Dynamical Systems Approach in Scalar Field Cosmologies

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ΜΕΘΟΔΟΙ ΔΥΝΑΜΙΚΩΝ ΣΥΣΤΗΜΑΤΩΝ ΣΕ ΣΥΓΧΡΟΝΑ ΠΡΟΒΛΗΜΑΤΑ ΜΑΘΗΜΑΤΙΚΗΣ ΚΟΣΜΟΛΟΓΙΑΣ

της

Κοραλίας Τζαννή

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Abstract

Dynamical Systems Approach in Scalar Field Cosmologies

In this thesis flat Friedmann-Lemaître-Robertson-Walker models with a perfect fluid matter source and a scalar field non-minimally coupled to matter are considered. In the first part of the thesis we study the case of double exponential potentials. It is shown that the scalar field almost always diverges to infinity. We find conditions on the parameter space such that the model is able to provide an acceptable cosmological history of our Universe, that is, a transient matter era followed by an accelerating future attractor. It is found that only a very weak coupling can lead to a viable cosmology. We study in the Einstein frame, the cosmological viability of the asymptotic form of a class of $f(R)$ theories predicting acceleration. Double exponential potentials could take negative values with respect to the parameters. We prove rigorously that a general class of bounded from above potentials which fall to minus infinity as the field goes to minus infinity, forces the Hubble function to diverge to $-\infty$ in a finite time.

In the second part of the thesis, we study more systematically scalar fields with potentials taking negative values. We prove that the Hubble function generically diverges to $-\infty$ in a finite time, except in case the potential exhibits a flat plateau at infinity, tending to zero from below. We find conditions on the parameter space which may give rise to ever expanding or collapsing Universes. To illustrate our results we revisit the double exponential potential.

Περίληψη

Μέθοδοι Δυναμικών Συστημάτων σε

Σύγχρονα Προβλήματα Μαθηματικής Κοσμολογίας

Στη διατριβή αυτή μελετούμε επίπεδα, ομογενή και ισότροπα Σύμπαντα που περιέχουν ως ύλη ένα τέλειο ρευστό και ένα βαθμωτό πεδίο συζευγμένο με την ύλη. Στο πρώτο μέρος μελετούμε την περίπτωση που η συνάρτηση δυναμικού είναι άθροισμα δύο εκθετικών. Αποδεικνύεται ότι το βαθμωτό πεδίο σχεδόν πάντα τείνει στο άπειρο. Βρίσκουμε τα διαστήματα των παραμέτρων για τα οποία το μοντέλο περιγράφει μία κοσμολογικά αποδεκτή ιστορία. Κοσμολογικά αποδεκτή ιστορία θεωρείται η λύση του δυναμικού συστήματος που περιλαμβάνει μια μεταβατική εποχή δόμησης ύλης η οποία ακολουθείται από μια επιταχυνόμενη εποχή που αντιστοιχεί σε ευσταθή λύση του συστήματος. Αποδεικνύεται ότι για να έχουμε βιώσιμη κοσμολογική ιστορία, το πεδίο πρέπει να είναι πολύ ασθενώς συζευγμένο με την ύλη. Συναρτήσεις με διπλά εκθετικά δυναμικά που παρουσιάζουν ολικό θετικό μέγιστο, τείνουν στο $-\infty$ όταν $\phi \to -\infty$ και στο 0^{+} όταν $\phi \to \infty$ μπορούν να προκύψουν και ως ασυμπτωτική μορφή δυναμικών θεωριών $f(R)$ που προβλέπουν επιτάχυνση. Τέτοια δυναμικά μπορούν να παίρνουν και αρνητικές τιμές.

Στο δεύτερο μέρος αποδεικνύουμε ότι για την γενική κατηγορία αυτών των δυναμικών, η συνάρτηση Hubble αποκλίνει σε πεπερασμένο χρόνο. Αποδεικνύουμε ότι διάφορες ομάδες δυναμικών που παίρνουν αρνητικές τιμές οδηγούν σε καταρρέοντα Σύμπαντα, εκτός από κάποια δυναμικά που τείνουν στο 0^{-} καθώς $\phi \to \infty$.

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- R. Giambò, J. Miritzis and K. Tzanni (2015) *Negative potentials and collapsing universes* Classical and Quantum Gravity **32**, 035009.
- R. Giambò, J. Miritzis and K. Tzanni (2015) *Negative potentials and collapsing universes II* Classical and Quantum Gravity **32**, 5017.

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"Good, he did not have enough imagination
to become a mathematician".

—Hilbert's response upon hearing
that one of his students
had dropped out to study poetry.

Chapter 1

Outline and Conventions

1.1 Introduction and Outline

Cosmology is the study of the origin, the evolution, the structure and the fate of our Universe. The basis of physical cosmology is the cosmological principle, an assumption justified by observations, which states that the Universe, as far as we can detect, is homogeneous and isotropic at sufficiently large scales. This simplifies the study of the large scale structure and dynamics of the Universe and allows the study of the Universe as an entity. The cosmological principle implies that there is a universal factor, the scale factor a, that relates the distance between two objects, e.g. a pair of galaxy clusters, at any arbitrary time t. The scale factor, $a = a(t)$, is a dimensionless function of time only, the exact form of which remains one of the major current problems of cosmology.

Observational data from 1998, [1,2], indicate that we live in a Universe that expands with an accelerated expansion rate. The visible Universe consists of stars, galaxies, interstellar and intergalactic gas, mentioned collectively as ordinary matter or just matter. Ordinary matter would rather cause the Universe to either collapse or to expand with a decelerated rate. There must be a force that does more than simply counter the mutual gravitational attraction of the galaxies and forces them to run away from each other [3, 4]. A variety of suggestions have been proposed the past two decades. These proposals can be roughly grouped into two categories [5–8]. First, a "fluid" of unknown nature, the dark energy, is responsible for the observed accelerating expansion of the Universe. Alternatively a modification of General Relativity at cosmological distance scales is required.

The thesis is organised as follows:

For the convenience of the reader, in Chapter 2 we present the basics of Friedman–Lemaître–Robertson–Walker (FLRW) Universes in the context of scalar field cosmologies. We briefly review the cosmological constant, scalar fields and modified gravity models.

In Chapter 3 we present the results of our study on scalar fields with a double exponential potential, $V(\phi) = V_1 e^{-\alpha\phi} + V_2 e^{-\beta\phi}$. We assume FLRW models with the scalar field non-minimally coupled to a perfect fluid. We show that the scalar field almost always diverges to infinity. We show that double exponential potentials arise as an asymptotic form of a class of $f(R)$ theories predicting acceleration. We define the acceptable cosmological history of the Universe as a trajectory of the dynamical system that passes near a point that represents a transient matter epoch and lands on a stable point that represents the accelerated expansion. We present conditions on the parameters under which the model provides an acceptable cosmological history of the Universe. We discuss the role of the coupling constant and prove that only a vanishing coupling constant can lead to a viable cosmology. Double exponential potentials may take negative values, In particular there is a class of double exponential potentials with potential function that exhibits a global positive maximum, tends to zero from above as $\phi \to +\infty$ and falls to $-\infty$ as $\phi \to -\infty$.

In Chapter 4 we complete the analysis, started in Chapter 3, of the class of potentials taking negative values. We prove rigorously that initially expanding Universes, eventually collapse in a finite time. These results apply to both cases: scalar fields coupled to matter and uncoupled models studied so far in the literature. We extend our analysis to other forms of potentials that take negative values. We classify these potentials in five main classes and we prove that the evolution almost always forces the Hubble function to diverge to $-\infty$ in a finite time. Only potentials which tend to zero from below may under certain conditions give rise to ever expanding cosmologies.

Some useful formulas are presented in the Appendices. In Appendix A we present general formulas related to conformal transformation used in theories of gravity. In Appendix B we derive in detail the field equations in $f(R)$ gravity. In Appendix C we provide terminology and theorems of the theory of dynamical systems we use in the present study.

1.2 Conventions and notations

In the whole thesis, we choose units in which $c = \hbar = 8\pi G = 1$, where c is the speed of light, $\hbar$ the reduced Plank's constant and G the gravitational constant. Greek indices run from 0 to 3 whereas latin indices run from 1 to 3; we follow the sign conventions, [9]. The Christoffel symbols are defined as

$$\Gamma^{\sigma}_{\mu\nu} = \frac{1}{2} g^{\sigma\rho} \left(\partial_\mu g_{\nu\rho} + \partial_\nu g_{\rho\mu} - \partial_\rho g_{\mu\nu} \right), \tag{1.1}$$

where $\partial_\mu = \partial / \partial x^\mu$. The Riemann tensor is

$$R^{\rho}{}_{\sigma\mu\nu} = \partial_\mu \Gamma^{\rho}_{\nu\sigma} - \partial_\nu \Gamma^{\rho}_{\mu\sigma} + \Gamma^{\rho}_{\mu\lambda}\Gamma^{\lambda}_{\nu\sigma} - \Gamma^{\rho}_{\nu\lambda}\Gamma^{\lambda}_{\mu\sigma}, \tag{1.2}$$

and the Ricci tensor is

$$R_{\mu\nu} = R^{\lambda}{}_{\mu\lambda\nu} = \partial_\lambda \Gamma^{\lambda}_{\mu\nu} - \partial_\nu \Gamma^{\lambda}_{\mu\lambda} + \Gamma^{\lambda}_{\mu\nu}\Gamma^{\sigma}_{\lambda\sigma} - \Gamma^{\lambda}_{\mu\sigma}\Gamma^{\sigma}_{\lambda\nu}. \tag{1.3}$$

The trace of the Ricci tensor is the Ricci scalar

$$R = R^{\mu}{}_{\mu} = g^{\mu\nu} R_{\mu\nu}. \tag{1.4}$$

The Einstein tensor is defined by

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu}. \tag{1.5}$$

The Friedmann–Lemaître–Robertson–Walker (FLRW) metric, takes the form

$$ds^2 = g_{\mu\nu} dx^{\mu} dx^{\nu} = -dt^2 + a^2 \left(\frac{dr^2}{1 - kr^2} + r^2 \left(d\theta^2 + \sin^2\theta d\phi^2 \right) \right), \tag{1.6}$$

where we have used pseudo-spherical coordinates (r, θ, ϕ), a is the scale factor and $k = -1, 0, 1$ is the spatial curvature representing an open, flat or closed Universe respectively.

The Christoffel symbols as well as the components of the Ricci and the Einstein tensors for the metric 1.6 are given in the Appendix A.

Chapter 2

Introduction to Scalar Field Cosmologies

In this introductory chapter we briefly review Scalar Field Cosmologies. For textbooks in Relativity and Cosmology we refer for example to [9–18].

2.1 FLRW Universes and Field Equations

Hilbert derived the Einstein equations in vacuum by varying the so-called Einstein-Hilbert action with respect to the metric tensor. The Einstein-Hilbert action is

$$S_{\mathrm{EH}} = \frac{1}{2}\int d^4x\sqrt{-g}R,$$

where g is the determinant of the metric tensor $g_{\mu\nu}$. In the presence of matter, by adding a matter action, $S_{\mathrm{M}} = S_{\mathrm{M}}[g_{\mu\nu}, \Psi]$, the total action is given by the Einstein-Hilbert term plus the Langrangian density of the matter, $\mathcal{L}_{\mathrm{M}}$, which contains all matter fields, Ψ, collectively. The total action is

$$S = S_{\mathrm{EH}} + S_{\mathrm{M}} = \int d^4x\sqrt{-g}\left(\frac{1}{2}R + \mathcal{L}_{\mathrm{M}}\right).$$

In order to construct a cosmological model compatible to the general belief of the time, namely that the Universe is static, Einstein introduced the cosmological constant, Λ

$$S = S_{\Lambda} + S_{\rm M} = \int d^4x\sqrt{-g}\left(\frac{1}{2}(R-2\Lambda)+\mathcal{L}_{\rm M}\right).$$

The Einstein field equations derived from this action are

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} + \Lambda g_{\mu\nu} = T_{\mu\nu}, \tag{2.1}$$

where the tensor $T_{\mu\nu}$ describes the distribution of energy, momentum and stresses associated to any force field.

We assume that the matter content of the Universe is described by a perfect fluid with an energy momentum tensor of the form

$$T_{\mu\nu} = (\rho + p)\, u_{\mu}u_{\nu} + pg_{\mu\nu}, \tag{2.2}$$

where u^{μ} denotes the four-velocity of a comoving observer, ρ is the energy density and p is the pressure in the rest frame of the fluid. Any fluid in FRLW Universe has to be a perfect fluid since isotropy imposes zero viscosity. Note that both ρ and p do not depend on the spatial coordinates, but are functions of time t, only. That is, only time-depended mathematical quantities exist in isotropic and homogeneous Universes. Density and pressure are linearly related by the equation of state

$$p = (\gamma - 1)\rho, \tag{2.3}$$

where γ is a parameter taking values in the integral $[0, 2]$. For dust, $\gamma = 1$, for a relativistic fluid, radiation, $\gamma = 4/3$ and for the cosmological constant, $\gamma = 0$. In general, the energy density might be the sum of more than one components, which evolve differently with respect to a. Denoting by ρ and

p the corresponding sums of density and pressure respectively, we have the effective equation of state

$$p = w_{\text{eff}}\rho,$$

where w_{eff} is called the effective equation of state parameter.

The Einstein's equations (2.1) in the FLRW metric provide the cosmological equations, as follows. The time-time component, $\mu\nu = 00$, is the Friedmann equation (see Appendix A),

$$\left(\frac{\dot{a}}{a}\right)^2 + \frac{k}{a^2} = \frac{1}{3}\rho + \frac{\Lambda}{3}, \tag{2.4}$$

and the ii components give the acceleration equation

$$2\frac{\ddot{a}}{a} = -\left(\frac{\dot{a}}{a}\right)^2 - p - \frac{k}{a^2} + \Lambda, \tag{2.5}$$

where an over-dot denotes differentiation with respect to the time t. Using Eq. (2.4) in Eq. (2.5) to eliminate the term $(\dot{a}/a)^2$, we derive the Raychaudhuri equation

$$\frac{\ddot{a}}{a} = -\frac{1}{6}(\rho + 3p) + \frac{\Lambda}{3}. \tag{2.6}$$

When a positive cosmological constant is present, the Universe is accelerated for

$$\Lambda > \frac{1}{2}(\rho + 3p).$$

When $\Lambda = 0$, combining Eqs (2.6) and (2.3), we get

$$\frac{\ddot{a}}{a} = -\frac{1 + 3w_{\text{eff}}}{6}\rho. \tag{2.7}$$

From the above we see that for $\Lambda = 0$, the acceleration of the Universe depends only on the matter constituents, and the Universe is accelerating

($\ddot{a} > 0$), when $1 + 3w_{\rm eff} < 0$. Then the condition for acceleration is

$$w_{\rm eff} \leq -\frac{1}{3}. \tag{2.8}$$

Note that the condition for acceleration implies that the pressure has the peculiar property $p < 0$.

From the definition (2.2) and the conservation of the energy momentum tensor, $\nabla_\mu T^{\mu\nu} = 0$, we get the conservation equation

$$\dot{\rho} = -3\frac{\dot{a}}{a}\left(\rho + p\right). \tag{2.9}$$

Using (2.3), one obtains

$$\rho \sim a^{-3\gamma}. \tag{2.10}$$

We define the density parameter Ω, as

$$\Omega = \frac{\rho}{\rho_{\rm c}}, \tag{2.11}$$

where ρ is the observed density and $\rho_{\rm c}$ is the critical density for which the spacial geometry of the Universe is Euclidean. The energy density for each of the different components also denoted by $\Omega_i \equiv \rho_i/\rho_c$ and the curvature density by $\Omega_k \equiv \rho_k/\rho_c$, hence the total energy density can be written as

$$\sum_i \Omega_i + \Omega_k = \Omega. \tag{2.12}$$

Consequently, the geometry of the spacetime is spherical if $\Omega > 1$, hyperbolic if $\Omega < 1$, and flat if $\Omega = 1$.

