{\operatorname{cl}\operatorname{co} T}$ is BO-maximal, we deduce

$$\langle x^*, y - x\rangle \leq G_T(x, y) \leq \sup_{z^* \in \operatorname{cl}\operatorname{co} T(x)} \langle z^*, y - x\rangle, \quad \forall y \in X. \tag{4.6}$$

Since $\operatorname{cl}\operatorname{co} T(x)$ is weak*-closed and convex, (4.6) together with the separation theorem imply that $x^* \in \operatorname{cl}\operatorname{co} T(x)$. ■

In particular, if T is an operator with weak*-closed convex values, then T is maximal monotone if and only if the monotone bifunction G_T "created" by T is BO-maximal monotone.

One the other hand, every monotone bifunction F gives rise to a monotone operator A^F. Exactly as in Chapter 2, Proposition 2.6 one can show:

Proposition 4.8 *Let F be a monotone bifunction. If A^F is maximal monotone, then F is BO-maximal monotone.*

We end this section with the following result.

Proposition 4.9 *Suppose that $T : X \to 2^{X^*}$ is monotone and $\operatorname{dom} T = X$. Then $A^{G_T} = \operatorname{cl}\operatorname{co} T$.*

Proof. For each $x \in X$ and $z^* \in A^{G_T}(x)$ since $G_T = G_{\operatorname{cl}\operatorname{co} T}$ we get

$$G_{\operatorname{cl}\operatorname{co} T}(x, y) = \sup_{x^* \in T(x)} \langle x^*, y - x\rangle \geq \langle z^*, y - x\rangle \quad \forall y \in X.$$

On the other hand $\operatorname{cl}\operatorname{co} T(x)$ is weak*-closed and convex, so the separation theorem implies that $z^* \in \operatorname{cl}\operatorname{co} T(x)$. Thus $A^{G_T} \subset \operatorname{cl}\operatorname{co} T$.

Conversely, assume that $z^* \in \operatorname{cl}\operatorname{co} T(x)$. Since $G_T = G_{\operatorname{cl}\operatorname{co} T}$ we deduce that for each $y \in X$

$$\langle z^*, y - x\rangle \leq \sup_{x^* \in \operatorname{cl}\operatorname{co} T(x)} \langle x^*, y - x\rangle = G_{\operatorname{cl}\operatorname{co} T}(x, y) = G_T(x, y).$$

This means that $\operatorname{cl}\operatorname{co} T(x) \subset A^{G_T}(x)$. Thus $\operatorname{cl}\operatorname{co} T \subset A^{G_T}$. ■

Note that the above proposition implies that if $T : X \to 2^{X^*}$ is monotone with weak*-closed convex values and $\operatorname{dom} T = X$, then $A^{G_T} = T$. In particular if T is single-valued with $\operatorname{dom} T = X$, then $A^{G_T} = T$.

4.3 Fitzpatrick Transform of Bifunctions

Definition 4.10 *Suppose that* $F : X \times X \to \overline{\mathbb{R}}$ *is a normal bifunction. Define its Fitzpatrick transform* $\varphi_F : X \times X^* \to \mathbb{R} \cup \{+\infty\}$ *by*

$$\varphi_F(x, x^*) = \sup_{y \in X} \left(\langle x^*, y \rangle + F(y, x) \right), \quad \forall (x, x^*) \in X \times X^*.$$

Whenever $F(y, \cdot)$ is lsc and convex for all $y \in \operatorname{dom} F$, then φ_F is also lsc and convex.

For every BO-maximal monotone bifunction we have the following theorem, which is similar to a corresponding theorem for the Fitzpatrick function of a maximal monotone operator; in case $F(x, \cdot)$ is lsc and convex, the theorem says that φ_F is a representative function for the operator A^F.

Theorem 4.11 *Assume that* F *is a BO-maximal monotone bifunction. For each* $(x, x^*) \in X \times X^*$ *one has* $\langle x^*, x \rangle \le \varphi_F(x, x^*)$. *Equality holds if and only if* $x^* \in A^F(x)$.

Proof. Suppose that for some $(x, x^*) \in X \times X^*$ one has

$$\varphi_F(x, x^*) \le \langle x^*, x \rangle . \tag{4.7}$$

Then $\sup_{y \in X} \left(\langle x^*, y \rangle + F(y, x) \right) \le \langle x^*, x \rangle$, thus

$$F(y, x) + \langle x^*, y - x \rangle \le 0, \quad \forall y \in X.$$

By assumption F is BO-maximal, therefore

$$\langle x^*, y - x \rangle \le F(x, y), \quad \forall y \in X.$$

By Remark 4.6, this implies that $x \in \operatorname{dom} F$ and $F(x, x) = 0$, thus

$$\varphi_F(x, x^*) = \sup_{y \in X} \left(\langle x^*, y \rangle + F(y, x) \right) \ge \langle x^*, x \rangle + F(x, x) = \langle x^*, x \rangle . \tag{4.8}$$

Now from (4.7) and (4.8) we get $\varphi_F(x, x^*) = \langle x^*, x \rangle$. So the strict inequality $\varphi_F(x, x^*) < \langle x^*, x \rangle$ is not possible, thus for all $(x, x^*) \in X \times X^*$ we have $\langle x^*, x \rangle \le \varphi_F(x, x^*)$.

In order to show the second assertion, let $\langle x^*, x \rangle = \varphi_F(x, x^*)$. We already showed that this implies $\langle x^*, y - x \rangle \le F(x, y)$ for all $y \in X$ which means that $x^* \in A^F(x)$.

Conversely, assume that $x^* \in A^F(x)$; then $\langle x^*, y - x \rangle \le F(x, y)$ for all $y \in X$. By monotonicity of F we obtain $\langle x^*, y - x \rangle \le -F(y, x)$ for all $y \in X$. This implies that $\langle x^*, y \rangle + F(y, x) \le \langle x^*, x \rangle$ for all $y \in X$. From here we conclude that $\varphi_F(x, x^*) \le \langle x^*, x \rangle$. By the first part of the proof, $\varphi_F(x, x^*) = \langle x^*, x \rangle$. ■

The Fitzpatrick transform of a normal bifunction and the Fitzpatrick function of an operator are related via the following proposition.

Proposition 4.12 *Let T be an operator. Then $\varphi_{G_T} = \mathcal{F}_T$, where $\mathcal{F}_T$ is the Fitzpatrick function of T.*

Proof. For each $(x, x^*) \in X \times X^*$,

$$\varphi_{G_T}(x, x^*) = \sup_{y \in X} (\langle x^*, y\rangle + G_T(y, x)) = \sup_{y \in X} \left(\langle x^*, y\rangle + \sup_{y^* \in T(y)} \langle y^*, x - y\rangle \right)$$
$$= \sup_{(y, y^*) \in \operatorname{gr} T} (\langle x^*, y\rangle + \langle y^*, x\rangle - \langle y^*, y\rangle) = \mathcal{F}_T(x, x^*).$$

This proves the desired statement. ■

In the following proposition we will show that when the variables of a bi-function F are separated by a function f on a set C, then the subdifferential of f is equal to A^F, the Fitzpatrick transform is nothing else than the sum of f and its conjugate. In addition, the Fitzpatrick transform of G_{A^F} is equal to the Fitzpatrick function of the subdifferential of f.

Proposition 4.13 *Let $f : X \to \mathbb{R} \cup \{+\infty\}$ be a function with* $\operatorname{dom} f = C = \{x \in X : f(x) < \infty\} \neq \emptyset$. *Suppose that $F : X \times X \to \mathbb{R}$ is defined by*

$$F(x, y) = \begin{cases} f(y) - f(x) & \text{if } x \in C, y \in X, \\ -\infty & \text{otherwise.} \end{cases}$$

Then

(i) $A^F = \partial f$.

(ii) $\varphi_F(x, x^*) = f(x) + f^*(x^*)$.

(iii) $\varphi_{G_{A^F}}(x, x^*) = \mathcal{F}_{\partial f}(x, x^*)$.

Proof. (i) It is clear that for $x \notin C$, $A^F(x) = \emptyset = \partial f(x)$. For $x \in C$ we have

$$A^F(x) = \{x^* \in X : F(x, y) \geq \langle x^*, y - x\rangle \quad \forall y \in X\}$$
$$= \{x^* \in X : F(x, y) \geq \langle x^*, y - x\rangle \quad \forall y \in C\}$$
$$= \{x^* \in X : f(y) - f(x) \geq \langle x^*, y - x\rangle \quad \forall y \in C\} = \partial f(x).$$

(ii) By our assumptions, we have

$$\varphi_F(x, x^*) = \sup_{y \in X} (\langle x^*, y\rangle + F(y, x))$$
$$= \sup_{y \in C} (\langle x^*, y\rangle - f(y) + f(x))$$
$$= f(x) + \sup_{y \in C} (\langle x^*, y\rangle - f(y)) = f(x) + f^*(x^*).$$

(iii) By part (i) we have $G_{A^F}(x,y) = G_{\partial f}(x,y)$ and so

$$\begin{aligned}
\varphi_{G_{A^F}}(x,x^*) &= \sup_{y\in X}\left(\langle x^*,y\rangle + G_{\partial f}(y,x)\right) \\
&= \sup_{y\in X}\left(\langle x^*,y\rangle + \sup_{y^*\in\partial f}\langle y^*,x-y\rangle\right) \\
&= \sup_{(y,y^*)\in \mathrm{gr}\,\partial f}\left(\langle x^*,y\rangle + \langle y^*,x-y\rangle\right) \\
&= \mathcal{F}_{\partial f}(x,x^*).
\end{aligned}$$

We are done. ■

Note that in the above proposition if f is lsc and convex then A^F is maximal monotone.