Observations [19] indicate that the value of Ω is currently close to 1, i.e., we live in a spatially flat Euclidean Universe.

It is straightforward to obtain the function $a(t)$ for the case of $\gamma = 1$ with respect to the different values of k, as shown in Fig. 2.1. That is, in

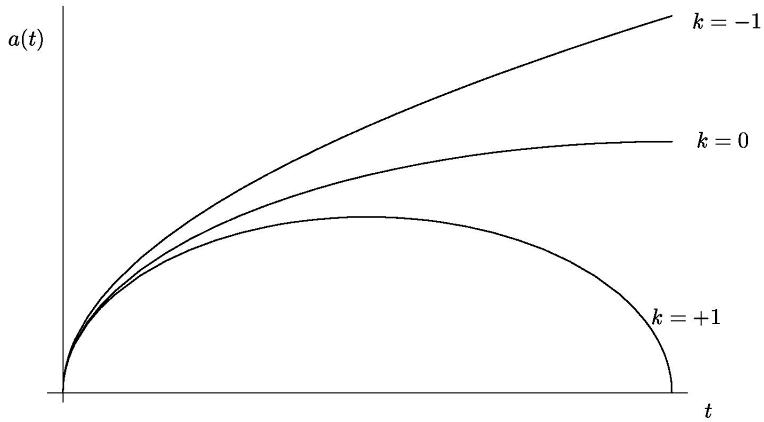


Figure 2.1: The scale factor in the case of dust.

the case of dust, $w_{\text{eff}} = \gamma - 1 = 0$, the condition for acceleration, (2.8), is not satisfied. A similar behaviour of the scale factor is true for a Universe filled with radiation, $w_{\text{eff}} = \gamma - 1 = 1/3$. Therefore a Universe filled with ordinary matter or radiation cannot be accelerated.

As mentioned in Chapter 1, there are two major approaches to obtain acceleration: (a) modify the right hand side of Eqs (2.4) and (2.6) (dark energy [20–23]), (b) modify the left hand side of (2.4) and (2.6) (modified gravity).

2.1.1 Dark Energy: The cosmological constant

The simplest model of dark energy is provided by the cosmological constant. From (2.4) and (2.5), we can see the contribution of Λ as a fluid of constant energy with an equation of state,

$$p_\Lambda = -\rho_\Lambda := -\Lambda,$$

i.e.,

$$w_\Lambda = -1.$$

Since $\rho_\Lambda > 0$, the cosmological constant can be thought as a matter component with positive constant energy and negative constant pressure. We define the Hubble function as $H := \dot{a}/a$. When the Λ term completely dominates the evolution of a flat Universe, then from (2.4)

$$H = \sqrt{\frac{\Lambda}{3}},$$

and by integrating we obtain

$$a(t) \sim e^{\frac{\Lambda}{3}t},$$

which describes the de Sitter Universe. de Sitter solution although satisfies the condition of acceleration, $\ddot{a} > 0$, can not be used as a realistic model of our Universe since we have neglected contributions from radiation and matter completely. However, it can be thought as an approximation of the evolution of the Universe at late times.

Although simple as an idea, the cosmological constant suffers from some fundamental problems. We briefly mention the cosmological constant problem and the coincidence problem. The cosmological constant problem is the discrepancy of more than 100 orders of magnitude between the small value of the cosmological constant measured and the theoretical value expected from quantum field theory. The coincidence problem is that the observed, extremely small value of the cosmological constant is the exact one needed in order to have a sufficient matter domination era. A slightly bigger value of Λ forces a direct transition from radiation to dark matter domination preventing the formation of matter. For the cosmological constant and problems regarding it, see [3, 5, 24–27].

2.1.2 Dark Energy: Quintessence

Another popular candidate of dark energy is scalar fields with positive potentials, [28,29]. Scalar fields have been already used in inflationary models, in which the scale factor undergoes extremely rapid quasi-exponential expansion, [30,31]. In the standard inflationary scenario [30–32], the Universe is dominated by a real scalar field ϕ, homogeneously distributed in space, with a potential function $V(\phi)$. The field that drives this expansion is the so called inflaton. The action is described by

$$S = \int d^4x\sqrt{-g}\left(\frac{R}{2} - \frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - V(\phi)\right) + S_{\mathrm{m}}, \tag{2.13}$$

where S_{m} is the action of ordinary matter. Hence the Lagrangian density of a scalar field is

$$\mathcal{L}_\phi = -\frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - V(\phi). \tag{2.14}$$

By varying (2.13) with respect to $g_{\mu\nu}$, we obtain the field equations

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = \left(T^{(\mathrm{m})}_{\mu\nu} + T^{(\phi)}_{\mu\nu}\right), \tag{2.15}$$

where

$$T^{(\phi)}_{\mu\nu} := \partial_\mu\phi\partial_\nu\phi - g_{\mu\nu}\left(\frac{1}{2}g^{\rho\sigma}\partial_\rho\phi\partial_\sigma\phi + V(\phi)\right), \tag{2.16}$$

is the energy- momentum tensor of the scalar field. Varying (2.13) with respect to ϕ, we obtain the equation of motion of the scalar field

$$\Box\phi - V'(\phi) = 0, \tag{2.17}$$

where the D'Alembertian is defined by $\Box := \nabla^\mu\nabla_\mu$, ∇_μ is the covariant derivative and $V'(\phi) = \partial V(\phi)/\partial\phi$.

In the FLRW metric, the field equations for a flat Universe become

$$3H^2 = \rho + \frac{1}{2}\dot{\phi}^2 + V(\phi), \tag{2.18}$$
$$2\dot{H} + 3H^2 = -p - \frac{1}{2}\dot{\phi}^2 + V(\phi), \tag{2.19}$$

while the equation of motion (2.17) takes the form

$$\ddot{\phi} + 3H\dot{\phi} + V'(\phi) = 0. \tag{2.20}$$

From (2.18) and (2.19) we can see that the contribution of the scalar field ϕ to the energy content of the Universe takes the perfect fluid form, $p_\phi = w_\phi \rho_\phi$, with energy density and pressure given by

$$\rho_\phi = \frac{1}{2}\dot{\phi}^2 + V(\phi), \tag{2.21}$$
$$p_\phi = \frac{1}{2}\dot{\phi}^2 - V(\phi), \tag{2.22}$$

respectively. The equation of state of the scalar field is then given by,

$$w_\phi = \frac{\dot{\phi}^2 - 2V(\phi)}{\dot{\phi}^2 + 2V(\phi)}. \tag{2.23}$$

Eq. (2.23) implies that slowly evolving scalar fields, $\dot{\phi} \to 0$, are hardly distinguished from the cosmological constant. From another point of view, the cosmological constant can be thought of as a special case of a constant scalar field. Models based on scalar fields to explain the late time cosmic acceleration are collectively called quintessence models. For a review see [33].

Scalar fields arise naturally in alternative theories of gravity which aim to extend General Relativity, as for example, higher order gravity theories [34], scalar-tensor theories with multiple scalar fields [35, 36] and string cosmologies [37, 38].

2.1.3 Modified gravity dark energy models

The most straightforward generalisation of General Relativity is obtained by replacing the term R in the action by a smooth, otherwise arbitrary function $f(R)$. Under certain conditions [39,40], these models may be cosmologically viable. There is a vast amount of studies in $f(R)$ models in the literature used for explaining acceleration, see for example [7, 41–44] and references therein; for discussions and reviews see [45–53].

The action of an $f(R)$ theory is given by

$$S = \int d^4x\sqrt{-g}f(R) + S_{\mathrm{m}}(g_{\mu\nu}, \Psi),$$

where Ψ denotes all matter fields collectively. Varying the above action with respect to the metric tensor, we obtain (see Appendix B)

$$f'(R)R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}f(R) + g_{\mu\nu}\Box f'(R) - \nabla_\mu\nabla_\nu f'(R) = T^{(\mathrm{m})}_{\mu\nu}.$$

The first modified dark energy models proposed in the literature, were constructed by adding a term $1/R$ to R in the Einstein–Hilbert action, [7, 54]. The term $1/R$ dominates as the Universe expands and produces acceleration [55]. However these models violate the local gravity constraints and become nonviable as general gravity theories, [8, 56].

Other modifications of General Relativity are scalar-tensor theories which include the original Brans-Dicke theory, see for example [57].

2.1.4 Acceleration in the context of General Relativity

There are also attempts to explain acceleration in the context of General Relativity. Some authors [58,59] attribute the observed, "apparent" as they claim, acceleration to the inhomogeneities of the Universe. They claim that "the accelerated expansion of the Universe is not an observed phenomenon, but an element of interpretation of observations, forced upon us by the FLRW models", see [60].

The authors of the investigations [61, 62], explored the possibility that the dark energy needed to accelerate the expansion of the Universe is attributed to the energy of the cosmic fluid internal motions. As the authors state, "in this framework, the Universe is filled with a perfect fluid, consisting mainly of self-interacting dark matter, the volume elements of which perform hydrodynamic flows".

2.2 Scalar-Tensor theories of Gravity

In the inflationary scenarios, there exists a period of slow-roll evolution of a scalar field, the inflaton, during which, its potential energy drives the Universe into expansion with an accelerated rate. Quintessential dark energy models provide the simplest mechanism to obtain accelerated expansion of the Universe within General Relativity. These models are described by an ordinary scalar field minimally coupled to gravity. For example in [63–65] are studied models that contain both a perfect fluid of ordinary matter and a scalar field with an exponential potential. It is proved that in these models Ω_ϕ is a constant fraction of the total density Ω, for that reason they are called "scaling" cosmologies. Inclusion of non-minimal couplings in scalar field cosmology is important to be considered although it increases the mathematical difficulty of the analysis, [66,67]. In fact, the introduction of non-minimal coupling is not a matter of taste [68]. In the string effective action, the dilaton field is generally coupled to matter in the Einstein frame [37]. In scalar-tensor theories of gravity [66, 67, 69], the action in the Einstein frame takes the form

$$S = \int d^4x\sqrt{-g}\left\{R - \left[(\partial\phi)^2 + 2V\left(\phi\right)\right] + 2\chi^{-2}\mathcal{L}_{\mathrm{m}}\left(\widetilde{g}_{\mu\nu}, \Psi\right)\right\}, \qquad (2.24)$$

with

$$\widetilde{g}_{\mu\nu} = \chi^{-1} g_{\mu\nu},$$

where $\chi = \chi(\phi)$ is the coupling function and matter fields are collectively denoted by Ψ. In particular, as mentioned before, for higher order gravity theories derived from Lagrangians of the form

$$f\left(\widetilde{R}\right) + 2\mathcal{L}_{\mathrm{m}}\left(\widetilde{g}_{\mu\nu}, \Psi\right), \tag{2.25}$$

it is well known [34] that under the conformal transformation

$$g_{\mu\nu} = f'\left(\widetilde{R}\right)\widetilde{g}_{\mu\nu},$$

the field equations reduce to the Einstein field equations with a scalar field ϕ as an additional matter source. The conformal equivalence can be formally obtained by conformally transforming the Lagrangian (2.25) and the resulting action becomes [70]

$$S = \int d^4x\sqrt{-g}\left\{R - \left[(\partial\phi)^2 + 2V(\phi)\right] + 2e^{-2\sqrt{2/3}\phi}\mathcal{L}_{\mathrm{m}}\left(e^{-\sqrt{2/3}\phi}g_{\mu\nu}, \Psi\right)\right\}.$$

Therefore the Lagrangian of HOG theories is a particular case of the general scalar-tensor Lagrangian with $\chi(\phi) = e^{\sqrt{2/3}\phi}$, in equation (2.24). Non-minimal coupling occurs also in models of chameleon gravity [71, 72]

$$S = \int d^4x\sqrt{-g}\left\{R - \left[(\partial\phi)^2 + 2V(\phi)\right] + 2\mathcal{L}_{\mathrm{m}}\left(\widetilde{g}_{\mu\nu}, \Psi\right)\right\},$$

with

$$\widetilde{g}_{\mu\nu} = e^{2\beta\phi} g_{\mu\nu},$$

where β is a coupling constant. The same form of coupling has been proposed in models of the so called coupled quintessence [73]. For more general couplings see also [74], and [75] for a generalisation involving a scalar field coupled both to matter and a vector field. For the consequences of a phantom field minimally coupled to gravity see for example [76].

In general, each component of the total energy momentum tensor may not be conserved. However, the general interactions between the scalar field and matter have to satisfy

$$\nabla^{\mu} T^{(\phi)}_{\mu\nu} = -Q_{\nu}, \tag{2.26}$$

$$\nabla^{\mu} T^{(\mathrm{m})}_{\mu\nu} = Q_{\nu}, \tag{2.27}$$

where $T^{(\phi)}_{\mu\nu}, T^{(\mathrm{m})}_{\mu\nu}$ are the energy momentum tensors of the scalar field ϕ and matter respectively. The term Q_{ν}, called the interaction term, denotes the energy momentum exchanged between the two fluids. For $Q_{\nu} = 0$, there is no interaction between the two fluids. The form of Q_{ν} is specified by the physical properties of the fluids and the coupling terms occurring in the Lagrangians.

Chapter 3

Coupled Dark Energy with Double Exponential Potentials

In this chapter we study flat FLRW models with a perfect fluid matter source and a scalar field non-minimally coupled to matter having a double exponential potential of the form

$$V(\phi) = V_1 e^{-\alpha\phi} + V_2 e^{-\beta\phi}, \tag{3.1}$$

where α, β are positive constants and V_1, V_2 are constants of arbitrary sign.

The chapter is organised as follows. In the next section we construct the dynamical system. In Section 3.2 we show some preliminary results for non negative potentials satisfying certain assumptions in an initially expanding Universe, the scalar field almost always diverges as $t \to \infty$. In Section 3.3 we present the double exponential potential and examine its different forms regarding the signs of the parameters V_1, V_2. In Section 3.4 we use expansion-normalised variables to write the system as a polynomial three-dimensional system. We study the equilibrium points and analyse their properties. In Section 3.5 we set the conditions on the parameter space, which allow for an acceptable cosmological history of our Universe: a transient matter era followed by an accelerating future attractor. In Section 3.6 we examine the asymptotic form of the potential in the Einstein frame of a popular class of $f(R)$ theories predicting acceleration.

3.1 Constructing the Dynamical System

Ordinary matter is described by a perfect fluid with equation of state

$$p = (\gamma - 1)\rho,$$

where γ is a constant value taking values in the interval $(0, 2)$. Varying the action (2.24) with respect to the metric, we obtain the field equations

$$G_{\mu\nu} = T^{(\phi)}_{\mu\nu} + T^{(\mathrm{m})}_{\mu\nu}, \tag{3.2}$$

where $T^{(\phi)}_{\mu\nu}$ is the scalar field energy momentum tensor and $T^{(\mathrm{m})}_{\mu\nu}$ is the matter energy momentum tensor. The Bianchi identities imply that the total energy-momentum tensor is conserved and therefore there is an energy exchange between the scalar field and ordinary matter. In all the above examples, the conservation of their sum is provided by the equations (compare to [73])

$$\nabla^{\mu} T^{(\mathrm{m})}_{\mu\nu}(g, \Psi) = Q T^{(\mathrm{m})} \nabla_{\nu}\phi, \qquad \nabla^{\mu} T^{(\phi)}_{\mu\nu}(g, \phi) = -Q T^{(\mathrm{m})} \nabla_{\nu}\phi,$$

where $Q := d\ln\chi/d\phi$, depends in general on ϕ and $T^{(\mathrm{m})}$ is the trace of the matter energy-momentum tensor, i.e.,

$$T^{(\mathrm{m})} = g^{\mu\nu} T^{(\mathrm{m})}_{\mu\nu}(g, \Psi).$$

Variation of S with respect to ϕ yields the equation of motion of the scalar field

$$\Box\phi - V'(\phi) = -QT^{(\mathrm{m})}. \tag{3.3}$$

For homogeneous and isotropic flat spacetimes the field equations (3.2) and (3.3), reduce to the Friedmann equation

$$3H^2 = \rho + \frac{1}{2}\dot{\phi}^2 + V(\phi); \tag{3.4}$$

the Raychaudhuri equation

$$\dot{H} = -\frac{1}{2}\dot{\phi}^2 - \frac{\gamma}{2}\rho; \tag{3.5}$$

the equation of motion of the scalar field

$$\ddot{\phi} + 3H\dot{\phi} + V'(\phi) = \frac{4-3\gamma}{2}Q\rho; \tag{3.6}$$

and the conservation equation

$$\dot{\rho} + 3\gamma\rho H = -\frac{4-3\gamma}{2}Q\rho\dot{\phi}. \tag{3.7}$$

Note that the presence of the trace of the energy-momentum tensor in the right-hand side of equations (3.6) and (3.7), implies that energy exchange between radiation and the scalar field does not exist. Of course, interaction between radiation and the scalar field is present during the warm inflation epoch. However, as stated before, we are interested in the late time evolution of the Universe, therefore radiation shall be neglected. As can be seen by the conservation equation, (3.7), the set, $\rho > 0$, is invariant under the flow of equations (3.5)-(3.7). Thus if we initially assume that for some t_0, $\rho(t_0)$ is nonzero, it shall remain nonzero, $\forall t > t_0$. Furthermore, if $\rho(t_0)$ is initially positive, it remains positive for the rest of the history. This property is not satisfied if we assume arbitrary interaction terms, see Ref. [77].

Setting $\dot{\phi} = y$, we write Eqs. (3.5)–(3.7) as an autonomous dynamical system

$$\dot{\phi} = y, \tag{3.8}$$

$$\dot{y} = -3Hy - V'(\phi) + \frac{4-3\gamma}{2} Q\rho, \tag{3.9}$$

$$\dot{\rho} = -\rho \left(3\gamma H + \frac{4-3\gamma}{2} Qy \right), \tag{3.10}$$

$$\dot{H} = -\frac{1}{2} y^2 - \frac{\gamma}{2} \rho, \tag{3.11}$$

subject to the constraint

$$3H^2 = \frac{1}{2} y^2 + V(\phi) + \rho. \tag{3.12}$$

We recall the remarkable property of the Einstein equations that, if Eq. (3.12) is satisfied at some initial time, then it is satisfied throughout the evolution. Also, in most quintessence models, the coupling coefficient $Q = Q(\phi)$, is postulated to be a positive constant, see for example [73].

Before attacking our main problem, some preliminary results are needed.

3.2 Non-negative Potentials

In this section we assume that the potential function of the scalar field is non-negative with either a minimum $V_{\min} \geq 0$ or bounded from below by a non-negative value. We will need the following Lemma, proven in [78].

Lemma 1 *Suppose a function $f \in C^1$, such that $f > 0$, the integral $\int_{t_0}^{\infty} f(t)dt$ converges and there exists a positive constant k such that $|\dot{f}| < k$, then*

$$\lim_{t \to \infty} f(t) = 0.$$

3.2.1 Potentials having non-negative minima

In this section we generalise Propositions proven in [78, 79], for our case where an interaction term between the two matter components is present.

Assumption 1 *We assume that $V(\phi) \in C^2$ is a potential function such that:*

1. *$V \geq 0$,*
2. *$V(\phi) = 0$ holds only for $\phi = 0$,*
3. *If $A \subset \mathbb{R}$ is such that V is bounded on A, then V' is bounded on A.*

Theorem 1 *Let $V(\phi) \in C^2$ be a potential function satisfying the Assumption 1. Then*

$$\lim_{t\to\infty} \rho = 0 = \lim_{t\to\infty} y.$$

If additionally $V'(\phi) > 0$ for $\phi > 0$ and $V'(\phi) < 0$ for $\phi < 0$,

then

$$\lim_{t\to+\infty} \phi = +\infty, \text{or } 0 \text{ or } -\infty.$$

Proof. From (3.10) it follows that $\rho = 0$ is an invariant set of the system. Hence, if initially $\rho > 0$, it has to remain positive. Consider a trajectory with $H > 0$ at some time $t = t_0$. From (3.11), $H(t)$ is decreasing and positive, it follows that the limit $\lim_{t\to\infty} H(t)$ exists and is a non-negative number, say H_∞. The monotonicity of H also indicates that for all $t \geq t_0$, $H(t) \leq H(t_0)$. Then from the constraint (3.12), we deduce that each of the terms ρ, $\frac{1}{2}y^2$, and V is bounded by $3H^2(t_0)$. Let $A = \{\phi : V(\phi) \leq 3H^2(t_0)\}$. Then a trajectory passing through any point $\mathbf{x}_0 = (\phi, y, \rho, H)$ with $H(t_0) > 0$, is such that ϕ stays inside A. From (3.11) we get

$$\frac{1}{2}\int_{t_0}^{+\infty} \left(y^2 + \gamma\rho\right) dt = H\left(t_0\right) - H_\infty.$$

Therefore

$$\int_{t_0}^{+\infty} \left(y^2(t) + \gamma\rho(t)\right) dt < +\infty.$$

We now prove that the derivative with respect to t of the quantity $y^2(t) + \gamma\rho(t)$ is bounded. Indeed

$$\begin{aligned}\frac{d}{dt}\left(y^2 + \gamma\rho\right) &= -6Hy^2 - 2yV'(\phi) - 3\gamma^2\rho H + \frac{4-3\gamma}{2}Q(2-\gamma)\rho y \\ &\leq -2yV'(\phi) + \frac{4-3\gamma}{2}Q(2-\gamma)\rho y.\end{aligned}$$

As we already remarked, y and ρ are bounded; also, by our assumption on V, $V'(\phi)$ is bounded. We conclude that the derivative of the function $y^2+\gamma\rho$ is bounded from above and therefore, Lemma 1 applies and $\lim_{t\to\infty} y(t)^2 = 0$ and $\lim_{t\to\infty}\rho(t) = 0$.

If additionally $V'(\phi) > 0$ for $\phi > 0$ and $V'(\phi) < 0$ for $\phi < 0$, then if $H_\infty = 0$, the constraint (3.12) implies that $\lim_{t\to\infty} V(\phi) = 0$, and by our assumption on the potential $\lim_{t\to\infty}\phi = 0$. Suppose now that $H_\infty > 0$. From the constraint $\lim_{t\to\infty} V(\phi(t)) = 3H_\infty^2$. Thus there exists t_1 such that $V(\phi) > \frac{3}{2}H_\infty^2$, for all $t > t_1$. Hence, since $V(0) = 0$, $\phi \neq 0$, for some $t > t_1$. So, suppose that $\phi > 0$, for all $t > t_1$. The monotonicity of the potential $V(\phi)$, implies $\lim_{t\to\infty} V(\phi(t)) = 3H_\infty^2 \leq \lim_{\phi\to\infty} V(\phi)$.

i. If $\lim_{t\to\infty} V(\phi(t)) = \lim_{\phi\to\infty} V(\phi)$, then $\lim_{t\to\infty}\phi = +\infty$.

ii. If $\lim_{t\to\infty} V(\phi(t)) < \lim_{\phi\to\infty} V(\phi)$, then there exists $\bar{\phi} \geq 0$ such that $\lim_{t\to\infty} V(\phi(t)) = V(\bar{\phi})$. From the monotonicity of the potential, it follows $\lim_{t\to\infty}\phi(t) = \bar{\phi}$. From Eq. (3.9), we have $\lim_{t\to\infty}\dot{y} = -V'(\bar{\phi}) < 0$. Hence, there exists $t_2 > t_1$ such that for all $t > t_2$, $\dot{y} < -\frac{1}{2}V'(\bar{\phi})$, i.e.,

$$y(t) - y(t_2) = \int_{t_2}^{t}\dot{y}(s)ds < -\frac{1}{2}V'\left(\bar{\phi}\right)(t - t_2),$$

that is as t increases, y takes arbitrary large negative values, which is contradictory since $\lim_{t\to\infty} y = 0$.

Thus, if $\phi > 0$ for all $t > t_1$, then $\lim_{t\to\infty}\phi = \infty$. We work similarly for the case $\phi < 0$ for all $t > t_1$, and we conclude that $\lim_{t\to\infty}\phi = -\infty$. ■

A large class of potentials used in scalar-field cosmological models have a non negative minimum. Below are some examples occurring in the literature. Polynomial potentials of the form

$$V(\phi) = m^2\phi^{2n}, \quad n \in \mathbb{N}, \quad n \geq 1,$$

exponential or logarithmic potentials [80, 81]

$$V(\phi) = \phi^n e^{-\lambda^2\phi^m}, \quad n, m \in \mathbb{N}, \quad n, m \geq 1, \quad \lambda \in \mathbb{R},$$

$$V(\phi) = \phi^n \ln^{2m}\phi, \quad n, m \in \mathbb{N}, \quad n, m \geq 1,$$

double exponential potentials [82]

$$V(\phi) = Ae^{-\lambda\phi} + Be^{\kappa\phi}, \quad A, B, \lambda, \kappa \in \mathbb{R}^+,$$

or chameleon effective potentials [71].

3.2.2 Decreasing non-negative potentials

In the following we shall consider potentials satisfying the following assumption.

Assumption 2 *We assume that $V \in C^2$ is such that:*

1. $V \geq 0$
2. $V'(\phi) < 0$.
3. *If $A \subseteq \mathbb{R}$ is such that V is bounded on A, then V' is bounded on A.*

The following Theorem generalises Proposition 4 proven in [78].

Theorem 2 *Let V be a potential function satisfying the Assumption 2. Then*

$$\lim_{t\to+\infty} y = 0 = \lim_{t\to+\infty} \rho,$$

and

$$\lim_{t\to+\infty} \phi = +\infty.$$

Proof. Since $V(\phi) \geq 0$, it follows from (3.4) that H is never zero, thus it cannot change sign. Hence, H is always non-negative if $H(t_0) > 0$. Furthermore, H is decreasing in view of (3.5), thus $H(t) \leq H(t_0)$, for all $t \geq t_0$. We then deduce from (3.4) that each of the terms $\rho, \frac{1}{2}y^2$ and V is bounded by $3H(t_0)^2$. Since H is decreasing, $\exists \lim_{t\to+\infty} H = \eta \geq 0$, therefore (3.5) implies that

$$\frac{1}{2}\int_{t_0}^{+\infty} \left(y^2 + \gamma\rho\right) dt = H(t_0) - \eta < +\infty. \tag{3.13}$$

From Lemma 1 we know that in general, if f is a non-negative function, the convergence of $\int_{t_0}^{\infty} f(t)\, dt$ does not imply that $\lim_{t\to\infty} f(t) = 0$, unless the derivative of f is bounded. In our case and setting $\lambda = (4 - 3\gamma) Q$,

$$\begin{aligned}\frac{d}{dt}\left(y^2 + \gamma\rho\right) &= -6Hy^2 - 2yV'(\phi) - 3\gamma^2\rho H + \lambda\left(1 - \frac{\gamma}{2}\right)\rho y \\ &\leq -2yV'(\phi) + \lambda\left(1 - \frac{\gamma}{2}\right)\rho y.\end{aligned}$$

As we already remarked, y and ρ are bounded; also, by our assumption on V, $V'(\phi)$ is bounded. We conclude that the derivative of the function $y^2 + \gamma\rho$ is bounded from above and therefore, (3.13) implies that $\lim_{t\to\infty} y(t)^2 = 0$ and $\lim_{t\to\infty} \rho(t) = 0$.

The proof that, $\lim_{t\to+\infty} \phi = +\infty$, follows after suitable adaptation of the arguments used in Proposition 4 in [78], and has been reproduced in the proof of Theorem 1. ■

If in addition, $\lim_{\phi\to+\infty} V(\phi) = 0$, we conclude that $H \to 0$ as $t \to \infty$.

An important example of decreasing non-negative potential is the exponential potential

$$V(\phi) = V_0 e^{-\lambda\phi}, \quad \lambda > 0, \tag{3.14}$$

which has been widely studied in the literature of scalar-field cosmologies, due to the variety of alternative theories of gravity with predict exponential potentials and also to the fact that V'/V =constant, which allows for the introduction of expansion-normalised variables [83] during the analysis of the dynamical system. Exponential potentials have been studied with dynamical system techniques in the context of inflation long before the discovery of cosmic acceleration [63, 84–87]. Potentials of the form (3.14) fall in the class of non-negative, decreasing potentials.

3.3 Double Exponential Potentials

The content of this section constitutes the heart of the first part of this thesis. We study the late time evolution of initially expanding flat FLRW models, with a scalar field coupled to matter and having a potential of the form

$$V(\phi) = V_1 e^{-\alpha\phi} + V_2 e^{-\beta\phi}, \tag{3.15}$$

where α, β are positive constants and V_1, V_2 are constants of arbitrary sign. The cases $0 < \beta < \alpha$ and $0 < \alpha < \beta$ are considered as "twin" cases and not treated separately since a mere renaming of the parameters yields to the same conclusions. For $0 < \alpha = \beta$ the case reduces to a single exponential potential, see for example [88]. The case where $\beta < \alpha < 0$ is simply a reverse $\phi \leftrightarrow -\phi$. For the rest of this chapter and without loss of generality, we assume $0 < \alpha < \beta$. For the general case of (3.15) where the parameters are of arbitrary sign, see Chapter 4. We also assume that the coupling coefficient is a constant, of order $Q \lesssim 1$.

Double exponential potential is usually the asymptotic form of other potentials. For example in Kaluza-Klein theories with d extra dimensions reformulated in the Einstein frame, α and β are [66]

$$\alpha = \sqrt{\frac{2d}{(d+2)}} \quad \text{and} \quad \beta = \sqrt{\frac{2\,(d+2)}{d}}.$$

The physical reason for the choice (3.15), is that in quintessence models, the dark energy is the energy of a slowly varying scalar field ϕ with equation of state

$$p_\phi = w\rho_\phi, \qquad w \simeq -1.$$

In most of the models of dark energy, it is assumed that the cosmological constant is zero and the potential energy, $V(\phi)$, of the scalar field driving the present stage of acceleration, slowly decreases and eventually vanishes as the field approaches the value $\phi = \infty$, [89]. In this case, after a transient accelerating stage, the speed of expansion of the Universe decreases and the Universe reaches Minkowski regime. Double exponential potentials of the form (3.15) were investigated in [90, 91]. Solutions were obtained in [92–94] with the ansatz $\dot{\phi} = \lambda H$; see also [95] for more general couplings. A scalar field with a double exponential potential without coupling to matter was investigated in [96]. For exact solutions of a scalar field non coupled to dust with single and double exponential potentials see [82]. Quintessence cosmologies of double exponential potentials in the absence of matter were studied in [97] with the techniques of phase space analysis. Coupled quintessence field with a double exponential potential and galileon like correction was considered in [98].

Interaction terms between the two matter components of the form $-\alpha\rho\dot{\phi}$ as in (3.7) with a simple exponential potential, were firstly considered in [99], see also [100]. Although there is an energy exchange between the fluid and the scalar field, it is easy to see that the set, $\rho > 0$, is invariant under the flow of (3.5)-(3.7), therefore ρ is nonzero if initially $\rho(t_0)$ is nonzero; this trivial physical demand is not satisfied if one assumes arbitrary interaction terms, cf. [77].