In a similar way as in [30], given a monotone bifunction F we define on $X \times X^*$ the *upper Fitzpatrick transform* φ^F of F by

$$\varphi^F(x,x^*) = \sup_{y\in X}(\langle x^*,y\rangle - F(x,y)), \quad \forall (x,x^*) \in X \times X^*.$$

Remark 4.14 It is easy to show that F is BO-maximal monotone if and only if for all $(x,x^*) \in X \times X^*$, the following equivalence holds:

$$\langle x^*,x\rangle \geq \varphi_F(x,x^*) \Longleftrightarrow \langle x^*,x\rangle \geq \varphi^F(x,x^*). \tag{4.9}$$

In fact, given that $\varphi_F \leq \varphi^F$, (4.9) is equivalent to

$$\langle x^*,x\rangle \geq \varphi_F(x,x^*) \Longrightarrow \langle x^*,x\rangle \geq \varphi^F(x,x^*)$$

or, successively,

$$\begin{gathered}
\langle x^*,x\rangle \geq \sup_{y\in X}\left(\langle x^*,y\rangle + F(y,x)\right) \Rightarrow \langle x^*,x\rangle \geq \sup_{y\in X}(\langle x^*,y\rangle - F(x,y)) \\
\langle x^*,y-x\rangle + F(y,x) \leq 0,\ \forall y\in X \Rightarrow \langle x^*,y-x\rangle \leq F(x,y), \quad \forall y \in X.
\end{gathered}$$

The last line means that F is BO-maximal monotone. Note that whenever F is BO-maximal monotone Theorem 4.11 implies

$$\langle x^*,x\rangle \leq \varphi_F(x,x^*) \leq \varphi^F(x,x^*),$$

so (4.9) can be rewritten as

$$\langle x^*,x\rangle = \varphi^F(x,x^*) \Longleftrightarrow \langle x^*,x\rangle = \varphi_F(x,x^*). \tag{4.10}$$

Note also that

$$\begin{aligned}
\varphi^F(x,x^*) &= \left(F(x,\cdot)\right)^*(x^*) \\
\varphi_F(x,x^*) &= (-F(\cdot,x))^*(x^*)
\end{aligned}$$

These equalities hold for each $(x,x^*) \in X \times X^*$. In case $x \notin \operatorname{dom} F$ then both sides of the first equality are equal to $+\infty$. Also,

$$\begin{aligned}\left(\varphi^F\right)^*(x^*,x) &= \sup_{(y,y^*)\in X\times X^*}\{\langle y^*,x\rangle + \langle x^*,y\rangle - \varphi^F(y,y^*)\}\\ &= \sup_{(y,y^*)\in X\times X^*}\{\langle y^*,x\rangle + \langle x^*,y\rangle - (F(y,\cdot))^*(y^*)\}\\ &= \sup_{y\in X}\{\langle x^*,y\rangle + (F(y,\cdot))^{**}(x)\}\end{aligned}$$

and

$$\begin{aligned}(\varphi_F)^*(x^*,x) &= \sup_{(y,y^*)\in X\times X^*}\{\langle y^*,x\rangle + \langle x^*,y\rangle - \varphi_F(y,y^*)\}\\ &= \sup_{(y,y^*)\in X\times X^*}\{\langle y^*,x\rangle + \langle x^*,y\rangle - (-F(\cdot,y))^*(y^*)\}\\ &= \sup_{y\in X}\{\langle y^*,x\rangle + (-F(\cdot,y))^{**}(x)\}.\end{aligned}$$

In the special case where $F(x,\cdot)$ is lsc and convex for all $x \in \operatorname{dom} F$, then $(F(y,\cdot))^{**} = F(y,\cdot)$ for every $y \in X$, so $\left(\varphi^F\right)^*(x^*,x) = \varphi_F(x,x^*)$. ♦

The following theorem, stated for the reflexive case for simplicity, shows that the arguments of [30] can be used in our framework to obtain the following result. As in [30] we will use the following theorem from [89].

Theorem 4.15 *Let X be a reflexive Banach space. If $h : X \times X^* \to \overline{\mathbb{R}}$ is a proper, lsc and convex function such that $h(x,x^*) \geq \langle x^*,x\rangle$ and $h^*(x^*,x) \geq \langle x^*,x\rangle$, then the operator with graph $\{(x,x^*) : h(x,x^*) = \langle x^*,x\rangle\}$ is maximal monotone.*

Theorem 4.16 *Let X be a reflexive Banach space. Assume that F is a BO-maximal monotone bifunction and $F(x,\cdot)$ is lsc and convex for each $x \in \operatorname{dom} F$. Then A^F is maximal monotone.*

Proof. The assumption that $F(x,\cdot)$ is lsc and convex implies that

$$\left(\varphi^F\right)^*(x^*,x) = \varphi_F(x,x^*).$$

Since $\varphi_F \leq \varphi^F$ and φ_F is lsc and convex, we deduce that

$$\begin{aligned}\varphi^F(x,x^*) &\geq \operatorname{cl}\operatorname{co}\varphi^F(x,x^*) \geq \varphi_F(x,x^*)\\ &= \left(\varphi^F\right)^*(x^*,x) = \left(\operatorname{cl}\operatorname{co}\varphi^F\right)^*(x^*,x).\end{aligned} \tag{4.11}$$

By Theorem 4.11 we know that $\varphi_F(x,x^*) \geq \langle x^*,x\rangle$ with equality if and only if $x^* \in A^F(x)$. This shows in particular that all functions appearing in (4.11) are proper, since $\varphi^F \equiv +\infty$ implies $\left(\varphi^F\right)^* \equiv -\infty$ which is impossible. By Remark 4.14, $\varphi^F(x,x^*) = \langle x^*,x\rangle$ if and only if $x^* \in A^F(x)$. Combining with (4.11) we obtain that $\operatorname{cl}\operatorname{co}\varphi^F(x,x^*) \geq \langle x^*,x\rangle$ and $\left(\operatorname{cl}\operatorname{co}\varphi^F\right)^*(x^*,x) \geq \langle x^*,x\rangle$, with equality if and only if $x^* \in A^F(x)$. Theorem 4.15 now implies that A^F is maximal monotone. ■

Note that it is not necessary to have $F(x,x) = 0$ for all $x \in \operatorname{dom} F$ or to have a closed and convex $\operatorname{dom} F$. Of course, in the case $F(x,y) = +\infty$ when $x \in \operatorname{dom} F$ and $y \notin \operatorname{dom} F$ that was considered in previous papers, the assumption on $F(x,\cdot)$ implies that $\operatorname{dom} F$ is convex. However, it does not imply that $\operatorname{dom} F$ is closed, or that $F(x,x) = 0$ for x in $\operatorname{dom} F$. Consequently, Theorem 4.16 generalizes the corresponding results in [1, 30, 64].

We will need the following result, which is a simple adaptation of Proposition 4.1 of [64] to our framework. Note that in [64] all bifunctions were supposed to satisfy $F(x,x) = 0$, $x \in \operatorname{dom} F$, but this property was actually not needed in Proposition 4.1 that we use.

Proposition 4.17 *Let X be a reflexive Banach space and F be a maximal monotone bifunction. Assume that for every $x \in \operatorname{dom} F$ and any converging sequence $\{x_n\} \subseteq \operatorname{dom} F$, the sequence $\{F(x,x_n)\}$ is bounded from below*[1]*. Then $\operatorname{dom} F \subseteq \operatorname{cl} \operatorname{dom} A^F$. In particular, $\operatorname{cl} \operatorname{dom} F$ is convex.*

Proof. Define the monotone bifunction F_1 by

$$F_1(x,y) = \begin{cases} F(x,y) & \text{if} \quad x \notin \operatorname{dom} F \text{ or } y \in \operatorname{dom} F, \\ +\infty & \text{if} \quad x \in \operatorname{dom} F \text{ and } y \notin \operatorname{dom} F. \end{cases}$$

Then $A^F(x) \subseteq A^{F_1}(x)$ for all $x \in X$ and by maximality of F, $A^F = A^{F_1}$. We apply Proposition 4.1 of [64] and get the result. ■

A trivial consequence is the following corollary.

Corollary 4.18 *Assume that X is reflexive Banach space, F is maximal and $F(x,\cdot)$ is lsc for every $x \in \operatorname{dom} F$. Then $\operatorname{dom} F \subseteq \operatorname{cl} \operatorname{dom} A^F$.*

Using the above, we now show that whenever $F(x,\cdot)$ is lsc and convex for all $x \in \operatorname{dom} F$, BO-maximal monotonicity is equivalent to a more general statement than (4.5).

Theorem 4.19 *Let X be a reflexive Banach space. Assume that $F(x,\cdot)$ is lsc and convex for every $x \in \operatorname{dom} F$. Then the following statements are equivalent:*

(i) F is BO-maximal monotone.