The field equations with (3.15) reduce to the Friedmann equation

$$3H^2 = \rho + \frac{1}{2}\dot{\phi}^2 + V_1 e^{-\alpha\phi} + V_2 e^{-\beta\phi}; \tag{3.16}$$

the Raychadhuri equation

$$\dot{H} = -\frac{1}{2}\dot{\phi}^2 - \frac{\gamma}{2}\rho; \tag{3.17}$$

the equation of motion of the scalar field

$$\ddot{\phi} + 3H\dot{\phi} - \alpha V_1 e^{-\alpha\phi} - \beta V_2 e^{-\beta\phi} = \frac{4-3\gamma}{2} Q\rho; \tag{3.18}$$

and the conservation equation

$$\dot{\rho} + 3\gamma\rho H = -\frac{4-3\gamma}{2} Q\rho\dot{\phi}. \tag{3.19}$$

3.3.1 Double exponential potentials with $V_1, V_2 > 0$

For potentials (3.15) with $V_1, V_2 > 0$, we have already shown in Theorem 2 of Section 3.2, the global result

$$\lim_{t\to+\infty} \dot{\phi} = 0, \ \lim_{t\to+\infty} \rho = 0, \quad \text{and} \ \lim_{t\to+\infty} \phi = +\infty.$$

If in addition, $\lim_{\phi\to+\infty} V(\phi) = 0$, as is the case of the double exponential potential (3.15) with $V_1, V_2 > 0$, then we conclude that

$$\lim_{t\to\infty} H = 0.$$

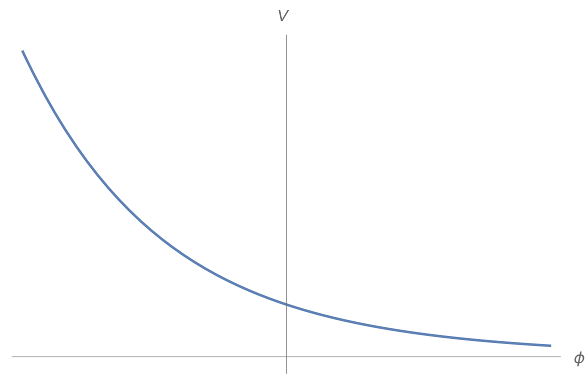


Figure 3.1: Potentials (3.15) with $V_1 > 0, V_2 > 0$.

3.3.2 Double exponential potentials with $V_1 > 0, V_2 < 0$

The case $V_1 > 0, V_2 < 0$, is more delicate and the asymptotic state depends on the initial conditions. The dynamical system (3.17)-(3.19) has for $V_1 > 0,\ V_2 < 0$, two finite equilibrium points

$$\left(\phi = \phi_m, \dot{\phi} = 0, \rho = 0, H = \pm\sqrt{\frac{V_{\max}}{3}}\right),$$

see Fig. 3.2. They represent de Sitter and anti-de Sitter solutions and it is easy to see that they are unstable. It is known that for potentials having a maximum, the field near the top of the potential corresponds to the tachyonic (unstable) mode with negative mass squared [89, 101–103]. The other asymptotic states of the system correspond to the points at infinity, $\phi \to \pm\infty$.

(i) If initially $\phi_0 > \phi_m$, and $3H(t_0)^2 < V_{\max}$, then from (3.16), $V(\phi)$ remains less than $V_{\max}$ since H is decreasing. We conclude that $V(\phi(t)) < V_{\max}$ for all $t \geq t_0$, thus ϕ cannot pass to the left of ϕ_m. In the interval $(\phi_m, +\infty)$ the potential satisfies the assumptions of Theorem 2 and therefore, $\phi \to \infty$ as $t \to \infty$.

(ii) If initially $\phi_0 < \phi_m$, and $\dot{\phi}_0$ is larger than the critical value $\dot{\phi}_{\text{crit}} > 0$,

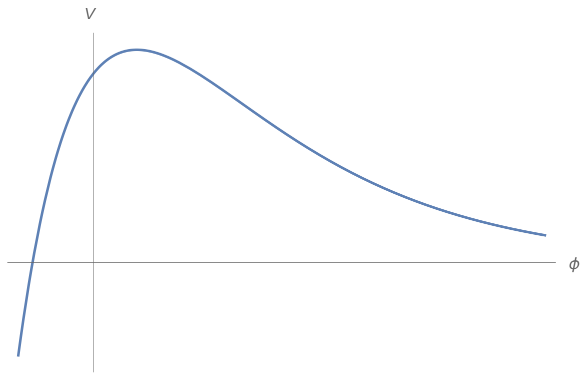


Figure 3.2: Potentials (3.15) with $V_1 > 0, V_2 < 0$ have a local maximum at some ϕ_m and diverge to minus infinity as $\phi \to -\infty$.

which allows for ϕ to pass on the right of ϕ_m, then the conclusions of case (i) hold.

(iii) Finally, suppose that initially $\phi_0 < \phi_m$, and $\dot{\phi}_0$ is less than the critical value $\dot{\phi}_{\text{crit}} > 0$, i.e., $-\infty < \dot{\phi}_0 < \dot{\phi}_{\text{crit}}$. From (3.17), H is monotonically decreasing and not bounded below from zero, hence eventually H may change sign. We cannot use the same argument as in Theorem 2 concerning the asymptotic behaviour of $\dot{\phi}(t)^2$ and $\rho(t)$, since V and V' are not bounded. A heuristic argument is the following. Suppose, firstly, that $\lim_{t\to+\infty} H = \eta$, where η is finite. But, an asymptotic state of the form, $\mathbf{p} = \left(H = \eta, \rho = \rho_*, \dot{\phi} = \dot{\phi}_*, \phi = \phi_*\right)$, is impossible, i.e., the point $\mathbf{p}$ cannot be an equilibrium point of the dynamical system (3.17)-(3.19) for $\phi_* < \phi_m$. Although we cannot exclude periodic orbits, or strange attractors as $\omega-$limit sets for our system, numerical experiments suggest that, H diverges to $-\infty$. If this is the case, it can be shown that H diverges to $-\infty$, in a finite time. Suppose on the contrary that, $\lim_{t\to+\infty} H = -\infty$. Since $\gamma < 2$

$$3H^2 = \frac{\dot{\phi}^2}{2} + \rho + V(\phi) < \frac{\dot{\phi}^2 + \gamma\rho}{\gamma} + V(\phi) = -\frac{2\dot{H}}{\gamma} + V(\phi),$$

hence,

$$3 < -\frac{2\dot{H}}{\gamma H^2} + \frac{V(\phi)}{H^2}. \tag{3.20}$$

Taking limits as $t \to +\infty$, and since $V(\phi)$ is bounded from above

$$\lim_{t\to+\infty} \frac{V(\phi)}{H^2} \leq 0.$$

Inequality (3.20) implies that

$$\lim_{t\to+\infty} \frac{-\dot{H}}{H^2} \geq 3\gamma/2,$$

which is impossible, since

$$-\frac{\dot{H}}{H^2} = \frac{d}{dt}\frac{1}{H} \quad \text{and} \quad \frac{1}{H} \to 0.$$

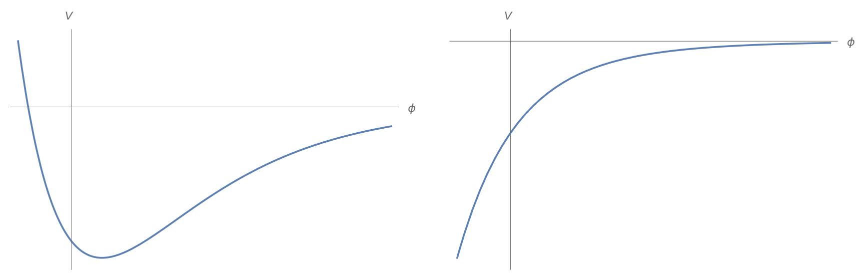
In view of (3.17), $\dot{\phi}^2 + \gamma\rho$ also diverges to infinity. Again, an asymptotic state of the form, $H = -\infty, \dot{\phi}^2 + \gamma\rho = \infty$ and $\phi =$ finite is impossible, therefore ϕ diverges to $-\infty$ in a finite time. The above qualitative arguments for potentials of the form (3.15) with $V_1 > 0, V_2 < 0$, establish the following result, which we prove rigorously in Chapter 4:

Theorem 3 *Let V be a C^1 potential function with the following properties: 1. V is negative and monotonically increasing for $\phi < 0$, with $\lim_{\phi\to-\infty} V(\phi) = -\infty$. 2. V has a global maximum at some $\phi_m > 0$. Suppose that the following initial conditions hold: $H(t_0) > 0$, $\phi(t_0) < \phi_m$, and $-\infty < \dot{\phi}(t_0) < \dot{\phi}_{\text{crit}}$, where $\dot{\phi}_{\text{crit}} > 0$, is the critical value which allows for ϕ to pass to the right of ϕ_m. Then H and ϕ diverge to $-\infty$ in a finite time.*

This result generalises previous investigations indicating that negative potentials may drive a flat initially expanding Universe to recollapse, see [104–106].

3.3.3 Double exponential potentials with $V_1 < 0$

Potentials falling into this class are either entirely negative or have a global negative minimum, see Fig. 3.3. As we show in the next section, these



(a) (b)

Figure 3.3: Left 3.3(a) potentials with $V_1 < 0, V_2 > 0$. Right 3.3(b) potentials with $V_1 < 0, V_2 < 0$.

are not physically interesting cases. Especially for potentials shown in Fig. 3.3(a), we prove in Chapter 4 that they collapse in finite time except in the case where special assumptions on the parameters α, β, γ and the coupling constant Q hold.

3.4 Expansion-Normalised Variables and Critical Points

There exists a well established mathematical procedure for the investigation of scalar field cosmologies with exponential potentials in the context of dynamical systems theory [63, 83]. It consists in the introduction of the so called, expansion-normalised variables by defining

$$x = \frac{\dot{\phi}}{\sqrt{6}H}, \qquad y = \sqrt{\frac{V_1 e^{-\alpha\phi}}{3H^2}}, \qquad z = \sqrt{\frac{V_2 e^{-\beta\phi}}{3H^2}}, \qquad \Omega = \frac{\rho}{3H^2}, \quad (3.21)$$

and a new time variable,

$$\tau = \ln a.$$

Note that while x and Ω could take only real values, variables y and z may take both real and pure imaginary values, depending on the sign of V_1, V_2. With this choice we avoid to have four different dynamical systems. The Friedmann equation (3.16) decouples and imposes the phase space

$$\Omega = 1 - \left(x^2 + y^2 + z^2\right), \tag{3.22}$$

to the state vector (x, y, z, Ω). Since

$$\frac{dt}{d\tau} = \frac{1}{H},$$

the evolution equations become

$$\begin{aligned}
\frac{dx}{d\tau} &= \frac{dx}{dt}\frac{dt}{d\tau} = \frac{1}{H}\left(\frac{\ddot{\phi}}{\sqrt{6}H} - \frac{\dot{\phi}\dot{H}}{\sqrt{6}H^2}\right) \\
&= \frac{-3\dot{\phi}}{\sqrt{6}H} + \frac{\alpha V_1 e^{-\alpha\phi}}{\sqrt{6}H^2} + \frac{\beta V_1 e^{-\beta\phi}}{\sqrt{6}H^2} + \frac{\frac{4-3\gamma}{2}Q\rho}{\sqrt{6}H^2} + \frac{\frac{1}{2}\dot{\phi}^3}{\sqrt{6}H^3} + \frac{\frac{\gamma}{2}\dot{\phi}\rho}{\sqrt{6}H^3} \\
&= -3x + \sqrt{\frac{3}{2}}\alpha y^2 + \sqrt{\frac{3}{2}}\beta z^2 + \sqrt{\frac{3}{2}}\frac{4-3\gamma}{2}Q\Omega + 3x^3 + \frac{3\gamma}{2}\Omega x,
\end{aligned} \tag{3.23}$$

$$\begin{aligned}
\frac{dy}{d\tau} &= \frac{dy}{dt}\frac{dt}{d\tau} = \frac{1}{H}\sqrt{\frac{V_1}{3}}\left(-\frac{\alpha\dot{\phi}e^{-\alpha\phi/2}}{2H} - \frac{e^{-\alpha\phi/2}\dot{H}}{H^2}\right) \\
&= \sqrt{\frac{V_1 e^{-\alpha\phi}}{3H^2}}\left(-\frac{\alpha}{2}\frac{\dot{\phi}}{H} - \frac{\dot{H}}{H^2}\right) \\
&= \sqrt{\frac{V_1 e^{-\alpha\phi}}{3H^2}}\left(-\frac{\alpha}{2}\frac{\dot{\phi}}{H} + \frac{1}{2}\frac{\dot{\phi}^2}{H^2} + \frac{\gamma}{2}\frac{\rho}{H^2}\right) \\
&= y\left(-\sqrt{\frac{3}{2}}\alpha x + 3x^2 + \frac{3\gamma}{2}\Omega\right),
\end{aligned} \tag{3.24}$$

$$
\begin{aligned}
\frac{dz}{d\tau} &= \frac{dz}{dt}\frac{dt}{d\tau} = \frac{1}{H}\sqrt{\frac{V_2}{3}}\left(-\frac{\beta\dot{\phi}e^{-\beta\phi/2}}{2H} - \frac{e^{-\beta\phi/2}\dot{H}}{H^2}\right) \\
&= \sqrt{\frac{V_2 e^{-\beta\phi}}{3H^2}}\left(-\frac{\beta}{2}\frac{\dot{\phi}}{H} - \frac{\dot{H}}{H^2}\right) \\
&= \sqrt{\frac{V_2 e^{-\beta\phi}}{3H^2}}\left(-\frac{\beta}{2}\frac{\dot{\phi}}{H} + \frac{1}{2}\frac{\dot{\phi}^2}{H^2} + \frac{\gamma}{2}\frac{\rho}{H^2}\right) \\
&= z\left(-\sqrt{\frac{3}{2}}\beta x + 3x^2 + \frac{3\gamma}{2}\Omega\right), \qquad (3.25)
\end{aligned}
$$

and the evolution of the density parameter Ω is

$$
\begin{aligned}
\frac{d\Omega}{d\tau} &= \frac{d\Omega}{dt}\frac{dt}{d\tau} = \frac{1}{H}\left(\frac{\dot{\rho}}{3H^2} - \frac{2\rho\dot{H}}{3H^3}\right) \\
&= \frac{1}{H}\left(-\frac{3\gamma\rho}{3H} - \frac{4-3\gamma}{2}Q\frac{\rho\dot{\phi}}{3H^2} + \frac{\rho\dot{\phi}^2}{3H^3} + \frac{\gamma\rho^2}{3H^3}\right) \\
&= \frac{\rho}{3H^2}\left(-3\gamma - \frac{4-3\gamma}{2}Q\frac{\dot{\phi}}{H} + \frac{\dot{\phi}^2}{H^2} + \frac{\gamma\rho}{H^2}\right) \\
&= \Omega\left(-3\gamma - \sqrt{6}\frac{4-3\gamma}{2}Qx + 6x^2 + 3\gamma\Omega\right). \qquad (3.26)
\end{aligned}
$$

The evolution of the Hubble function is

$$
\begin{aligned}
\frac{dH}{d\tau} &= \frac{dH}{dt}\frac{dt}{d\tau} = \frac{\dot{H}}{H} \\
&= \frac{1}{H}\left(-\frac{1}{2}\dot{\phi}^2 - \frac{\gamma}{2}\rho\right) \\
&= -H\left(\frac{1}{2}\frac{\dot{\phi}^2}{H^2} + \frac{\gamma}{2}\frac{\rho}{H^2}\right) \\
&= -H\left(3x^2 + \frac{3\gamma}{2}\Omega\right),
\end{aligned}
$$

which decouples from the rest of the evolution equations. This is one of the merits of the introduction of the variables (3.21); it allows for the reduction of the dimension of the dynamical system by one.

Using (3.22) we can eliminate Ω from (3.23)-(3.25) and we end up with a three-dimensional dynamical system, (x, y, z),

$$
\begin{aligned}
\frac{dx}{d\tau} =& \sqrt{6}Q - \frac{3}{2}\sqrt{\frac{3}{2}}\gamma Q + \left(\frac{3\gamma}{2} - 3\right) x + \left(\frac{3}{2}\sqrt{\frac{3}{2}}\gamma - \sqrt{6}\right) Qx^2 \\
&+ \left(3 - \frac{3\gamma}{2}\right) x^3 + \left(\sqrt{\frac{3}{2}}\alpha - \sqrt{6}Q + \frac{3}{2}\sqrt{\frac{3}{2}}\gamma Q\right) y^2 \\
&+ \left(\sqrt{\frac{3}{2}}\beta - \sqrt{6}Q + \frac{3}{2}\sqrt{\frac{3}{2}}\gamma Q\right) z^2 - \frac{3}{2}\gamma x y^2 - \frac{3}{2}\gamma x z^2, \\
\frac{dy}{d\tau} =& y\left(\frac{3\gamma}{2} - \sqrt{\frac{3}{2}}\alpha x + \left(3 - \frac{3\gamma}{2}\right) x^2 - \frac{3\gamma}{2} y^2 - \frac{3\gamma}{2} z^2\right), \\
\frac{dz}{d\tau} =& z\left(\frac{3\gamma}{2} - \sqrt{\frac{3}{2}}\beta x + \left(3 - \frac{3\gamma}{2}\right) x^2 - \frac{3\gamma}{2} y^2 - \frac{3\gamma}{2} z^2\right),
\end{aligned}
\tag{3.27}
$$

where

$$x^2 + y^2 + z^2 \leq 1. \tag{3.28}$$

The phase space depends significantly on the signs of V_1, V_2. For $V_1, V_2 > 0$, the phase space (3.28) is the intersection of a closed unit ball in $\mathbb{R}^3$ with the octants $y > 0, z > 0$. For $V_1 > 0$ and $V_2 < 0$, the phase space is the intersection of the one sheet hyperboloid $x^2 + y^2 - (\operatorname{Im} z)^2 = 1$ and its interior with the octants $y > 0, z > 0$. For $V_1 < 0$ and $V_2 > 0$, the phase space is the intersection of one sheet hyperboloid $x^2 - (\operatorname{Im} y)^2 + z^2 = 1$ and its interior with the octants $y > 0, z > 0$ and for $V_1, V_2 < 0$ the phase space is the intersection of the two sheet hyperboloid $x^2 - (\operatorname{Im} y)^2 - (\operatorname{Im} z)^2 = 1$ and its exterior with the octants $y > 0, z > 0$. The resulting dynamical system depends on four parameters $(\gamma, \alpha, \beta, Q)$. Using (3.17), the effective equation of state parameter

$$w_{\text{eff}} = -1 - \frac{2\dot{H}}{3H^2}, \tag{3.29}$$

is written in terms of the new variables as

$$\begin{aligned} w_{\text{eff}} &= -1 + \frac{\dot{\phi}^2}{3H^2} + \frac{\gamma\rho}{3H^2} \\ &= -1 + 2x^2 + \gamma\Omega. \end{aligned} \tag{3.30}$$

At an equilibrium point we may integrate Eq. (3.29) to obtain,

$$H \sim \frac{2}{3(w_{\text{eff}}+1)t}, \qquad \text{if} \quad w_{\text{eff}} \neq -1,$$

and

$$H = \text{constant}, \qquad \text{if} \quad w_{\text{eff}} = -1.$$

Integrating again the above equations we find that the scale factor evolves as

$$a \sim t^{\frac{2}{3(w_{\text{eff}}+1)}}, \qquad \text{if} \quad w_{\text{eff}} \neq -1,$$

and

$$a \sim e^t, \qquad \text{if} \quad w_{\text{eff}} = -1.$$

For a trajectory to be cosmologically acceptable, it has to pass near a matter point, slow enough such as to allow the construction of matter, the matter era, and to land to an accelerated point. A critical point is a good candidate for a matter point if it

(i) satisfies the matter condition, $\Omega > 0$,

(ii) satisfies the "right"scale factor condition, $a \sim t^{2/3}$, (or equivalently, w_{eff} close to zero), and

(iii) represents a transient phase, i.e., in the language of dynamical systems has to be a saddle point.

On the other hand, an acceptable late attractor has to be

(iv) accelerated, $w_{\text{eff}} < -1/3$, and

(v) stable.

We start the study of the system by determining its critical points. We solve the system of equations $dx/d\tau = 0, dy/d\tau = 0, dz/d\tau = 0$, determined by (3.27), and we get 15 critical points listed in Table 3.9 at the end of the chapter. The eigenvalues are presented in Table 3.10. According to the definition (3.21), if y, z are real, then $y, z \geq 0$ and if they are complex, (pure imaginary numbers), then $\operatorname{Im} y, \operatorname{Im} z$ is non negative. This means that the points $\mathcal{C}_-, \mathcal{D}_-, \mathcal{D}'_\pm, \mathcal{E}_-, \mathcal{F}_-$ and $\mathcal{G}_-$ are not acceptable. Furthermore, not all of the remaining points are present in all forms of the potential with respect to the values of α, β, V_1, V_2. We will examine the different cases below.

The different forms of the potentials with respect to the different signs of V_1, V_2 are shown in Fig. 3.4.

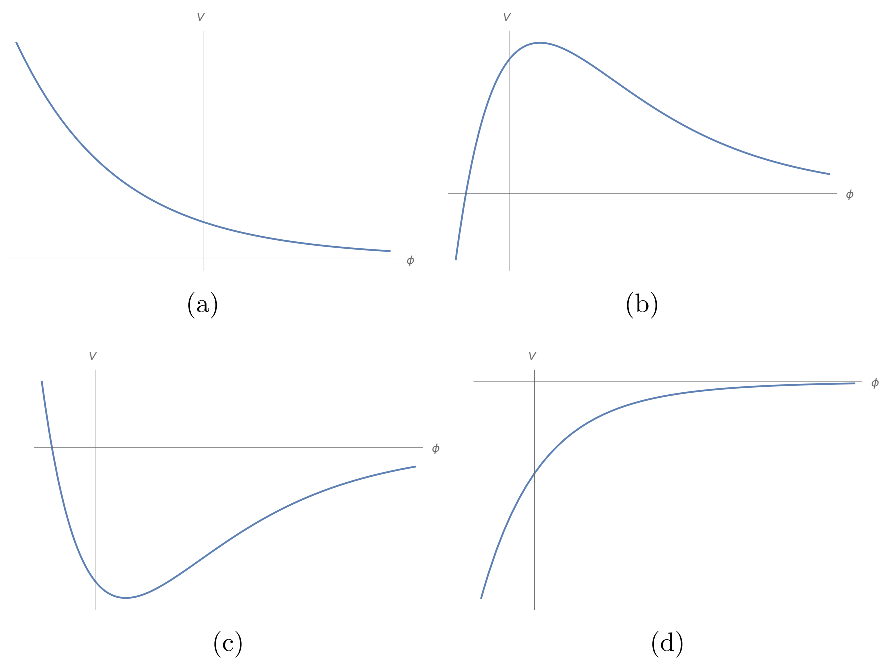


Figure 3.4: Potentials with $0 < \alpha < \beta$. (a) $V_1, V_2 > 0$, (b) $V_1 > 0, V_2 < 0$, (c) $V_1 < 0, V_2 > 0$, (d) $V_1, V_2 < 0$

Below we list all the physically acceptable equilibria. In the next section, we resume the cosmologically interesting cases.

$\mathcal{A}_\pm$ Following the usual terminology (see for example [104]), points $\mathcal{A}_\pm$

correspond to kinetic-dominated solutions and exist for any potential (3.15). The eigenvalues are

$$3-\sqrt{\frac{3}{2}}\alpha,\ 3-\sqrt{\frac{3}{2}}\beta,\ 3(2-\gamma)-\sqrt{6}\frac{4-3\gamma}{2}Q,$$

and

$$3+\sqrt{\frac{3}{2}}\alpha,\ 3+\sqrt{\frac{3}{2}}\beta,\ 3(2-\gamma)+\sqrt{6}\frac{4-3\gamma}{2}Q,$$

for $\mathcal{A}_+$ and $\mathcal{A}_-$ respectively. Hence, point $\mathcal{A}_-$ is always unstable and point $\mathcal{A}_+$ is stable only for

$$\alpha>\sqrt{6},\ Q>\sqrt{6}\frac{2-\gamma}{4-3\gamma}\ \text{and}\ \gamma<\frac{4}{3},$$

but unstable otherwise. The effective equation of state is $w_{\rm eff}=1$ and the density parameter $\Omega=0$. Hence these points, although exist for all forms of the potential as shown in Fig. 3.4, cannot be used neither as matter points nor as accelerated attractors.

$\mathcal{B}$ This is a fluid-kinetic scaling solution. It exists for all different signs of V_1, V_2. The eigenvalues are

$$\frac{(4-3\gamma)^2 Q^2-2\alpha(4-3\gamma)Q+6\gamma(2-\gamma)}{4(2-\gamma)},$$
$$\frac{(4-3\gamma)^2 Q^2-2\beta(4-3\gamma)Q+6\gamma(2-\gamma)}{4(2-\gamma)},$$
$$\frac{(4-3\gamma)^2 Q^2-6(2-\gamma)^2}{4(2-\gamma)}.$$

Point $\mathcal{B}$ enters the phase space when

$$Q\leq\sqrt{6}\frac{2-\gamma}{|4-3\gamma|}, \tag{3.31}$$

for $\gamma\neq 4/3$ and lies always in the phase space for $\gamma=4/3$, irrespectively of the nature of the potential. For $\gamma<4/3$, condition (3.31) is always satisfied for sufficiently small values of Q, e.g., $Q \lesssim 1$. Matter point conditions (i), (ii) and (iii) are satisfied whenever

$$Q = \frac{\sqrt{6\left(2-\gamma\right)\left(1-\gamma\right)}}{4-3\gamma}, \;\gamma \leq 1, \;\alpha < \sqrt{\frac{3}{2}}\sqrt{\frac{2-\gamma}{1-\gamma}}. \tag{3.32}$$

On the other hand, point $\mathcal{B}$ may be an accelerated attractor if (iv) and (v) hold, provided that (3.31) is satisfied. The condition for acceleration (iv) gives

$$Q < \frac{\sqrt{2\left(2-\gamma\right)\left(2-3\gamma\right)}}{4-3\gamma}, \tag{3.33}$$

with $\gamma < 2/3$. Assuming (3.33), the stability condition, (v), gives

$$\left(4-3\gamma\right)^2 Q^2 - 2\alpha\left(4-3\gamma\right)Q + 6\gamma\left(2-\gamma\right) < 0.$$

$\mathcal{C}_+$ This is a kinetic-potential scaling solution and exists in potentials with $V_1 > 0$ for $\alpha < \sqrt{6}$ and in potentials with $V_1 < 0$ for $\alpha > \sqrt{6}$. It cannot be used as a matter point since $\Omega = 0$. Point $\mathcal{C}_+$ is accelerated for $\alpha < \sqrt{2}$. The eigenvalues are

$$\frac{\alpha^2-6}{2}, \;\frac{\alpha\left(\alpha-\beta\right)}{2}, \;\frac{2\alpha^2-6\gamma-\alpha\left(4-3\gamma\right)Q}{2}.$$

Thus, it is stable and accelerated whenever

$$\left(4-3\gamma\right)Q > \frac{2\left(\alpha^2-3\gamma\right)}{\alpha} \text{ and } \alpha < \sqrt{2}. \tag{3.34}$$

Hence, it is a good candidate as an accelerated late attractor only in potentials with $V_1 > 0$.

$\mathcal{D}_+$ This is a potential solution and exist only in potentials with $V_1 >$

$0,\ V_2 < 0$. The eigenvalues are

$$\frac{-3+\sqrt{9+12\alpha\beta}}{2},\ \frac{-3-\sqrt{9+12\alpha\beta}}{2},\ -3\gamma,$$

therefore, this point is unstable and represents de Sitter solutions.

$\mathcal{E}_+$ Point $\mathcal{E}_+$ is a fluid-kinetic-potential scaling solution (see also Ref. [104] for the uncoupled case). It enters the phase space when

$$\begin{aligned} Q \leq \frac{2}{\alpha}\frac{\alpha^2-3\gamma}{4-3\gamma},\ \alpha \geq \sqrt{3\gamma},\ \text{for } \gamma < \frac{4}{3}, \\ Q \geq \frac{2}{\alpha}\frac{\alpha^2-3\gamma}{4-3\gamma},\ \text{for } \gamma > \frac{4}{3}, \\ \alpha \geq \sqrt{3\gamma},\ \text{for } \gamma = \frac{4}{3}. \end{aligned}$$

The eigenvalues are

$$\frac{3\left(\alpha-\beta\right)\gamma}{2\alpha-\left(4-3\gamma\right)Q},\ \frac{\sigma \pm \sqrt{\sigma^2-4\delta}}{2\left(2\alpha-\left(4-3\gamma\right)Q\right)^2},$$

where

$$\begin{aligned} \sigma =& 3\left(2\alpha-\left(4-3\gamma\right)Q\right)\left(\left(4-3\gamma\right)Q-\alpha\left(2-\gamma\right)\right), \\ \delta =& \frac{3}{2}\left(2\alpha-\left(4-3\gamma\right)Q\right)^2\left(2\alpha^2-6\gamma-\alpha\left(4-3\gamma\right)Q\right) \\ & \left(\left(4-3\gamma\right)^2Q^2-2\alpha\left(4-3\gamma\right)Q+6\gamma\left(2-\gamma\right)\right) \end{aligned}$$

Point $\mathcal{E}_+$ may be used for the matter epoch if it satisfies conditions (i), (ii) and (iii). For $\gamma < 4/3$, (i) is satisfied for

$$Q < \frac{2}{\alpha}\frac{\alpha^2-3\gamma}{4-3\gamma},\ \alpha > \sqrt{3\gamma}. \tag{3.35}$$

Under the assumption (3.35), the scale factor condition, (ii), is satisfied for

$$Q = 2\alpha\frac{1-\gamma}{4-3\gamma},\ \gamma \leq 1,\ \alpha > \sqrt{3} \tag{3.36}$$

and using the value of Q given in (3.36), condition (iii) is satisfied for

$$\alpha > \sqrt{\frac{3}{2}}\sqrt{\frac{2-\gamma}{1-\gamma}}. \tag{3.37}$$

For $\gamma > 4/3$, whenever the scale factor evolves as $t^{2/3}$, point $\mathcal{E}_+$ is stable and therefore, does not represent transient solutions. For $\gamma = 4/3$, $\mathcal{E}_+$ represents radiation solutions and the scale factor evolves as $t^{1/2}$. Hence, $\mathcal{E}_+$ is a good candidate for the matter era if it satisfies

$$\gamma \leq 1,\ Q = 2\alpha\frac{1-\gamma}{4-3\gamma},\ \alpha > \sqrt{\frac{3}{2}}\sqrt{\frac{2-\gamma}{1-\gamma}}. \tag{3.38}$$

In that case, point $\mathcal{E}_+$ exists only for potentials with $V_1 < 0$. Hence, when $\mathcal{E}_+$ is used as a matter point, point $\mathcal{C}_+$ cannot be used as the accelerated attractor. Only point $\mathcal{B}$ is left as a candidate for the accelerated era, but $\mathcal{B}$ does not satisfy conditions (iv) and (v), as long as the parameters γ, Q and α take values in the ranges defined by (3.38). Therefore, point $\mathcal{E}_+$ cannot describe the transient matter phase.