(ii) For each given $\overline{x} \in X$ and for every lsc and convex function ψ with $\psi(\overline{x}) = 0$ and $\operatorname{int}(\operatorname{dom}\psi) \cap \operatorname{dom} F \neq \emptyset$, the following implication holds:

$$F(y,\overline{x}) \leq \psi(y), \quad \forall y \in \operatorname{dom} F \implies$$
$$\exists x^* \in \partial\psi(\overline{x}) : 0 \leq F(\overline{x},y) + \langle x^*, y - \overline{x}\rangle, \quad \forall y \in X .$$

(iii) For each given $\overline{x} \in X$ and for every lsc and convex function ψ with $\psi(\overline{x}) = 0$ and $\operatorname{int}(\operatorname{dom}\psi) \cap \operatorname{dom} F \neq \emptyset$, the following implication holds:

$$F(y,\overline{x}) \leq \psi(y), \quad \forall y \in \operatorname{dom} F \implies 0 \leq F(\overline{x},y) + \psi(y), \quad \forall y \in X .$$

[1] The bound may depend on x.

Proof. Implications (ii)⇒(iii) and (iii)⇒(i) are obvious, so we prove only implication (i)⇒(ii). Let $\overline{x} \in X$ and $\psi(\overline{x}) = 0$. Suppose that

$$F(y, \overline{x}) \leq \psi(y) \quad \forall y \in \operatorname{dom} F\ . \tag{4.12}$$

By Theorem 4.16 the operator A^F is maximal monotone. By assumption, $\operatorname{int}(\operatorname{dom}\psi) \cap \operatorname{dom} F \neq \emptyset$. Since $\operatorname{int}(\operatorname{dom}\psi) = \operatorname{int}(\operatorname{dom}\partial\psi)$ and $\operatorname{dom} F \subseteq \operatorname{cl}\operatorname{dom} A^F$ by Corollary 4.18, we infer that $\operatorname{int}(\operatorname{dom}\partial\psi) \cap \operatorname{dom} A^F \neq \emptyset$. It follows from the well-known theorem of Rockafellar (see [100, Theorem 1]) that $A^F + \partial\psi$ is maximal monotone.

For every $y \in (\operatorname{dom}\partial\psi \cap \operatorname{dom} A^F)$ and every $y_1^* \in A^F(y)$ and $y_2^* \in \partial\psi(y)$, relation (4.12) implies

$$\langle y_1^*, \overline{x} - y\rangle \leq F(y, \overline{x}) \leq \psi(y) = \psi(y) - \psi(\overline{x}) \leq -\langle y_2^*, \overline{x} - y\rangle$$

so

$$\langle y_1^* + y_2^*, y - \overline{x}\rangle \geq 0. \tag{4.13}$$

Relation (4.13) can be written as

$$\langle y^* - 0, y - \overline{x}\rangle \geq 0, \quad \forall (y, y^*) \in \operatorname{gr}\left(A^F + \partial\psi\right).$$

Hence $0 \in (A^F + \partial\psi)(\overline{x})$, i.e., there exists $x^* \in \partial\psi(\overline{x})$ such that $-x^* \in A^F(\overline{x})$. This means that

$$\langle -x^*, y - \overline{x}\rangle \leq F(\overline{x}, y), \quad \forall y \in X$$

i.e., (ii) holds. ■

This result was proved by other methods in [23], assuming in addition that ψ is continuously Gâteaux differentiable, $\operatorname{dom} F$ is convex and contained in $\operatorname{dom}\psi$, and $F(x, y) = +\infty$ for $x \in \operatorname{dom} F$, $y \notin \operatorname{dom} F$.

4.4 Fitzpatrick Transform of Sum

In this section we will redefine pair and partial convolutions and then we will establish various kinds of inequalities.

Fitzpatrick in [52, Problem 5.4] proposed a question for characterization of $\mathcal{F}_{T_1+T_2}$. This problem is still open. Penot and Zalinescu in [89, page 15] and also Bauschke, McLaren and Sendov in [16, Proposition 4.2] have found an upper bound for $\mathcal{F}_{T_1+T_2}$ where T_1 and T_2 are maximal monotone.

Definition 4.20 *Assume that $f, g : X \times Y \to \mathbb{R} \cup \{+\infty\}$ are two functions. Then the* pair convolution *of f, g is defined by*

$$f \Box g(x, y) := \inf\{f(x_1, y_1) + g(x_2, y_2) : x_1 + x_2 = x, y_1 + y_2 = y\}.$$

As in [89] and [98], the *partial convolutions* of f and g are defined by

$$f \Box_1 g(x, y) = \inf\{f(x_1, y) + g(x_2, y) : x_1, x_2 \in X \text{ and } x_1 + x_2 = x\},$$

and

$$f\Box_2 g(x,y) = \inf\{f(x,y_1) + g(x,y_2) : y_1, y_2 \in Y \text{ and } y_1 + y_2 = y\}.$$

In the next proposition we will find an upper bound for $\varphi_{F_1+F_2}$.

Proposition 4.21 *Let $F_1, F_2 : X \times X \to \overline{\mathbb{R}}$ be monotone bifunctions such that* $\operatorname{dom} F_1 = \operatorname{dom} F_2$. *Then*

$$\varphi_{F_1+F_2} \leq \varphi_{F_1}\Box_2\varphi_{F_2}.$$

Proof. Assume that $(x, x^*) \in X \times X^*$ and set $x^* = x_1^* + x_2^*$. Then for all $y \in X$ we have

$$\begin{aligned}\langle x^*, y\rangle + (F_1 + F_2)(y,x) &= (\langle x_1^*, y\rangle + F_1(y,x)) + (\langle x_2^*, y\rangle + F_2(y,x)) \\ &\leq \sup_{y\in X}(\langle x_1^*, y\rangle + F_1(y,x)) + \sup_{y\in X}(\langle x_2^*, y\rangle + F_2(y,x)) \\ &= \varphi_{F_1}(x, x_1^*) + \varphi_{F_2}(x, x_2^*).\end{aligned}$$

By taking the supremum over all $y \in X$ we conclude that

$$\varphi_{F_1+F_2}(x, x^*) \leq \varphi_{F_1}(x, x_1^*) + \varphi_{F_2}(x, x_2^*).$$

Now by taking infimum over all $x_1^* + x_2^* = x^*$ we get

$$\varphi_{F_1+F_2}(x, x^*) \leq \varphi_{F_1}\Box_2\varphi_{F_2}(x, x^*).$$

We are done. ■

Remark 4.22 Assume that $T_1, T_2 : X \to 2^{X^*}$ are two monotone operators such that $\operatorname{dom} T_1 \cap \operatorname{dom} T_2 \neq \emptyset$. Then $G_{T_1+T_2} \leq G_{T_1} + G_{T_2}$. In fact,

$$\begin{aligned}G_{T_1+T_2}(x,y) &= \sup_{x^*\in(T_1+T_2)(x)} \langle x^*, y-x\rangle \\ &= \sup_{x_1^*\in T(x), x_2^*\in T(x), x^*=x_1^*+x_2^*} \langle x_1^* + x_2^*, y-x\rangle \\ &\leq \sup_{x_1^*\in T_1(x)} \langle x_1^*, y-x\rangle + \sup_{x_2^*\in T_2(x)} \langle x_2^*, y-x\rangle \\ &= G_{T_1}(x,y) + G_{T_2}(x,y). \quad \blacklozenge\end{aligned}$$

An immediate consequence of the Proposition 4.21 is the following corollary (see also [16, Proposition 4.2]).

Corollary 4.23 *Let $T_1, T_2 : X \to 2^{X^*}$ be two monotone operators such that* $\operatorname{dom} G_{T_1} = \operatorname{dom} G_{T_2}$. *Then*

$$\mathcal{F}_{T_1+T_2} \leq \mathcal{F}_{T_1}\Box_2\mathcal{F}_{T_2}.$$

Proof. By the above remark $G_{T_1+T_2} \leq G_{T_1} + G_{T_2}$ This implies that

$$\varphi_{G_{T_1+T_2}} \leq \varphi_{G_{T_1}+G_{T_2}}.$$

Now by Applying Proposition 4.12 and Proposition 4.21 we get the desired inequality. ■

Note that this inequality can be strict. See Example 1 in [89]. However in [16] there are some examples where was shown that the equality can become true. In Proposition 4.25, we will show that equality holds for a special type of bifunctions. First we prove a lemma.

Lemma 4.24 *Let $f, g : X \to \mathbb{R} \cup \{+\infty\}$ be two functions such that $C = \operatorname{dom} f = \operatorname{dom} g$. Define $F : X \times X \to \overline{\mathbb{R}}$ by*

$$F(x, y) = \begin{cases} f(y) - g(x) & \text{if } \ x \in C, y \in X, \\ -\infty & \text{otherwise.} \end{cases}$$

Then F is a normal bifunction and $\varphi_F(x, x^*) = f(x) + g^*(x^*)$.

Proof. The normality of F is obvious. Given $(x, x^*) \in X \times X^*$,

$$\begin{aligned} \varphi_F(x, x^*) &= \sup_{y \in X} \{\langle x^*, y\rangle + F(y, x)\} \\ &= \sup_{y \in C} \{\langle x^*, y\rangle + f(x) - g(y)\} \\ &= f(x) + \sup_{y \in C} \{\langle x^*, y\rangle - g(y)\} = f(x) + g^*(x^*). \end{aligned}$$

This proves the lemma. ■

We note that in the above lemma if $f = g$, then F is monotone. Moreover, if f is lsc and convex, then ∂f is maximal monotone and $\varphi_{G_{A^F}} = \mathcal{F}_{\partial f}$.

Proposition 4.25 *Let $F_i : X \times X \to \overline{\mathbb{R}}$ for $i = 1, 2$ be normal bifunctions defined by*

$$F_i(x, y) = \begin{cases} f_i(y) - g_i(x) & \text{if } \ x \in C, y \in X, \\ -\infty & \text{otherwise.} \end{cases}$$

where $f_i, g_i : X \to \mathbb{R} \cup \{+\infty\}$ and $C = \operatorname{dom} f_i = \operatorname{dom} g_i$ for $i = 1, 2$. Then $\varphi_{F_1+F_2} = \varphi_{F_1} \Box_2 \varphi_2$.