In order for $\mathcal{E}_\pm$ to be used for the accelerated epoch it has to satisfy conditions (iv) and (v). For $\gamma \geq 2/3$, when the point enters the phase space, it is either unstable or non accelerated. For $\gamma < 2/3$, it satisfies condition (iv) for

$$Q < \frac{2-3\gamma}{4-3\gamma}\alpha, \tag{3.39}$$

and it is stable, (v), when

$$\alpha < \sqrt{6\gamma\,(2-\gamma)}, \tag{3.40}$$

and when

$$Q < \frac{\alpha - \sqrt{\alpha^2 - 6\gamma\,(2-\gamma)}}{4-3\gamma} \text{ or } Q > \frac{\alpha + \sqrt{\alpha^2 - 6\gamma\,(2-\gamma)}}{4-3\gamma} \tag{3.41}$$

otherwise. Whenever $\mathcal{E}_+$ is an accelerated attractor, the only remaining candidate for the matter epoch is point $\mathcal{B}$, but $\mathcal{B}$ does not satisfy the conditions (i), (ii) and (iii) for the range of the parameters (3.39)-(3.41).

$\mathcal{F}_+$, $\mathcal{G}_+$ Since $0 < \alpha < \beta$, then $y \to 0$ means $z \to 0$. Therefore, in the case of $0 < \alpha < \beta$, these points are not acceptable. Points $\mathcal{F}_+$ and $\mathcal{G}_+$ substitute points $\mathcal{C}_+$ and $\mathcal{E}_+$ respectively, in the "twin" case where $0 < \beta < \alpha$.

3.5 Cosmologically acceptable solutions

In this section, we discuss only these equilibria which allow for a viable cosmological history of the Universe. In Table 3.1 are shown the equilibria for $V_1 > 0$ and

$$\alpha < \sqrt{2},\ \gamma \leq 1,\ (4-3\gamma)\,Q \in \left(\max\left\{0, 2\frac{\alpha^2-3\gamma}{\alpha}\right\}, \sqrt{6}\,(2-\gamma)\right).$$

The two critical points $\mathcal{A}_\pm$ correspond to kinetic dominated solutions which are unstable and are only expected to be relevant at early times. Point $\mathcal{B}$ represents a type of scaling solution, i.e., the kinetic energy density of the scalar field remains proportional to that of the perfect fluid. Point $\mathcal{C}_+$ is

Table 3.1: Equilibrium Points for viable cosmology

Label	(x, y, z)	Ω	Stability	$a(t)$
$\mathcal{A}_\pm$	$(\pm 1, 0, 0)$	0	Unstable	$t^{1/3}$
$\mathcal{B}$	$\left(\frac{(4-3\gamma)Q}{\sqrt{6}(2-\gamma)}, 0, 0\right)$	$1 - \frac{(4-3\gamma)^2 Q^2}{6(2-\gamma)^2}$	Saddle	$t^{4(2-\gamma)/\left(6\gamma(2-\gamma)+(4-3\gamma)^2 Q^2\right)}$
$\mathcal{C}_+$	$\left(\frac{\alpha}{\sqrt{6}}, \sqrt{1-\frac{\alpha^2}{6}}, 0\right)$	0	Stable	t^{2/α^2}
$\mathcal{D}_+$	$\left(0, \sqrt{\frac{\beta}{\beta-\alpha}}, \sqrt{\frac{\alpha}{\alpha-\beta}}\right)$	0	Saddle	e^t

accelerated only in potentials with $V_1 > 0$. It corresponds to scalar field dominated solutions which exist for sufficiently flat potentials, $\alpha < \sqrt{6}$. These are the same conclusions as in [79] for an exponential potential and $Q = \sqrt{2/3}$, and also in [63], [104] and [96] and in the case of a scalar field non coupled to matter, although the ranges of the parameters (α, γ) are different. Point $\mathcal{D}_+$ exists only in models with $V_1 > 0$, $V_2 < 0$. It corresponds to the unstable state $\left(\phi = \phi_m, \dot{\phi} = 0, \rho = 0, H = \sqrt{V_{\max}/3}\right)$ and represents de Sitter solutions.

A successful cosmological model should comprise an accelerating solution as a future attractor. It is evident that point $\mathcal{C}_+$, could satisfy the condition for acceleration, $w_{\text{eff}} < -1/3$, provided that $\alpha < \sqrt{2}$, compare with the conclusions in [63]. From now on we assume this range for the parameter α. Moreover, the equilibrium $\mathcal{C}_+$, is stable for all physically interesting values of γ. For a cosmological theory to be acceptable, it has to possess a matter dominated epoch followed by a late time accelerated attractor. The saddle character of point $\mathcal{B}$, implies that it represents a transient phase and therefore, it is a good candidate for a matter point, provided that Ω is close to one. This happens only for very small values of the coupling parameter Q and for γ close to one. Another way to see this, is the following. During the matter era, the scale factor has to expand approximately as $a \sim t^{2/3}$. The scale factor near $\mathcal{B}$ evolves as $a \sim t^{\frac{2}{3(w_{\text{eff}}+1)}}$, therefore, w_{eff}, has to be close to zero. As seen in Table 3.1, $a\,(t)$ at $\mathcal{B}$, evolves as $t^{2/3}$ when Q takes the values

$$Q = \frac{\sqrt{6\,(2-\gamma)\,(1-\gamma)}}{(4-3\gamma)}, \ \gamma \leq 1. \tag{3.42}$$

Therefore, the realistic value $\gamma = 1$, corresponding to dust, is incompatible to scalar field coupled to matter, i.e., the coupling parameter Q must be zero, see also [107]. On the other hand, (3.6) and (3.7) imply that for $\gamma = 4/3$, the value of Q is undetermined. Below we summarize our results for the particular values $\gamma = 1, 4/3, 2/3$.

A. Dust ($\gamma = 1$). The critical points of our system are those of Table 3.1 for $\alpha < \sqrt{2}$, $\beta > \alpha$, $Q = 0$. Note that the future attractor $\mathcal{C}_+$ has non phantom acceleration for every value of α in the interval $(0, \sqrt{2})$. A cosmologically acceptable trajectory should pass near $\mathcal{B}$ and finally land on point $\mathcal{C}_+$, depending on the initial conditions. Note that $\mathcal{A}_\pm$, $\mathcal{B}$ and $\mathcal{C}_+$ lie on the invariant plane $z = 0$ and under our assumption on α, point $\mathcal{C}_+$ exists only in potentials with $V_1 > 0$. We consider the projection of the system (3.27) on the invariant set $z = 0$. The phase portrait is shown in Fig. 3.5 and is the same in both cases where the phase space is the intersection of the octants $y > 0, z > 0$ with a sphere ($V_2 > 0$), or with an one sheet hyperboloid ($V_2 < 0$). The 3-D phase portrait is depicted in Figs 3.6. For the case of dust and $\alpha = 1, \beta = 2, Q = 0$, see Tables 3.2, 3.3.

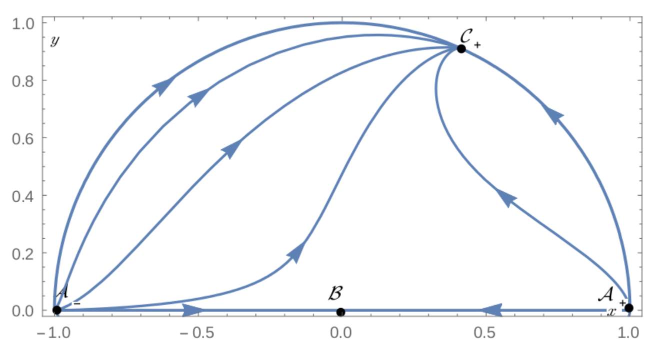


Figure 3.5: Phase portrait of the projected three-dimensional system on the invariant set $z = 0$, for the case of dust, with $\alpha = 1, \beta = 2, Q = 0$.

Table 3.2: Case of dust and $V_1 > 0, V_2 > 0$, $Q = 0$, $\alpha = 1$, $\beta = 2$.

Label	(x, y, z)	Ω	Stability	$a(t)$
$\mathcal{A}_\pm$	$(\pm 1, 0, 0)$	0	sources	$t^{1/3}$
$\mathcal{B}$	$(0, 0, 0)$	1	saddle	$t^{2/3}$
$\mathcal{C}_+$	$\left(\sqrt{\frac{1}{6}}, \sqrt{\frac{5}{6}}, 0\right)$	0	sink	t^2

Table 3.3: Case of dust and $V_1 > 0, V_2 < 0$, $Q = 0$, $\alpha = 1$, $\beta = 2$.

Label	(x, y, z)	Ω	Stability	$a(t)$
$\mathcal{A}_\pm$	$(\pm 1, 0, 0)$	0	sources	$t^{1/3}$
$\mathcal{B}$	$(0, 0, 0)$	1	saddle	$t^{2/3}$
$\mathcal{C}_+$	$\left(\sqrt{\frac{1}{6}}, \sqrt{\frac{5}{6}}, 0\right)$	0	sink	t^2
$\mathcal{D}_+$	$(0, \sqrt{2}, i)$	0	saddle	e^t

B. Radiation ($\gamma = 4/3$). The case of $\gamma = 4/3$ corresponds to radiation, and therefore there is no matter point with a scale factor $a \sim t^{2/3}$. Instead, point $\mathcal{B}$, which coincides with the origin $(0, 0, 0)$, now represents the well-known radiation dominated solution, $a \sim t^{1/2}$, as a transient phase. $\mathcal{C}_+$ is a future attractor for $\alpha < \sqrt{2}$. For the case of radiation and $\alpha = 1, \beta = 2$, see Tables 3.4, 3.5.

Table 3.4: Case of radiation and $V_1 > 0, V_2 > 0$, $\alpha = 1$, $\beta = 2$.

Label	(x, y, z)	Ω	Stability	$a(t)$
$\mathcal{A}_\pm$	$(\pm 1, 0, 0)$	0	sources	$t^{1/3}$
$\mathcal{B}$	$(0, 0, 0)$	1	saddle	$t^{1/2}$
$\mathcal{C}_+$	$\left(\sqrt{\frac{1}{6}}, \sqrt{\frac{5}{6}}, 0\right)$	0	sink	t^2

C. The value $\gamma = 2/3$ corresponds to ordinary matter marginally satisfying the strong energy condition. Eq. (3.42) implies $Q = \sqrt{2/3}$. An acceptable trajectory exists for $\alpha < \sqrt{2}$. For these values of α and Q, points $\mathcal{A}_\pm$ are always unstable. Point $\mathcal{B} \equiv (1/2, 0, 0)$, corresponds to the

Table 3.5: Case of radiation and $V_1 > 0, V_2 < 0$, $\alpha = 1$, $\beta = 2$.

Label	(x, y, z)	Ω	Stability	$a(t)$
$\mathcal{A}_\pm$	$(\pm 1, 0, 0)$	0	sources	$t^{1/3}$
$\mathcal{B}$	$(0, 0, 0)$	1	saddle	$t^{1/2}$
$\mathcal{C}_+$	$\left(\sqrt{\frac{1}{6}}, \sqrt{\frac{5}{6}}, 0\right)$	0	sink	t^2
$\mathcal{D}_+$	$(0, \sqrt{2}, i)$	0	saddle	e^t

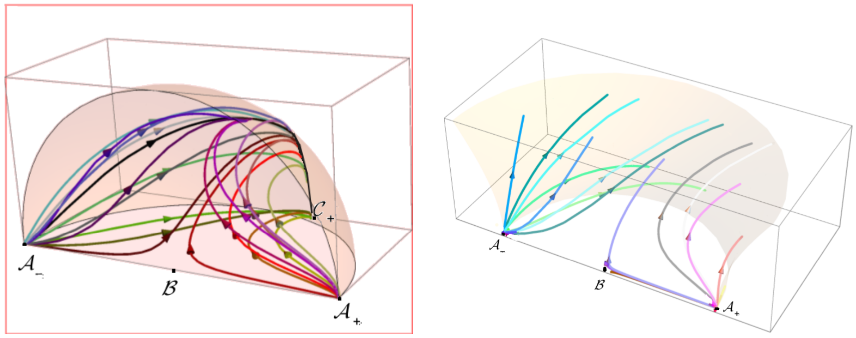
transient matter era, with $\Omega = 3/4$. The accelerated point $\mathcal{C}_+$ is a future attractor. For the case of ordinary matter and $\alpha = 1, \beta = 2$, see Tables 3.6, 3.7.

Table 3.6: Case where $\gamma = 2/3$ and $V_1 > 0, V_2 > 0$, $Q = \sqrt{2/3}$, $\alpha = 1$, $\beta = 2$.

Label	(x, y, z)	Ω	Stability	$a(t)$
$\mathcal{A}_\pm$	$(\pm 1, 0, 0)$	0	sources	$t^{1/3}$
$\mathcal{B}$	$\left(\frac{1}{2}, 0, 0\right)$	$\frac{3}{4}$	saddle	$t^{2/3}$
$\mathcal{C}_+$	$\left(\sqrt{\frac{1}{6}}, \sqrt{\frac{5}{6}}, 0\right)$	0	sink	t^2

Table 3.7: Case where $\gamma = 2/3$ and $V_1 > 0, V_2 < 0$, $Q = \sqrt{2/3}$, $\alpha = 1$, $\beta = 2$.

Label	(x, y, z)	Ω	Stability	$a(t)$
$\mathcal{A}_\pm$	$(\pm 1, 0, 0)$	0	sources	$t^{1/3}$
$\mathcal{B}$	$\left(\frac{1}{2}, 0, 0\right)$	$\frac{3}{4}$	saddle	$t^{2/3}$
$\mathcal{C}_+$	$\left(\sqrt{\frac{1}{6}}, \sqrt{\frac{5}{6}}, 0\right)$	0	sink	t^2
$\mathcal{D}_+$	$(0, \sqrt{2}, i)$	0	saddle	e^t



(a) (b)

Figure 3.6: Phase plots indicating cosmologically viable trajectories, with $0 < \alpha < \beta$. (a) $V_1, V_2 > 0$, (b) $V_1 > 0, V_2 < 0$

3.6 Asymptotic form of some $f(R)$ theories predicting acceleration

A large class of dynamical dark energy models, [48, 108–111], is based on the large-distance modification of gravity. We consider higher order gravity theories in vacuum derived from Lagrangians of the form

$$\mathcal{L} = f(R)\sqrt{-g}, \tag{3.43}$$

in the Einstein frame. The corresponding potential (see Appendix B) is given by

$$V(\phi) = \frac{1}{2(f')^2}\left(Rf' - f\right).$$

Conformal transformation yields the field equations in the Einstein frame

$$\tilde{G}_{\mu\nu} = T_{\mu\nu}(\tilde{g}, \phi) + \tilde{T}_{\mu\nu}(\tilde{g}, \Psi). \tag{3.44}$$

Models of the form

$$f(R) = R - \frac{\mu^{2(n+1)}}{R^n},$$

where $\mu > 0,\ n > 1$, were proposed to explain the late-time cosmic acceleration in the context of $f(R)$ gravity theories, [7,54]. The obvious idea is the introduction of modifications to the Einstein-Hilbert Lagrangian which become important at low curvatures. For these models the potential functions in the Einstein frame have the form

$$V_n(\phi) = \frac{\mu^2(n+1)n^{1/(n+1)}(e^{\sqrt{2/3}\phi} - 1)^{n/(n+1)}}{2ne^{2\sqrt{2/3}\phi}}. \tag{3.45}$$

These functions are defined only for $\phi \geq 0$, and their behaviour is similar to that indicated in Figure 3.2, i.e., they have a local maximum at some ϕ_m depending on n, and for large ϕ they approach zero exponentially. As $n \to \infty$ the potentials (3.45) approach the function

$$V(\phi) = \frac{\mu^2}{2}\left(e^{-\sqrt{2/3}\phi} - e^{-2\sqrt{2/3}\phi}\right), \tag{3.46}$$

corresponding to the asymptotic form of these theories, [7]. Thus, (3.46) is a particular case of the double exponential (3.15), with

$$\beta = 2\alpha = 2\sqrt{\frac{2}{3}},\ V_1 = -V_2 = \frac{\mu^2}{2} > 0,$$

cf. Figure 3.2.

Note that for large ϕ, V in (3.46) behaves similarly to V_n in (3.45). In contrast to the family (3.45), V in (3.46) has the nice property that it is defined for all $\phi \in \mathbb{R}$.

For the viable conditions of the asymptotic form of the potentials (3.45), the constraint (3.28) implies that the phase space is the set

$$x^2 + y^2 - (\operatorname{Im} z)^2 \leq 1.$$

There are up to five critical points for that system, depending on the value of γ. Point $\mathcal{C}_+$ is a future attractor and has non phantom acceleration with

Table 3.8: Equilibrium points for potentials (3.46)

Label	(x, y, z)	Ω	Existence	Stability	$a(t)$
$\mathcal{A}_\pm$	$(\pm 1, 0, 0)$	0	always	unstable	$t^{1/3}$
$\mathcal{B}$	$\left(\frac{4-3\gamma}{3(2-\gamma)}, 0, 0\right)$	$\frac{4(5-3\gamma)}{9(2-\gamma)^2}$	$\gamma \leq 5/3$	saddle	$t^{3(2-\gamma)/(8-3\gamma)}$
$\mathcal{C}_+$	$\left(\frac{1}{3}, \frac{2\sqrt{2}}{3}, 0\right)$	0	always	stable	t^3
$\mathcal{D}_+$	$(0, \sqrt{2}, i)$	0	always	saddle	e^t

$w_{\text{eff}} = -7/9$. However, in the case of dust, $\gamma = 1$, the scale factor at matter point $\mathcal{B}$ evolves as $a \sim t^{3/5}$, rather than the usual $a \sim t^{2/3}$. The scale factor evolves "correctly" only for $\gamma = 2/3$. The absence of the standard matter epoch is associated with the fact that matter is strongly coupled to gravity. This result is in agreement with the general conclusions in [39,40,112], that these $f(R)$ dark energy models are not cosmologically viable. In Table 3.6 we summarise the properties of the equilibria.

3.7 Conclusion

In this chapter we have focused on a general treatment of scalar fields with a double exponential potential non-minimally coupled to a perfect fluid. A full analysis of the equilibrium points of the resulted dynamical system is quite complicated, yet it revealed that the model predicts a late accelerated phase of the Universe for a wide range of the parameters, α, β, γ and Q. Moreover, there exists transient solutions representing a matter era, preceding the accelerating attractor. However, in most cases the scale factor near these transient phases evolves as $a(t) \sim t^{q(Q)}$, where the exponent q is in general different from the usual $2/3$. The "wrong" matter epoch is associated with the fact that for values of Q of order unity, matter is strongly coupled to gravity. A coupling constant of order unity means that matter feels an additional scalar force as strong as gravity itself, cf. [39]. Assuming that ordinary matter satisfies plausible energy conditions, i.e., $\gamma \gtrsim 1$, the coupling constant, Q, has to be very small; more precisely, $q\left(Q\right) \rightarrow 2/3$, only for $Q \rightarrow 0$. Therefore, only a very weak coupling of the scalar field to ordinary matter can lead to acceptable cosmological histories of the Universe. This surprising result, indicates that cosmological evolution imposes strict constraints on the choice of the correct Lagrangian of a gravity theory. In this study we restricted ourselves to constant couplings; had we let Q to be a function of ϕ obeying a proper evolution equation, the dimension of the dynamical system would have increased by one. In that case, it would be very interesting to see if the dynamics leads to a very tiny value of Q at late times. Such a result could lead to a generalisation of the attractor mechanism of scalar-tensor theories towards General Relativity, found by Damour and Nordtvedt in the case of a massless scalar field [113, 114].

Table 3.9: Critical Points of system (3.27)

Label	(x, y, z)	Ω	w_{eff}
$\mathcal{A}_{\pm}$	$(\pm 1, 0, 0)$	0	1
$\mathcal{B}$	$\left(\frac{(4-3\gamma)Q}{\sqrt{6}(2-\gamma)}, 0, 0\right)$	$1 - \frac{(4-3\gamma)^2 Q^2}{6(2-\gamma)^2}$	$-1 + \gamma + \frac{(4-3\gamma)^2 Q^2}{6(2-\gamma)}$
$\mathcal{C}_{\pm}$	$\left(\frac{\alpha}{\sqrt{6}}, \pm\sqrt{1 - \frac{\alpha^2}{6}}, 0\right)$	0	$-1 + \frac{\alpha^2}{3}$
$\mathcal{D}_{\pm}$	$\left(0, \pm\sqrt{\frac{\beta}{\beta-\alpha}}, \pm\sqrt{\frac{\alpha}{\alpha-\beta}}\right)$	0	-1
$\mathcal{D}'_{\pm}$	$\left(0, \pm\sqrt{\frac{\beta}{\beta-\alpha}}, \mp\sqrt{\frac{\alpha}{\alpha-\beta}}\right)$	0	-1
$\mathcal{E}_{\pm}$	$\left(\frac{\sqrt{6}\gamma}{2\alpha-(4-3\gamma)Q}, \pm\sqrt{\frac{(4-3\gamma)^2Q^2-2\alpha(4-3\gamma)Q+6\gamma(2-\gamma)}{(2\alpha-(4-3\gamma)Q)^2}}, 0\right)$	$\frac{2\left(2\alpha^2-6\gamma-\alpha(4-3\gamma)Q\right)}{(2\alpha-(4-3\gamma)Q)^2}$	$-1 + \sqrt{\frac{2}{3}}\alpha\frac{\sqrt{6}\gamma}{2\alpha-(4-3\gamma)Q}$
$\mathcal{F}_{\pm}$	$\left(\frac{\beta}{\sqrt{6}}, 0, \pm\sqrt{1 - \frac{\beta^2}{6}}\right)$	0	$-1 + \frac{\beta^2}{3}$
$\mathcal{G}_{\pm}$	$\left(\frac{\sqrt{6}\gamma}{2\beta-(4-3\gamma)Q}, 0, \pm\sqrt{\frac{(4-3\gamma)^2Q^2-2\beta(4-3\gamma)Q+6\gamma(2-\gamma)}{(2\beta-(4-3\gamma)Q)^2}}\right)$	$\frac{2\left(2\beta^2-6\gamma-\beta(4-3\gamma)Q\right)}{(2\beta-(4-3\gamma)Q)^2}$	$-1 + \sqrt{\frac{2}{3}}\beta\frac{\sqrt{6}\gamma}{2\beta-(4-3\gamma)Q}$

Table 3.10: Eigenvalues of system (3.27)

Label	Eigenvalues
$\mathcal{A}_{+}$	$3 - \sqrt{\frac{3}{2}}\alpha,\ 3 - \sqrt{\frac{3}{2}}\beta,\ 6 - 3\gamma - \sqrt{\frac{3}{2}}\left(4 - 3\gamma\right)Q,$
$\mathcal{A}_{-}$	$3 + \sqrt{\frac{3}{2}}\alpha,\ 3 + \sqrt{\frac{3}{2}}\beta,\ 6 - 3\gamma + \sqrt{\frac{3}{2}}\left(4 - 3\gamma\right)Q,$
$\mathcal{B}$	$\frac{(4-3\gamma)^2Q^2-2\alpha(4-3\gamma)Q+6\gamma(2-\gamma)}{4(2-\gamma)},\ \frac{(4-3\gamma)^2Q^2-2\beta(4-3\gamma)Q+6\gamma(2-\gamma)}{4(2-\gamma)},\ \frac{(4-3\gamma)^2Q^2-6(2-\gamma)^2}{4(2-\gamma)}$
$\mathcal{C}_{\pm}$	$\frac{\alpha^2-6}{2},\ \frac{\alpha(\alpha-\beta)}{2},\ \frac{2\alpha^2-6\gamma-\alpha(4-3\gamma)Q}{2}$
$\mathcal{D}_{\pm}$	$\frac{-3+\sqrt{9+12\alpha\beta}}{2},\ \frac{-3-\sqrt{9+12\alpha\beta}}{2},\ -3\gamma$
$\mathcal{D}'_{\pm}$	$\frac{-3+\sqrt{9+12\alpha\beta}}{2},\ \frac{-3-\sqrt{9+12\alpha\beta}}{2},\ -3\gamma$
$\mathcal{E}_{\pm}$	$\frac{3(\alpha-\beta)\gamma}{2\alpha-(4-3\gamma)Q},\ \frac{3(2\alpha-(4-3\gamma)Q)((4-3\gamma)Q-\alpha(2-\gamma))\pm\sqrt{(3(2\alpha-(4-3\gamma)Q)((4-3\gamma)Q-\alpha(2-\gamma)))^2-4\frac{3}{2}(2\alpha-(4-3\gamma)Q)^2(2\alpha^2-6\gamma-\alpha(4-3\gamma)Q)\left((4-3\gamma)^2Q^2-2\alpha(4-3\gamma)Q+6\gamma(2-\gamma)\right)}}{2(2\alpha-(4-3\gamma)Q)^2}$
$\mathcal{F}_{\pm}$	$\frac{\beta^2-6}{2},\ \frac{\beta(\beta-\alpha)}{2},\ \frac{2\beta^2-6\gamma-\beta(4-3\gamma)Q}{2}$
$\mathcal{G}_{\pm}$	$\frac{3(\beta-\alpha)\gamma}{2\beta-(4-3\gamma)Q},\ \frac{3(2\beta-(4-3\gamma)Q)((4-3\gamma)Q-\beta(2-\gamma))\pm\sqrt{(3(2\beta-(4-3\gamma)Q)((4-3\gamma)Q-\beta(2-\gamma)))^2-4\frac{3}{2}(2\beta-(4-3\gamma)Q)^2(2\beta^2-6\gamma-\beta(4-3\gamma)Q)\left((4-3\gamma)^2Q^2-2\beta(4-3\gamma)Q+6\gamma(2-\gamma)\right)}}{2(2\beta-(4-3\gamma)Q)^2}$

Chapter 4

Negative Potentials

Although scalar fields having non-negative potentials in FLRW models have been studied by several authors, there is a small number of papers with mathematically rigorous results [78, 79, 115–123]. On the other hand, up to our knowledge, there is no rigorous mathematical treatment of cosmological models with negative potentials apart from [124, 125].

As we shall see, almost always initially expanding Universes recollapse. The physical reason to understand why the Universe eventually collapses when $V < 0$, is that in the Friedmann equation,

$$3H^2 = \rho_{\text{total}},$$

the positive energy density of ordinary matter, as well as the positive kinetic energy density of the scalar field, decreases in an expanding Universe. At some moment, the total energy density ρ_{total}, including the negative contribution $V(\phi) < 0$, vanishes. Once it happens, the Universe, stops expanding and enters the stage of irreversible collapse [89].

In this chapter the investigation of scalar fields is extended to models with negative potentials. It is shown rigorously that almost always initially expanding Universes eventually collapse, independently of the particular functional form of the potential. Collapsing models were built using homogeneous scalar field solutions in [115, 121, 123, 126, 127]. The case where a scalar field is coupled to a perfect fluid was studied in [122,128]. The chapter is organized as follows. In the next section we classify negative potentials studied in the literature into five general classes. We analyse all possible limit sets of the dynamical system and prove a number of propositions that lead to the proof of our main result, that the Hubble function almost always diverges to $-\infty$ in a finite time. In Section 4.2 we consider the remaining forms of the double exponential potentials as an example to our results.

4.1 Potentials taking negative values

There are several reasons to study cosmology with negative potentials, see for example [105]. First of all, cosmology with negative potentials is related to the cosmological constant problem. The simplest potential used in inflationary cosmology has the form of a quadratic polynomial, $V(\phi) = \alpha^2\phi^2$, [129]. Adding a small positive constant of order of magnitude in Planck units 10^{-120}, does not change any features of inflation and can be used to describe the current observed acceleration of the Universe. As stated in [105], surprisingly enough, the same conclusion does not hold if we add a negative constant instead. After a long period of inflation the Universe with $V_0 < 0$ does not behave like anti-de Sitter space as expected, but it collapses, [101, 102, 130].

A second reason to study cosmology with negative potentials is that potentials with a global positive value of maximum and unbounded from below, under certain conditions are able to describe the present stage of inflation, [101, 102]. These include cosmological models in $N = 2, 4, 8$ gauged supergravity [101, 131], as well as double exponential potentials studied by several authors [82, 90–98, 132]; double exponential potentials with non-minimal coupling were studied in [133]. The physical interest of these potentials is described in [89], where it is shown that if initially the field ϕ_0 is near the value corresponding to the maximum of the potential, it takes time $t \sim 0.7 H_0^{-1} \ln \phi_0^{-1}$, until the field rolls down from ϕ_0 to the region where $V(\phi)$ becomes negative and the Universe collapses. This time is comparable to the age of our Universe, H_0^{-1}, and therefore it is possible that the present Universe is into an accelerated phase, yet it will collapse in about 18 billion years. For detailed cosmological implications see [89, 102, 103].

Other reasons include the relation between cosmology with negative potentials to the cyclic Universe scenario.

We classify potentials taking negative values into the following five families.

A. Potentials having a global positive maximum, $\lim_{\phi\to\infty} V(\phi) = 0$, and are free to fall to $-\infty$ as $\phi \to -\infty$. An example is the double exponential potential

$$V(\phi) = V_1 e^{-\alpha\phi} + V_2 e^{-\beta\phi}, \quad V_1, \alpha, \beta > 0, \quad V_2 < 0,$$

considered in Chapter 3, see Fig. 4.1

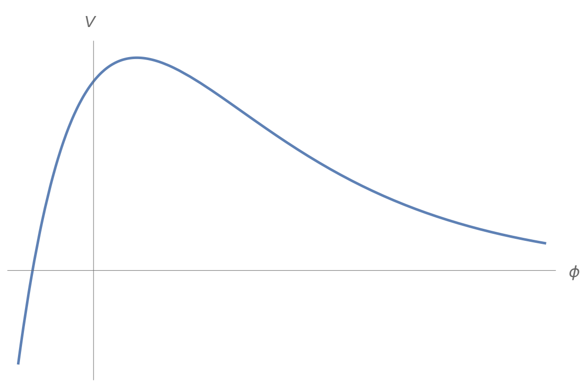


Figure 4.1: Potential falling into Class A

B. Potentials having a global positive maximum and $\lim_{\phi\to\pm\infty} V(\phi) = -\infty$.

Near the maximum, say at $\phi = 0$, they can be represented as

$$V(\phi) = V_0 - \frac{m^2}{2}\phi^2,$$

cf. [105]. An example is the potential

$$V(\phi) = V_0\left(2 - \cosh\left(\sqrt{2}\phi\right)\right), \quad V_0 > 0,$$

considered in [89]. Potentials of this class appear in cosmological models in $N = 2, 4, 8$ gauged supergravity [101, 131]. For detailed cosmological implications see [89, 102, 103], see for example Fig. 4.2

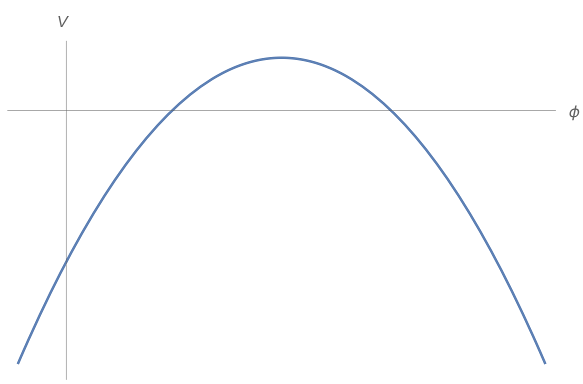


Figure 4.2: Potential falling into Class B

C. Potentials having a negative minimum. Two important examples include the ekpyrotic potentials and those used in models of cyclic Universes, see for example Fig. 4.3; for reviews see Refs. [134], [135, 136].