Proof. For every $(x, y) \in X \times X$, we have

$$(F_1 + F_2)(x, y) = \begin{cases} (f_1 + f_2)(y) - (g_1 + g_2)(x) & \text{if } \ x \in C, y \in X, \\ -\infty & \text{otherwise.} \end{cases}$$

Then for each $(x, x^*) \in X \times X^*$, using $(g_1 + g_2)^*(x^*) = (g_1^* \Box g_2^*)(x^*)$ (see

Theorem 1.20) and Lemma 4.24 we have

$$\begin{aligned}\varphi_{F_1+F_2}(x,x^*) &= (f_1+f_2)(x)+(g_1+g_2)^*(x^*)\\ &= (f_1+f_2)(x)+(g_1^*\Box g_2^*)(x^*)\\ &= (f_1+f_2)(x)+\inf\{g_1^*(x_1^*)+g_2^*(x_2^*) : x^*=x_1^*+x_2^*\}\\ &= \inf\{f_1(x)+g_1^*(x_1^*)+f_2(x)+g_2^*(x_2^*) : x^*=x_1^*+x_2^*\}\\ &= \inf\{\varphi_{F_1}(x,x_1^*)+\varphi_{F_2}(x,x_2^*) : x^*=x_1^*+x_2^*\}\\ &= \varphi_{F_1}\Box_2\varphi_2(x,x^*).\end{aligned}$$

This proves the desired statement. ■

Let $F : X \times X \to \overline{\mathbb{R}}$ be a bifunction. We say that F is subadditive with respect to the second variable if

$$F(x,y+z) \le F(x,y)+F(x,z) \quad \forall x,y,z \in X.$$

In the next proposition we derive an inequality for the Fitzpatrick transform of such a bifunction.

Proposition 4.26 *Suppose that $F : X \times X \to \overline{\mathbb{R}}$ is a monotone bifunction which is subadditive with respect to its second variable. Then*

$$\varphi_F \le \varphi_F \Box \varphi_F.$$

Proof. For all $x = x_1 + x_2, z \in X$ and $x_1^* + x_2^* = x^* \in X^*$, by using our assumptions, we have

$$\begin{aligned}\langle x_1^*+x_2^*,z\rangle + F(z,x_1+x_2) &\le \langle x_1^*,z\rangle + F(z,x_1)+\langle x_2^*,z\rangle + F(z,x_2)\\ &\le \sup_{z\in X}(\langle x_1^*,z\rangle + F(z,x_1)) + \sup_{z\in X}(\langle x_2^*,z\rangle + F(z,x_2))\\ &= \varphi_F(x_1,x_1^*)+\varphi_F(x_2,x_2^*).\end{aligned}$$

By taking the supremum over all $z \in X$ we get

$$\varphi_F(x,x^*) = \varphi_F(x_1+x_2,x_1^*+x_2^*) \le \varphi_F(x_1,x_1^*)+\varphi_F(x_2,x_2^*) \tag{4.14}$$

Now from the definition of the pair convolution and (4.14) we conclude that the desired inequality. ■

Fitzpatrick inequality of Fitzpatrick transform:

Let F_1 and F_2 be any two BO-maximal monotone bifunctions. Then for each pair $(x,x^*) \in X \times X^*$ we have $\varphi_{F_1}(x,x^*) \ge \langle x^*,x\rangle$ and $\varphi_{F_2}(x,-x^*) \ge \langle -x^*,x\rangle$, thus

$$\varphi_{F_1}(x,x^*)+\varphi_{F_2}(x,-x^*) \ge 0.$$

This inequality corresponds to the well-known *Fitzpatrick inequality* [25, Section 4.1].

4.5 Existence Results

The ideas, and most of the results of this section, originated in a paper of Blum and Oettli [23] for BO-maximality. We will extend their results to BO-maximality in our framework, i.e., for normal bifunctions defined on $X \times X$. Our results generalize the results of Blum and Oettli.

Theorem 4.27 *Assume that X is reflexive and F is BO-maximal monotone, $\operatorname{dom} F$ is closed and convex, and for each $x \in \operatorname{dom} F$, $F(x,x) = 0$ and $F(x,\cdot)$ is lsc and convex. Let $H : X \times X \to \mathbb{R}$ be a function such that $H(\cdot, y)$ is weakly usc for each $y \in \operatorname{dom} F$. Assume that for every $x \in \operatorname{dom} F$, $H(x,x) = 0$ and $H(x,\cdot)$ is lsc and convex. Furthermore, assume that for some $a \in \operatorname{dom} F$ the following implication holds*

$$\|x\| \to +\infty, \quad x \in \operatorname{dom} F \implies -F(a,x) + H(x,a) \to -\infty.$$

Then there exists $\overline{x} \in \operatorname{dom} F$ such that

$$F(y, \overline{x}) \le H(\overline{x}, y) \quad \forall y \in \operatorname{dom} F \tag{4.15}$$

and

$$0 \le F(\overline{x}, y) + H(\overline{x}, y) \quad \forall y \in X. \tag{4.16}$$

Proof. Let us equip X with the weak topology and let $g(x,y) = F(x,y)$, $h(x,y) = H(x,y)$ and $K = \operatorname{dom} F$. Observe that all assumptions of Theorem 1A in [23] are satisfied. By the proof of this theorem there exists $\overline{x} \in \operatorname{dom} F$ such that inequality (4.15) holds. If we set $\psi(y) = H(\overline{x}, y)$ and apply Proposition 4.19, then we conclude that inequality (4.16) also holds. ■

Proposition 4.28 *Let X be a reflexive Banach space and F be a BO-maximal monotone bifunction. Assume that* $\operatorname{dom} F$ *is a nonempty closed convex subset of X and $F(x,x) = 0$ for all $x \in \operatorname{dom} F$, $F(x,\cdot)$ is lsc and convex for each x in* $\operatorname{dom} F$. *Then for every $x^* \in X^*$ there exists $\overline{x} \in \operatorname{dom} F$ such that*

$$F(y, \overline{x}) \le \frac{1}{2}||y||^2 - \frac{1}{2}||\overline{x}||^2 - \langle x^*, y - \overline{x} \rangle \quad \forall y \in \operatorname{dom} F. \tag{4.17}$$

and

$$0 \le F(\overline{x}, y) + \frac{1}{2}||y||^2 - \frac{1}{2}||\overline{x}||^2 - \langle x^*, y - \overline{x} \rangle \quad \forall y \in X. \tag{4.18}$$

Proof. Set $H(x,y) = \frac{1}{2}||y||^2 - \frac{1}{2}||x||^2 - \langle x^*, y - x \rangle$. Then H satisfies all assumptions of Theorem 4.27, since $-\|\cdot\|^2$ is weakly usc. Now if we choose any $a \in \operatorname{dom} F$, then $F(a,\cdot)$ is lsc and convex, thus it is minorized by a continuous affine function. It follows that there exist $z^* \in X^*$ and $k \in \mathbb{R}$ such that

$$F(a,x) \ge \langle z^*, x \rangle + k.$$

Then we see that

$$-F(a,x) + H(x,a) \leq -k - \langle z^*, x\rangle + \frac{1}{2}||a||^2 - \frac{1}{2}||x||^2 - \langle x^*, a - x\rangle$$

From here we conclude that $-F(a,x) + H(x,a) \to -\infty$ as $\|x\| \to +\infty$. The result follows from Theorem 4.27. ■

Proposition 4.29 *Suppose that X is a reflexive Banach space and F is BO-maximal monotone. Assume that* $\operatorname{dom} F$ *is closed and convex, and $F(x,x) = 0$ for all $x \in \operatorname{dom} F$. If $F(x,\cdot)$ is lsc and convex for each $x \in \operatorname{dom} F$, then for every $x^* \in X^*$ there exist $\overline{x} \in \operatorname{dom} F$ and $\overline{x}^* \in \mathcal{J}\overline{x}$ such that*

$$0 \leq F(\overline{x}, y) + \langle \overline{x}^* - x^*, y - \overline{x}\rangle \quad \forall y \in X. \tag{4.19}$$

If in addition X is strictly convex, then $\overline{x}$ is uniquely determined.

Proof. For a given $x^* \in X^*$ by Proposition 4.28 there exists $\overline{x} \in \operatorname{dom} F$ such that

$$0 \leq F(\overline{x}, y) + \frac{1}{2}||y||^2 - \frac{1}{2}||\overline{x}||^2 - \langle x^*, y - \overline{x}\rangle \quad \forall y \in X.$$

Since $\overline{x} \in \operatorname{dom} F$ if $y \in \operatorname{dom} F$ then by Remark 4.3, $F(y, \overline{x}) \in \mathbb{R}$. From the monotonicity of F we get

$$F(y, \overline{x}) \leq \frac{1}{2}||y||^2 - \frac{1}{2}||\overline{x}||^2 - \langle x^*, y - \overline{x}\rangle \quad \forall y \in X. \tag{4.20}$$

Setting $\psi(y) = \frac{1}{2}||y||^2 - \frac{1}{2}||x||^2 - \langle x^*, y - x\rangle$ we note that $\partial\psi(\overline{x}) = \mathcal{J}\overline{x} - x^*$. Applying Theorem 4.19 we deduce the existence of $\overline{x}^* \in \mathcal{J}\overline{x}$ such that (4.19) holds.

Now assume that X is strictly convex. If $\overline{x}$ is not unique, then there exists $\overline{x}_1 \in \operatorname{dom} F$, $\overline{x}_1 \neq \overline{x}$ and $\overline{x}_1^* \in \mathcal{J}\overline{x}_1$ such that $0 \leq F(\overline{x}_1, y) + \langle \overline{x}_1^* - x^*, y - \overline{x_1}\rangle$ for all $y \in X$. From this inequality and (4.19) we obtain

$$0 \leq F(\overline{x}_1, \overline{x}) + \langle \overline{x}_1^* - x^*, \overline{x} - \overline{x}_1\rangle$$
$$0 \leq F(\overline{x}, \overline{x}_1) + \langle \overline{x}^* - x^*, \overline{x}_1 - \overline{x}\rangle.$$

By adding these inequalities and using monotonicity of F we get

$$\langle \overline{x}_1^* - \overline{x}^*, \overline{x} - \overline{x}_1\rangle \geq 0.$$

However, since X is strictly convex, $\mathcal{J}$ is strictly monotone, so we arrived to a contradiction. ■

Corollary 4.30 *Suppose that X is a reflexive, smooth and strictly convex Banach space and $F : X \times X \to \overline{\mathbb{R}}$ is BO-maximal monotone. Assume that* $\operatorname{dom} F$ *is a nonempty closed convex subset of X and $F(x,x) = 0$, for all x in* $\operatorname{dom} F$. *Furthermore, let $F(x,\cdot)$ be lsc and convex for all $x \in \operatorname{dom} F$. Then for every $x \in X$ and $\lambda > 0$ there exists a unique $x_\lambda \in \operatorname{dom} F$ such that*

$$0 \leq F(x_\lambda, y) + \frac{1}{\lambda}\langle \mathcal{J}x_\lambda - \mathcal{J}x, y - x_\lambda\rangle \quad \forall y \in X.$$

Proof. Fix x in X. For a given $\lambda > 0$, F and λF have the same properties. Thus for $\mathcal{J}x \in X^*$ there exists a unique $x_\lambda \in \operatorname{dom} F$ such that

$$0 \leq \lambda F(x_\lambda, y) + \langle \mathcal{J}x_\lambda - \mathcal{J}x, y - x_\lambda \rangle \quad \forall y \in X.$$

We are done. ■

Let F satisfy the assumptions of the above corollary. The single-valued operator $R_\lambda^F : X \to \operatorname{dom} F$ defined by $R_\lambda^F(x) = x_\lambda$ generalizes the notion of of a monotone bifunction defined on a subset C of X [64, 86], to normal bifunctions.

4.6 Illustrations and Examples

We will see several examples in this section. Throughout this section we set

$$\infty - \infty = -\infty + \infty = -\infty.$$

One can easily check that for each $\lambda > 0$ and every normal bifunction F we have

$$\varphi_{\lambda F}(x, x^*) = \lambda \varphi_F\left(x, \frac{x^*}{\lambda}\right).$$

Example 4.31 Let $f, g : X \to \mathbb{R}$ be two functions such that $X = \operatorname{dom} f = \operatorname{dom} g$ and $f(x) > 0$ for all $x \in X$. Define $F : X \times X \to \overline{\mathbb{R}}$ by

$$F(x, y) = -f(y)\, g(x).$$

Then F is a normal bifunction and

$$\begin{aligned} \varphi_F(x, x^*) &= \sup_{y \in X} \{\langle x^*, y\rangle + F(y, x)\} \\ &= \sup_{y \in X} \{\langle x^*, y\rangle - f(x)\, g(y)\} \\ &= f(x) \sup_{y \in X} \left\{\left\langle \frac{x^*}{f(x)}, y \right\rangle - g(y)\right\} = f(x)\, g^*\left(\frac{x^*}{f(x)}\right). \end{aligned}$$ ▲

In the following we provide an example which shows that φ_F differs from $\varphi_{G_{A^F}}$ even if F is maximal and $\operatorname{dom} F = X$.

Example 4.32 Let X be a Banach space and $F(x, y) = \frac{1}{2}\|y\|^2 - \frac{1}{2}\|x\|^2$. Then $\operatorname{dom} F = X$ and F is monotone. Proposition 4.13 implies $A^F(x) = \mathcal{J}(x)$. Since $\mathcal{J}$ is maximal monotone operator we conclude that F is maximal monotone and so $\varphi_F(x, x^*) = \frac{1}{2}\|x\|^2 + \frac{1}{2}\|x^*\|^2$. Furthermore,

$$\begin{aligned} \varphi_{G_{A^F}}(x, x^*) = \mathcal{F}_J(x, x^*) &\leq \frac{1}{4}(\|x\| + \|x^*\|)^2 \qquad (4.21) \\ &\leq \frac{1}{2}\|x\|^2 + \frac{1}{2}\|x^*\|^2 = \varphi_F(x, x^*). \end{aligned}$$

The first inequality in (4.21) is a consequence of Proposition 4.1 in [33]. Note that when X is a real Hilbert space, Example 3.10 in [16] implies that

$$\varphi_{G_{A^F}}(x, x^*) = \frac{1}{4}\|x + x^*\|^2. \qquad \blacktriangle$$

Example 4.33 Let X be a real Hilbert space and C is a closed convex subset of X. Define $F : X \times X \to \overline{\mathbb{R}}$ by $F(x, y) = \iota_C(y) - \iota_C(x)$ where ι_C is the indicator function of C. Then by Proposition 4.13

$$\varphi_F(x, x^*) = \iota_C(x) + \iota_C^*(x^*) \text{ and } A^F(x) = \partial\iota_C(x) = N_C(x).$$

Now if $x, y \in C$, then $G_{N_C}(x, y) = \sup_{x^* \in N_C(x)} \langle x^*, y - x\rangle = 0$ and so

$$\begin{aligned}\varphi_{G_{A^F}}(x, x^*) &= \sup_{y\in C}\left(\langle x^*, y\rangle + G_{N_C}(y, x)\right) \\ &= \sup_{y\in C}\langle x^*, y\rangle = \sigma_C(x^*) = \iota_C^*(x^*).\end{aligned}$$

If $x \notin C$, then $P_C(x) \in C$. Take $y = x - P_C(x) \in N_C(P_C(x)) \setminus \{0\}$. Then

$$G_{N_C}(x, y) = \sup_{x^* \in N_C(x)} \langle x^*, y - P_C(x)\rangle \geq \sup_{\lambda\in[0,\infty)} \langle y, \lambda y\rangle = +\infty.$$

Therefore

$$\begin{aligned}\varphi_{G_{A^F}}(x, x^*) &= \begin{cases} \sigma_C(x^*) & \text{if} \quad x \in C, \\ +\infty & \text{if} \quad x \notin C \end{cases} \\ &= \iota_C(x) + \iota_C^*(x^*).\end{aligned}$$

Note that this example also shows that $\varphi_F = \varphi_{G_{A^F}}$. $\blacktriangle$

Example 4.34 Assume that X is a real Hilbert space and $f(x) = \|x\|$. Define $F : X \times X \to \mathbb{R}$ by $F(x, y) = \|y\| - \|x\|$, then $f^*(x^*) = \iota_{\overline{B(0^*,1)}}$, that is, f^* is the indicator function of the closed unit ball in X^*. Now Example 3.3 in [16] and Proposition 4.13 imply that

$$\varphi_F(x, x^*) = \varphi_{G_{A^F}}(x, x^*) = \|x\| + \iota_{\overline{B(0^*,1)}} = \begin{cases} \|x\| & \text{if} \quad \|x^*\| \leq 1, \\ +\infty & \text{otherwise.} \end{cases}$$

We observe that also in this example $\varphi_F = \varphi_{G_{A^F}}$. $\blacktriangle$

Example 4.35 Let $X = \mathbb{R}$ and define $f : X \to \mathbb{R} \cup \{+\infty\}$ by

$$f(x) = \begin{cases} +\infty & \text{if} \quad x < 0, \\ 0 & \text{if} \quad x = 0, \\ x\ln(x) - x & \text{if} \quad x > 0. \end{cases}$$

Note that $f^*(x^*) = \exp(x^*)$. Now define $F : X \times X \to \overline{\mathbb{R}}$ by $F(x,y) = f(y) - f(x)$. Then by Proposition 4.13, $\varphi_F(x,x^*) = f(x) + f^*(x^*)$. From this and Example 3.6 in [16] we obtain

$$\varphi_{G_{A^F}}(x,x^*) = \begin{cases} +\infty & \text{if} \quad x < 0, \\ \exp(x^* - 1) & \text{if} \quad x = 0, \\ xx^* + x\left(W\left(xe^{1-x^*}\right) + \frac{1}{W\left(xe^{1-x^*}\right)} - 2\right) & \text{if} \quad x > 0. \end{cases}$$

Here, $W : [0,+\infty) \to [0,+\infty)$ is the Lambert function i.e., the function $W^{-1} : [0,+\infty) \to [0,+\infty)$ is defined by $W^{-1}(x) = x\exp(x)$. ▲

Note that generally $\varphi_{G_{A^F}} \le \varphi_F$ for each monotone bifunction F, because for each $x^* \in A^F(x)$ we have $\langle x^*, y - x\rangle \le F(x,y)$ and so

$$G_{A^F}(x,y) = \sup_{x^* \in A^F(x)} \langle x^*, y - x\rangle \le F(x,y).$$

Now by the definition of Fitzpatrick transform, $\varphi_{G_{A^F}}(x,x^*) \le \varphi_F(x,x^*)$. Next example shows that the inequality $\varphi_{G_{A^F}} \le \varphi_F$ can be strict even if X is finite dimensional, $\operatorname{dom} F = X$, F is continuous and maximal monotone.

Example 4.36 Define $F : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ by $F(x,y) = \frac{1}{2}y^2 - \frac{1}{2}x^2$. In this case we have $A^F(x) = \{x\}$,

$$G_{A^F}(x,y) = \sup_{x^* \in A^F(x)} \{\langle x^*, y - x\rangle\} = x(y - x),$$

and by Proposition 4.13

$$\varphi_F(x,x^*) = \frac{1}{2}x^2 + \frac{1}{2}(x^*)^2.$$

Also

$$\begin{aligned} \varphi_{G_{A^F}}(x,x^*) &= \sup_{y \in \mathbb{R}} \{x^* y + G(y,x)\} \\ &= \sup_{y \in \mathbb{R}} \left\{x^* y + yx - y^2\right\} = \frac{1}{4}(x^* - x)^2. \end{aligned}$$

Thus the inequality $\varphi_{G_{A^F}} \le \varphi_F$ can be strict. ▲

4.7 n-Cyclically Monotone Bifunctions

For $n = 2, 3, \ldots$ an operator $T : X \to 2^{X^*}$ is *n-cyclically monotone* [13, Definition 1.1] if for each cycle $x_1, x_2, \ldots, x_n, x_{n+1} = x_1$ and every $x_i^* \in T(x_i)$, $i = 1, 2, \ldots, n$

$$\sum_{i=1}^{n} \langle x_i^*, x_{i+1} - x_i\rangle \le 0.$$

Also T is called cyclically monotone (see also Chapter 1) if for every $n \in \{2, 3, ...\}$ and each cycle $x_1, x_2, ..., x_n, x_{n+1} = x_1$ so that $x_i^* \in T(x_i)$, $i = 1, 2, ..., n$

$$\sum_{i=1}^{n} \langle x_i^*, x_{i+1} - x_i \rangle \leq 0.$$

Every cyclically monotone and each 2-cyclically monotone operator is monotone.

An operator T is called *maximal n-cyclically monotone* (maximal cyclically monotone) if it has no n-cyclically monotone (cyclically monotone) extension other than itself, i.e., whenever $T_1 : X \to 2^{X^*}$ is a n-cyclically monotone (cyclically monotone) map such for all $x \in X$, $T(x) \subset T_1(x)$, then $T_1 = T$.

Notation 4.37 *In this section we set* $\infty - \infty = -\infty + \infty = -\infty$.

We reproduce the following definition from [13].

Definition 4.38 *Let* $T : X \to 2^{X^*}$ *be an operator and* $n \in \{2, 3, ...\}$. *For* $n = 2$, *define* $\mathcal{F}_{T,2} : X \times X^* \to \mathbb{R} \cup \{+\infty\}$ *by*

$$\mathcal{F}_{T,2}(x, x^*) = \sup_{(x_1, x_1^*) \in \operatorname{gr} T} \left(\langle x_1^*, x \rangle + \langle x^*, x_1 \rangle - \langle x_1^*, x_1 \rangle \right).$$

Now suppose that $n \in \{3, 4, ...\}$. *Then the* Fitzpatrick function of T of order n, *is the function* $\mathcal{F}_{T,n} : X \times X^* \to \overline{\mathbb{R}}$ *defined by*

$$\mathcal{F}_{T,n}(x, x^*) = \sup \left(\sum_{i=1}^{n-2} \langle x_i^*, x_{i+1} - x_i \rangle + \left\langle x_{n-1}^*, x - x_{n-1} \right\rangle + \langle x^*, x_1 \rangle \right) \quad (4.22)$$

where the supremum is taken over all families $(x_1, x_1^*), (x_2, x_2^*), ..., \left(x_{n-1}, x_{n-1}^*\right)$ *in* $\operatorname{gr} T$.
The Fitzpatrick function of T of infinite order *is defined by*

$$\mathcal{F}_{T,\infty} = \sup_{n \in \{2,3,...\}} \mathcal{F}_{T,n}.$$

We note that the Fitzpatrick function of T of order n is equal to

$$\sup \left\{ \langle x^*, x \rangle + \left(\sum_{i=1}^{n-2} \langle x_i^*, x_{i+1} - x_i \rangle + \left\langle x_{n-1}^*, x - x_{n-1} \right\rangle + \langle x^*, x_1 - x \rangle \right) \right\}. \quad (4.23)$$

Again the supremum is taken over all families $(x_1, x_1^*), (x_2, x_2^*), ..., \left(x_{n-1}, x_{n-1}^*\right)$ in $\operatorname{gr} T$.

Note that for $n \in \{2, 3, ...\}$, being the supremum of affine functions, $\mathcal{F}_{T,n}$ is lsc and convex. Also $\mathcal{F}_{T,2}$ is nothing else than the Fitzpatrick function.

Assume that $F : X \times X \to \overline{\mathbb{R}}$ is a monotone bifunction. For $n = 2, 3, ...$ we say that F is n-cyclically monotone if for every $x_1, x_2, ..., x_n \in X$

$$\sum_{i=1}^{n} F(x_i, x_{i+1}) \leq 0$$

where $x_{n+1} = x_1$.

We remind that F is cyclically monotone (see also Chapter 2), if it is n-cyclically monotone for every $n \in \mathbb{N}$.

Assume that $F : X \times X \to \overline{\mathbb{R}}$ is a monotone bifunction. If $n = 2$, we set

$$\varphi_{F,2}(x, x^*) = \sup_{x_1 \in X} (F(x_1, x) + \langle x^*, x_1 \rangle) = \varphi_F(x, x^*)$$

the original definition of Fitzpatrick transform (see Section 3, Definition 4.10). Let now $n \in \{3, 4, ...\}$. We define the *Fitzpatrick transform of F of order n* by

$$\varphi_{F,n}(x, x^*) = \sup_{x_1, \ldots, x_{n-1} \in X} \left[\left(\sum_{i=1}^{n-2} F(x_i, x_{i+1}) \right) + F(x_{n-1}, x) + \langle x^*, x_1 \rangle \right].$$

Equivalently, the Fitzpatrick transform of F of order n is equal to

$$\sup_{x_1, \ldots, x_{n-1} \in X} [\langle x^*, x \rangle + \left(\sum_{i=1}^{n-2} F(x_i, x_{i+1}) \right) + F(x_{n-1}, x) + \langle x^*, x_1 - x \rangle].$$

The *Fitzpatrick transform of infinite order* is defined by

$$\varphi_{F,\infty} = \sup_{n \in \{2,3,\ldots\}} \varphi_{F,n}.$$

It should be noticed that if $F(x, \cdot)$ is lsc and convex, then for each $n \in \{2, 3, ...\}$, $\varphi_{F,n}$ is also lsc and convex. Moreover $\varphi_{F,\infty}$ is lsc and convex if $F(x, \cdot)$ is lsc and convex.

We mention that $(\varphi_{F,n})$, $n \in \{2, 3, ...\}$ is a sequence of increasing functions and that $\varphi_{F,n} \to \varphi_{F,\infty}$ pointwise.

The Fitzpatrick transform of order n of a monotone bifunction and the Fitzpatrick function of order n of an operator are related via the following proposition.

Proposition 4.39 *Suppose that T is an operator. Then for all $n \in \{2, 3, ...\}$ $\varphi_{G_T,n} = \mathcal{F}_{T,n}$.*

Proof. For each $(x, x^*) \in X \times X^*$ and for every $n \in \{3, 4, ...\}$, we have

$$\begin{aligned}
\varphi_{G_T,n}(x, x^*) &= \sup_{x_1, \ldots, x_{n-1} \in X} \left(\sum_{i=1}^{n-2} G_T(x_i, x_{i+1}) \right) + G_T(x_{n-1}, x) + \langle x^*, x_1 \rangle \\
&= \sup_{x_1, \ldots, x_{n-1} \in X} [\left(\sum_{i=1}^{n-2} \sup_{x_i^* \in T(x_i)} \langle x_i^*, x_{i+1} - x_i \rangle \right) \\
&\qquad + \sup_{x_{n-1}^* \in T(x_{n-1})} \left\langle x_{n-1}^*, x - x_{n-1} \right\rangle + \langle x^*, x_1 \rangle] \\
&= \mathcal{F}_{T,n}(x, x^*).
\end{aligned}$$

Thus we observe that the $\varphi_{G_T,n}$ is the Fitzpatrick function of order $n \in \{3, 4, ...\}$. In particular, if $n = 2$ then Proposition 4.12 implies that $\varphi_{G_T,2}(x, x^*) = \mathcal{F}_T(x, x^*)$, i.e., $\varphi_{G_T,2}$ is the Fitzpatrick function. ■

Note that from the above proposition we conclude that $\varphi_{G_T,\infty} = \mathcal{F}_{T,\infty}$.

Definition 4.40 *A n-cyclically monotone bifunction $F : X \times X \to \overline{\mathbb{R}}$ is called* BO-n-cyclically maximal monotone *if for every $(x, x^*) \in X \times X^*$ the following implication holds:*

$$\left(\sum_{i=1}^{n-2} F(x_i, x_{i+1})\right) + F(x_{n-1}, x) + \langle x^*, x_1 - x\rangle \leq 0 \quad \forall x_1, x_2, ..., x_{n-1} \in X$$
$$\Longrightarrow \langle x^*, x_1 - x\rangle \leq F(x, x_1) \quad \forall x_1 \in X.$$

Theorem 4.41 *Assume that F is a BO-n-cyclically maximal monotone. Then for $n \in \{2, 3, ...\}$*

(i) $\langle x^*, x\rangle \leq \varphi_{F,n}(x, x^*)$ for all $(x, x^*) \in X \times X^*$;

(ii) $\langle x^*, x\rangle = \varphi_{F,n}(x, x^*)$ if and only if $x^* \in A^F(x)$.

Proof. For every $n \in \{2, 3, 4, ...\}$ we have $\varphi_{F,n}(x, x^*) \geq \varphi_{F,2}(x, x^*)$, so (i) is an obvious consequence of Theorem 4.11.

To show (ii), we remark first that if $\langle x^*, x\rangle = \varphi_{F,n}(x, x^*)$, then

$$\langle x^*, x\rangle \leq \varphi_{F,2}(x, x^*) \leq \varphi_{F,n}(x, x^*) = \langle x^*, x\rangle$$

so again by Theorem 4.11 we deduce $x^* \in A^F(x)$.

Conversely, suppose that $x^* \in A^F(x)$. Then $\langle x^*, x_1 - x\rangle \leq F(x, x_1)$ for every x_1 in X and so

$$-F(x, x_1) + \langle x^*, x_1\rangle \leq \langle x^*, x\rangle . \tag{4.24}$$

By hypothesis F is n-cyclically monotone. Thus for all $x_1, ..., x_{n-1} \in X$

$$\left(\sum_{i=1}^{n-2} F(x_i, x_{i+1})\right) + F(x_{n-1}, x) + F(x, x_1) \leq 0. \tag{4.25}$$

The following inequality can be read off from (4.24) and (4.25)

$$\left(\sum_{i=1}^{n-2} F(x_i, x_{i+1})\right) + F(x_{n-1}, x) + \langle x^*, x_1\rangle \leq \langle x^*, x\rangle .$$

Now by taking the supremum over all $x_1, x_2, ..., x_{n-1}$ in X it follows that $\varphi_{F,n}(x, x^*) \leq \langle x^*, x\rangle$. From this and part (i) we obtain $\varphi_{F,n}(x, x^*) = \langle x^*, x\rangle$. ■

Remark 4.42 (i) If $T : X \to 2^{X^*}$ is a monotone operator, then T is n-cyclically monotone operator if and only if G_T is n-cyclically monotone bifunction. Indeed, for a given cycle $x_1, x_2, ..., x_n$ we have

$$\sum_{i=1}^{n} \langle x_i^*, x_{i+1} - x_i\rangle \leq 0 \quad \forall x_i^* \in T(x_i)$$

if and only if

$$\sum_{i=1}^{n} \sup_{x_i^* \in T(x_i)} \langle x_i^*, x_{i+1} - x_i \rangle \leq 0$$

which is equivalent to $\sum_{i=1}^{n} G_T(x_i, x_{i+1}) \leq 0$.

(ii) *If* T is *maximal* n-cyclically *monotone, then* G_T is BO-n-cyclically *maximal monotone. Assume that* $(x_0, x_0^*) \in X \times X^*$ and

$$\left(\sum_{i=1}^{n-2} G_T(x_i, x_{i+1})\right) + G_T(x_{n-1}, x_0) + \langle x_0^*, x_1 - x_0 \rangle \leq 0$$

for all $x_1, x_2, ..., x_{n-1} \in X$. Then for each $x_i^* \in T(x_i)\,, i = 1, 2, ..., n-1$ we have

$$\left(\sum_{i=1}^{n-2} \langle x_i^*, x_{i+1} - x_i \rangle\right) + \left\langle x_{n-1}^*, x_0 - x_{n-1} \right\rangle + \langle x_0^*, x_1 - x_0 \rangle \leq 0 \tag{4.26}$$

for all $x_1, x_2, ..., x_{n-1} \in X$. Now define $\operatorname{gr} \hat{T} = \operatorname{gr} T \cup \{((x_0, x_0^*))\}$. According to relation (4.26), $\hat{T}$ is n-cyclically monotone and $\operatorname{gr} T \subset \operatorname{gr} \hat{T}$. By assumption T is *maximal* n-cyclically *monotone, so* $\operatorname{gr} \hat{T} = \operatorname{gr} T$. Therefore $(x_0, x_0^*) \in \operatorname{gr} T$, thus

$$\langle x_0^*, x_1 - x_0 \rangle \leq \sup_{x^* \in T(x_0)} \langle x^*, x_1 - x_0 \rangle = G_T(x_0, x_1)\,.$$

This means that G_T is BO-maximal monotone. ♦

In the following proposition we will find a recursion formula for the Fitzpatrick transform of order n. Bauschke, Borwein, and Wang in [18, Theorem 6.5] proved this formula for single valued monotone operators. Here we generalize it to monotone bifunctions.

Proposition 4.43 *Assume that* $F : X \times X \to \overline{\mathbb{R}}$ *is a monotone bifunction and* $n \in \{2, 3, ...\}$. *Then*

$$\varphi_{F,n+1}(x, x^*) = \sup_{y \in X} \{\varphi_{F,n}(y, x^*) + F(y, x)\} \quad \forall (x, x^*) \in X \times X^*. \tag{4.27}$$

Proof. Given $(x, x^*) \in X \times X^*$. By the definition of Fitzpatrick transform of order $n+1$ we have

$$\begin{aligned}
\varphi_{F,n+1}(x, x^*) &= \sup_{x_1,\dots,x_n \in X} \left[\left(\sum_{i=1}^{n-1} F(x_i, x_{i+1})\right) + F(x_n, x) + \langle x^*, x_1 \rangle\right] \\
&= \sup_{x_n} \{ \sup_{x_1,\dots,x_{n-1} \in X} \left[\left(\sum_{i=1}^{n-2} F(x_i, x_{i+1})\right) + F(x_{n-1}, x_n) + \langle x^*, x_1 \rangle\right] \\
&\qquad + F(x_n, x)\} \\
&= \sup_{x_n} \{\varphi_{F,n}(x_n, x^*) + F(x_n, x)\}\,.
\end{aligned}$$

This proves (4.27). ■

Corollary 4.44 *Assume that $T : X \to 2^{X^*}$ is monotone and $n \in \{2, 3, ...\}$. Then*

$$\mathcal{F}_{T,n+1}(x, x^*) = \sup_{y \in X} \{\mathcal{F}_{T,n}(y, x^*) + G_T(y, x)\} \quad \forall (x, x^*) \in X \times X^*.$$

Proof. Apply Propositions 4.43 and 4.39. ■

Example 4.45 (**Rotations**) *According to Example 4.6 in [13], let $X = \mathbb{R}^2$* and $n \in \{2, 3, ...\}$. Define R_n by

$$R_n = \begin{bmatrix} \cos(\pi/n) & -\sin(\pi/n) \\ \sin(\pi/n) & \cos(\pi/n) \end{bmatrix}.$$

*Then R_n is maximal monotone and n-*cyclically *monotone, but it is not $(n+1)$-cyclically monotone; see also [10]. The above remark implies that G_{R_n} is* BO-n-cyclically maximal *monotone bifunction, nevertheless it is not $(n+1)$-cyclically monotone bifunction.* ▲

Example 4.46 Define $T : \mathbb{R}^2 \to \mathbb{R}^2$ by

$$T(a, b) = (b, -a).$$

Then T is maximal monotone and so

$$G_T((a, b), (c, d)) = bc - ad$$

is BO-maximal monotone bifunction. However, it is not 3-cyclically monotone bifunction; for instance, if we consider the cycle $x_1 = (0, 1), x_2 = \left(\frac{\sqrt{3}}{2}, -\frac{1}{2}\right)$ and $x_3 = \left(-\frac{\sqrt{3}}{2}, -\frac{1}{2}\right)$, then

$$G_T(x_1, x_2) + G_T(x_2, x_3) + G_T(x_3, x_1) = 3\frac{\sqrt{3}}{2}.$$

Similarly, if we define (see also [62]) $T : \mathbb{R}^2 \to \mathbb{R}^2$ by $T(a, b) = \left(\frac{a}{4.} + b, \frac{b}{4} - a\right)$, then T is strictly monotone, maximal monotone and

$$G_T((a, b), (c, d)) = (bc - ad) - \frac{1}{4}\left(a^2 + b^2 - ac - bd\right)$$

is strictly monotone, BO-maximal monotone bifunction. However,

$$G_T(x_1, x_2) + G_T(x_2, x_3) + G_T(x_3, x_1) = \left(\frac{\sqrt{3}}{2} - \frac{3}{8}\right) > 0.$$

Therefore it is not a 3-cyclically monotone bifunction. ▲

Proposition 4.47 *Suppose that F is BO-n-cyclically maximal monotone. Then for each $n \in \{2, 3, ...\}$*

$$\varphi^*_{F,n}(x^*, x) \geq \mathcal{F}_{A^F}(x, x^*).$$

Proof. For a given $(x,x^*) \in X \times X^*$, by using Theorem 4.41, we have

$$\begin{aligned}\varphi^*_{F,n}(x^*,x) &= \sup_{(y,y^*)\in X\times X^*} \{\langle (x^*,x),(y,y^*)\rangle - \varphi_{F,n}(y,y^*)\} \\ &\geq \sup_{(y,y^*)\in \operatorname{gr} A^F} \{\langle x^*,y\rangle + \langle x,y^*\rangle - \langle y,y^*\rangle\} = \mathcal{F}_{A^F}(x,x^*)\,.\end{aligned}$$

This proves the desired inequality. ■

It should be noted that if A^F is maximal monotone, then from the above proposition we infer that

$$\varphi^*_{F,n}(x^*,x) \geq \langle x^*,x\rangle\,.$$

Proposition 4.48 *Assume that $F : X \times X \to \overline{\mathbb{R}}$ is a cyclically monotone bifunction such that $F(x,\cdot)$ is lsc and convex for every $x \in \operatorname{dom} F$. Then there exists a proper, lsc and convex function f such that*

$$F(x,y) \leq f(y) - f(x), \quad \forall x,y \in X. \tag{4.28}$$

If in addition F is BO-maximal monotone, then f is unique up to a constant and $A^F = \partial f$. In particular, A^F is maximal cyclically monotone.

Proof. The proof follows similar steps as in Proposition 2.29, but here F may take the values $\pm\infty$, so some extra care is necessary, taking into account Notation 4.37. As in Proposition 2.29, choose $x_0 \in \operatorname{dom} F$ and define $f : X \to \mathbb{R} \cup \{+\infty\}$ by

$$\begin{aligned}f(x) = \sup\{&F(x_0,x_1) + F(x_1,x_2) + \cdots + F(x_{n-1},x_n) \\ &+F(x_n,x) : x_1,x_2,\ldots x_n \in X\}.\end{aligned}$$

Note that the above supremum can be equivalently taken over $x_1,x_2,\ldots x_n \in \operatorname{dom} F$, and f is lsc and convex as supremum of lsc convex functions.

Let $x_1,x_2,\ldots x_n, x \in X$. Since F is cyclically monotone,

$$F(x_0,x_1) + F(x_1,x_2) + \cdots + F(x_{n-1},x_n) + F(x_n,x) + F(x,x_0) \leq 0$$

which implies

$$F(x_0,x_1) + F(x_1,x_2) + \cdots + F(x_{n-1},x_n) + F(x_n,x) \leq -F(x,x_0).$$

By taking the supremum over $x_1,x_2,\ldots x_n \in \operatorname{dom} F$ we obtain $f(x) \leq -F(x,x_0)$ for all $x \in X$. In particular, $f(x_0) \leq -F(x_0,x_0) < +\infty$; since also $f(x) \geq F(x_0,x) > -\infty$ for all $x \in X$, f is proper.

For every $x,y \in X$ and $x_1,x_2,\ldots x_n \in \operatorname{dom} F$ we have by the definition of f:

$$F(x_0,x_1) + F(x_1,x_2) + \cdots + F(x_{n-1},x_n) + F(x_n,x) + F(x,y) \leq f(y).$$

Taking the supremum over all $x_1, x_2, \ldots x_n \in X$ we deduce

$$f(x) + F(x,y) \leq f(y)$$

that is, inequality (4.28) holds.

Now assume that F is also BO-maximal monotone. Let $(x, x^*) \in \operatorname{gr} \partial f$. Then for all $y \in X$,

$$F(y,x) + \langle x^*, y - x \rangle \leq f(x) - f(y) + \langle x^*, y - x \rangle \leq 0.$$

Using that F is BO-maximal monotone we obtain

$$\langle x^*, y - x \rangle \leq F(x,y).$$

This implies that $x^* \in A^F(x)$. Since ∂f is maximal monotone, we deduce that $\partial f = A^F$ and A^F is maximal monotone.

Now assume that g is another lsc and convex function such that

$$F(x,y) \leq g(y) - g(x), \quad \forall x, y \in X.$$

For every $(x, x^*) \in \operatorname{gr} \partial f = \operatorname{gr} A^F$ and $y \in X$ we obtain

$$\langle x^*, y - x \rangle \leq F(x,y) \leq g(y) - g(x).$$

It follows that $\partial f \subseteq \partial g$, thus $\partial f = \partial g$. This implies that g differs from f by a constant [98]. ■

The following results are to be compared with Proposition 4.13.

Lemma 4.49 *Let F be a BO-maximal monotone bifunction. If there exists some proper, lsc and convex function f such that for all $(x, x^*) \in X \times X^*$: $\varphi_F(x, x^*) \leq f(x) + f^*(x^*)$, then $\partial f = A^F$ so that f is uniquely determined up to a constant.*

Proof. Let $x^* \in \partial f(x)$; then $\langle x^*, x \rangle = f(x) + f^*(x^*)$, so $\langle x^*, x \rangle \geq \varphi_F(x, x^*)$. By Theorem 4.11, this implies that $x^* \in A^F(x)$. Thus, $\partial f(x) \subseteq A^F(x)$. By maximal monotonicity of ∂f, we obtain that $\partial f = A^F$. ■

Proposition 4.50 *Suppose that F is a cyclically monotone bifunction such that $F(x, \cdot)$ is lsc and convex for every $x \in \operatorname{dom} F$. Then there exists a proper, lsc and convex function f such that*

$$\varphi_{F,n}(x, x^*) \leq f(x) + f^*(x^*) \quad \forall (x, x^*) \in X \times X^*, \forall n \in \{3, 4, ...\}$$

and

$$\varphi_{F,\infty}(x, x^*) \leq f(x) + f^*(x^*) \quad \forall (x, x^*) \in X \times X^*.$$

Furthermore, if F is BO-maximal monotone, then f is unique up to a constant.

Proof. By Proposition 4.48 there exist a proper, lsc and convex function f such that

$$F(x,y) \leq f(y) - f(x) \quad \forall x,y \in X.$$

Hence for each $n \in \{3,4,...\}$ we have

$$\left(\sum_{i=1}^{n-2} F(x_i, x_{i+1})\right) + F(x_{n-1}, x) + \langle x^*, x_1 \rangle \leq f(x) - f(x_1) + \langle x^*, x_1 \rangle.$$

By taking the supremum over $x_1, ..., x_{n-1}$, for each $n \in \{3,4,...\}$ we get

$$\begin{aligned}\varphi_{F,n}(x,x^*) &\leq \sup_{x_1 \in X} (f(x) - f(x_1) + \langle x^*, x_1 \rangle) \\ &= f(x) + \sup_{x_1 \in X} (\langle x^*, x_1 \rangle - f(x_1)) = f(x) + f^*(x^*).\end{aligned}$$

Now by taking supremum over n, we obtain

$$\varphi_{F,\infty}(x,x^*) \leq f(x) + f^*(x^*).$$

The uniqueness of f up to constant is an immediate consequence of the above lemma and the fact that $\varphi_F \leq \varphi_{F,n} \leq \varphi_{F,\infty}$. ■

Proposition 4.51 *Let $f : X \to \mathbb{R} \cup \{+\infty\}$ be a proper, lsc and convex function with* $\operatorname{dom} f = \{x \in X : f(x) < \infty\}$. *Define $F : X \times X \to \overline{\mathbb{R}}$ by*

$$F(x,y) = f(y) - f(x).$$

Then F is cyclically monotone and for each $n \in \{2,3,...\}$ and every (x,x^*) in $X \times X^*$,

$$\varphi_{F,n}(x,x^*) = \varphi_{F,\infty}(x,x^*) = f(x) + f^*(x^*).$$

That is, the sequence $\{\varphi_{F,n}\}$ is a constant sequence. Moreover, for each $n \in \{2,3,...\}$

$$\varphi_{G_{A^F},n} = \mathcal{F}_{\partial f,n}.$$

Proof. For the proof of first assertion we will apply the recursion formula in Proposition 4.43 and induction on n.
The base case $n = 2$ is proved in Proposition 4.13.
Induction step: Suppose the result is true for $n = k$. This says:

$$\varphi_{F,k}(x,x^*) = f(x) + f^*(x^*) \quad \forall (x,x^*) \in X \times X^*.$$

We need to prove is the result for $n = k+1$. By Proposition 4.43 for all (x,x^*) in $X \times X^*$ we have

$$\begin{aligned}\varphi_{F,k+1}(x,x^*) &= \sup_{y \in X} \{\varphi_{F,k}(y,x^*) + F(y,x)\} \\ &= \sup_{y \in \operatorname{dom} f} \{\varphi_{F,k}(y,x^*) + F(y,x)\}\end{aligned}$$

Thus

$$\begin{aligned}\varphi_{F,k+1}(x,x^*) &= \sup_{y\in \operatorname{dom} f}\{(f(y)+f^*(x^*))+(f(x)-f(y))\}\\ &= f(x)+f^*(x^*).\end{aligned}$$

Also by taking the supremum over n, from the definition we deduce that

$$\varphi_{F,\infty}(x,x^*) = f(x)+f^*(x^*).$$

The proof of second statement is also by induction.
Base case: Consider the case $n=2$. This case, also is proved in Proposition 4.13.
Induction step: Suppose the result is true for $n=k$. In other words, we have

$$\varphi_{G_{A^F,k}}(x,x^*) = \mathcal{F}_{\partial f,k}(x,x^*).$$

Again by using Propositions 4.43, 4.13, and Corollary 4.44 we have

$$\begin{aligned}\varphi_{G_{A^F,k+1}}(x,x^*) &= \sup_{y\in X}\left\{\varphi_{G_{A^F,k}}(y,x^*)+G_{\partial f}(y,x)\right\}\\ &= \sup_{y\in X}\{\mathcal{F}_{\partial f,k}(y,x^*)+G_{\partial f}(y,x)\}\\ &= \mathcal{F}_{\partial f,k+1}(x,x^*).\end{aligned}$$

We are done. ∎

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