D. Bounded from below potentials with no minimum, Fig. 4.4. As an example, we mention the potentials

$$V(\phi) = V_0 e^{-\lambda\phi} - C, \quad V_0, C, \lambda > 0,$$

which were considered in the context of supersymmetry theories, see for

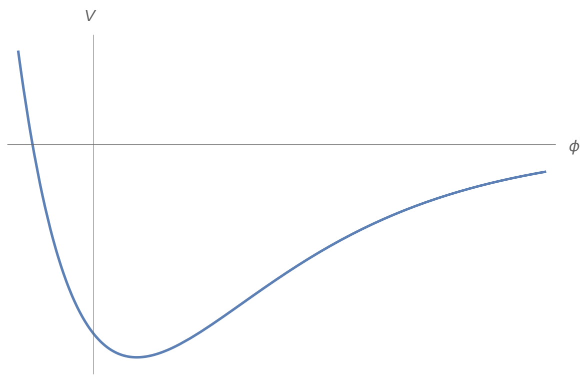


Figure 4.3: Potential falling into Class C

example Ref. [89].

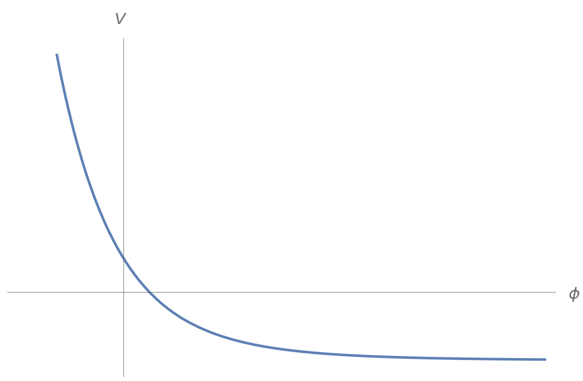


Figure 4.4: Potential falling into Class D

E. Potentials with $V(\phi)$ decreasing from $+\infty$ to $-\infty$, for example

$$V(\phi) = W_0 - V_0 \sinh(\lambda\phi), \qquad \lambda, V_0 > 0,$$

see Fig. 4.5, (see [106] where an exact solution was obtained in the absence of matter).

Up to now we have considered only constant coupling coefficients. In this chapter we consider more general couplings by assuming the coupling

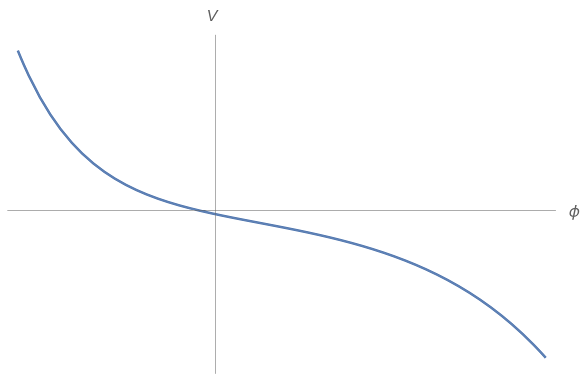


Figure 4.5: Potential falling into Class E

coefficient Q to be a positive and bounded function of class C^1 such that,

$$Q_{\pm} := \lim_{\phi \to \pm\infty} Q(\phi) > 0. \tag{4.1}$$

4.1.1 Potentials falling into Class A

In this section we prove the Theorem 3 stated in the previous chapter. Our system is again (3.8)–(3.11) with $Q = Q(\phi)$ satisfying (4.1). We suppose that a potential $V(\phi)$ satisfies the following assumption.

Assumption 3 *We assume that $V(\phi) \in C^2$ is such that*

1. $\lim_{\phi \to -\infty} V(\phi) = -\infty$ *and* $\lim_{\phi \to +\infty} V(\phi) = 0$.

2. *The potential has a unique critical point $\phi_m > 0$, with $V(\phi_m) > 0$, i.e., in view of (1), the ϕ_m has to be a global maximum. Moreover ϕ_m is non degenerate, i.e., $V''(\phi_m) < 0$.*

3. *There exist $\lambda > 0$ and $M < 0$ such that*

$$V'(\phi) \leq -\lambda V(\phi), \qquad \textit{for all } \phi < M. \tag{4.2}$$

Motivation for the Assumption 3 comes from the double exponential potential of Chapter 3

$$V(\phi) = V_1 e^{-\alpha\phi} + V_2 e^{\beta\phi}, \quad 0 < \alpha < \beta, \quad V_1 > 0, V_2 < 0,$$

see Fig. 3.2. In particular, condition (4.2) establish a bound for the growth of $|V(\phi)|$ to infinity, that must be at most exponential.

We will also assume that the function $Q(\phi)$, is bounded for all $\phi \in \mathbb{R}$; in particular, we suppose the existence of a constant A such as

$$|\tfrac{4-3\gamma}{2} Q(\phi)| \leq A. \tag{4.3}$$

The dynamical system (3.8)–(3.11) has only two finite equilibrium points,

$$\left(\phi = \phi_m, \dot{\phi} = 0, \rho = 0, H = \pm\sqrt{V_{\max}/3}\right).$$

They represent de Sitter and anti-de Sitter solutions and it is easy to see that are unstable. It is known that for potentials with a maximum, the field near the top of the potential corresponds to the tachyonic (unstable) mode with negative mass squared [101, 131]. The other asymptotic states of the system correspond to the points at infinity, $\phi \to \pm\infty$.

As stated in Chapter 3, it can be seen that if initially $\phi(0) \equiv \phi_0 < \phi_m$, then there is a critical value $\dot{\phi}_{\text{crit}} > 0$, which allows for ϕ to pass on the right of ϕ_m. More precisely, in the case of zero coupling, $Q = 0$, it is easy to show that there exists a critical value of $\dot{\phi}$, say $\dot{\phi}_{\text{crit}}$, such that if $\phi_0 < \phi_m$ and $\dot{\phi}(0) < \dot{\phi}_{\text{crit}}$, then $\phi(t)$ remains less than ϕ_m for all $t \geq 0$. The argument is similar to the mechanical analogue of the motion of a particle in the potential $V(\phi)$, according to Eq. (3.6). In the case of non-minimal coupling, the energy density of the scalar field is not necessarily decreasing, because there is an energy exchange between the scalar field and the fluid.

An estimation of the maximum allowable value of $\dot{\phi}_0$ can be obtained from (3.4), supposing that initially $3H^2(t_0) \leq V(\phi_m) = V_{\max}$. Indeed, since by Eq. (3.5) H is decreasing

$$\rho(t) + \frac{1}{2}\dot{\phi}(t)^2 + V(\phi(t)) \leq V_{\max}, \quad \text{for all } t \geq 0, \tag{4.4}$$

which implies that $V(\phi(t)) \leq V_{\max}$ for all $t \geq 0$, and therefore $\phi(t) < \phi_m$ for all $t \geq 0$. Moreover, inequality (4.4) and initial condition on $H(t)$ establish a maximum allowable value of $\dot{\phi}_{\text{crit}}$,

$$\dot{\phi}_{\text{crit}} \leq \sqrt{2V_{\max}}.$$

Therefore *if* $0 < H(t_0) \leq \sqrt{V_{\max}/3}$, *then* $\phi(t)$ *never crosses the maximum of* $V(\phi)$ *throughout the evolution.*

The following results are crucial for our study.

Lemma 2 *Let* $\gamma(t) = (\phi(t), y(t), \rho(t), H(t))$ *be a bounded solution such that* $\rho(t_0) > 0$. *Then* $\gamma(t) \in W^s(\mathbf{q})$, *where* $W^s(\mathbf{q})$ *is the stable manifold of an equilibrium point* $\mathbf{q}$ *and* $\mathbf{q}_\pm = \left(\phi_m, 0, 0, \pm\sqrt{\frac{V(\phi_m)}{3}}\right)$ *are the equilibria of the system.*

Proof. Let $\mathbb{I} = [t_0, t_m)$, where t_m is the supremum of the maximal right extension of γ, and $\Omega(t) = \{\gamma(t) : t \in \mathbb{I}\} \cup L^+(\gamma)$, where $L^+(\gamma)$ is the positive limit set of γ. Equation (3.11) implies that $\dot{H} \leq 0$ on Ω. Let $E = \{x \in \Omega : \dot{H} = 0\} = \{x \in \Omega : y = \rho = 0\}$, and let $\eta(t)$ be a solution to the system such that $\eta(t_0) \in E$ and $\eta(t) \in E$, for all $t \geq t_0$. It follows that $y(t) = \rho(t) = 0$, for all $t \geq t_0$ and, from Eq. (3.8), $\phi(t) = \phi_0$ constant for all $t \geq t_0$. From Eq. (3.9) we have that $V'(\phi_0) = 0$ and then $\phi_0 = \phi_m$. Since $\dot{H} = 0$, $H(t)$ is constant and from Eq. (3.12) it must be $H = \pm\sqrt{\frac{V(\phi_m)}{3}}$. Therefore $E = \{\mathbf{q}_\pm\}$, i.e. is made by the two equilibria of the system. LaSalle invariance principle (see Appendix C) and monotonicity of $H(t)$

ensure that $\gamma(t)$ converges to either $\mathbf{q}_+$ or $\mathbf{q}_-$, and then it belongs to the stable manifold of one of the two equilibria. ■

Remark 1 *As a consequence of the above fact, we can show that future bounded trajectories of the system* (3.8)–(3.12) *with* $\rho(t_0) > 0$ *are non generic. Indeed, let us first observe that, using Eq.* (3.12)*, we can rewrite Eq.* (3.11) *as follows*

$$\dot{H} = -3H^2 + \left(1 - \frac{\gamma}{2}\right)\rho + V(\phi). \tag{4.5}$$

Then, let us consider the equivalent system (3.8)–(3.10) *with Eq.* (4.5)*, and study the Jacobi matrix computed at the equilibria* $\mathbf{q}_\pm$*. Since* ϕ_m *is a non degenerate critical point for* $V(\phi)$*, we obtain that the stable manifold of* $\mathbf{q}_+$ *is* 3*–dimensional and the stable manifold of* $\mathbf{q}_-$ *is* 1*–dimensional. In the latter case the result straightly follows from the previous proposition. Also for the equilibrium* $\mathbf{q}_+$*, the result follows, taking some more care due to the fact that, actually, Eq.* (3.12) *selects a* 3*–dimensional submanifold of initial data, which anyway can be easily checked to be transversal to* $W^+(\mathbf{q}_+)$ *at* $\mathbf{q}_+$.

By the above result one can expect in principle that solutions of the system (3.8)–(3.12) are generically, i.e., up to a zero–measured set of initial data, unbounded, and our aim is now to study their qualitative behaviour. The following result will provide sufficient conditions for a future singularity.

Lemma 3 *Let* $\gamma(t)$ *be a solution to the system* (3.8)–(3.12)*. If there exists* $t_1 \geq t_0$ *and* $\bar{V} \in \mathbb{R}$ *such that, for all* $t \geq t_1$*,* $V(\phi(t)) \leq \bar{V}$*, and either (i)* $\bar{V} < 0$*, or (ii)* $H(t_1) < -\sqrt{\frac{\bar{V}}{3}}$*, then* $H(t)$ *negatively diverges in a finite time, i.e. the property*

$$\exists t_* > 0 \text{ such that } \lim_{t \to t_*^-} H(t) = -\infty.$$

holds.

Proof. To show the above, we use Eq. (3.11) and recalling that $0 \leq \gamma \leq 2$, we have for $t \geq t_1$

$$\dot{H} \leq \frac{\gamma}{2}\left(-3H^2 + \bar{V}\right). \tag{4.6}$$

Therefore, considering the Cauchy problem

$$\dot{Z}(t) = \frac{\gamma}{2}\left(-3Z(t)^2 + \bar{V}\right), \qquad Z(t_1) = H(t_1),$$

its solution $Z(t)$ is easily seen to diverge to $-\infty$ in a finite time. The result follows from comparison theorems in ODE theory. ■

We now prove the Theorem 3 conjectured in Chapter 3.

Theorem 4 *Let* $\gamma(t) = (\phi(t), y(t), \rho(t), H(t))$ *a solution to the system* (3.8)–(3.12) *such that* $\phi(t_0) < \phi_m$*,* $\rho(t_0) > 0$*,* $H(t_0) > 0$*, and* $y(t_0) < \dot{\phi}_{crit}$*, where* $\dot{\phi}_{crit}$ *is the critical value that allows for* ϕ *to pass to the right of* ϕ_m*. Then* $H(t)$ *generically negatively diverges in a finite time:*

$$\exists t_* > 0 \text{ such that } \lim_{t \to t_*^-} H(t) = -\infty. \tag{4.7}$$

Proof. According to Remark 1 bounded trajectories of the system (3.8)–(3.12) are non generic, we can only consider unbounded solutions without losing genericity. Then at least one of the components of $\gamma(t)$ is unbounded. If $H(t)$ is unbounded, then since by Eq. (3.11), $H(t)$ is decreasing, then it must be negatively unbounded, and then Lemma 3 immediately gives the result, recalling that $V(\phi)$ is bounded from above. For the rest of the proof we will argue by contradiction, and show that $H(t)$ must be necessarily unbounded.

So, suppose by contradiction that $H(t)$ is bounded. Then, assuming for sake of simplicity that $t_0 = 0$, Eq. (3.11) implies that there exists a constant

$K > 0$ such that $|H(t)| < K$, and also

$$\int_0^t \frac{1}{2} y^2(s)\, ds < K, \int_0^t \rho(s)\, ds < K, \qquad \text{for all } t \geq 0. \tag{4.8}$$

Moreover, since

$$3H^2 - V(\phi) = \frac{1}{2} y^2 + \rho,$$

and the solution must be unbounded, then either y^2 or ρ (or both) are unbounded (otherwise, $V(\phi)$ would be bounded that implies that ϕ is positively unbounded, which is excluded since $\phi(t) < \phi_m$) and then from Eq. (3.12) also $V(\phi)$ is negatively unbounded.

Suppose that y^2 is bounded and ρ is unbounded. If ρ diverges to ∞ then by Eq. (3.12) also $V(\phi)$ diverges (to $-\infty$). Therefore, hypotheses from Lemma 3 are satisfied which would imply that $H(t)$ is unbounded, contradiction. Then ρ cannot diverge to ∞, and as a consequence there exists an increasing sequence $\{t_n\}$ such that $\rho(t_{2n}) \to +\infty$ and $\rho(t_{2n-1}) < \bar{\rho}$ for some fixed $\bar{\rho}$. Moreover,

$$\rho(t_{2n}) - \rho(t_{2n-1}) = \int_{t_{2n-1}}^{t_{2n}} \dot{\rho}(t)\, dt = \int_{t_{2n-1}}^{t_{2n}} -\rho(t)(3\gamma H(t) + \frac{4-3\gamma}{2} Q y(t))\, dt,$$

and boundedness of both y and H implies the existence of some positive constant C such that

$$\rho(t_{2n}) - \rho(t_{2n-1}) \leq C \int_{t_{2n-1}}^{t_{2n}} \rho(t)\, dt \leq C\, K,$$

that is a contradiction because the left hand side diverges. Then y^2 must necessarily be unbounded, and let us now show that even in this case we get a contradiction. To begin, observe that Eq. (3.12) implies

$$\tfrac{1}{2} y^2 = 3H^2 - \rho - V(\phi) \leq -V(\phi) + 3K^2, \tag{4.9}$$

where we have also used $|H(t)| < K$, for all $t \geq 0$. Let t_n an increasing sequence such that $y^2(t_n) \to +\infty$. Then $\phi(t_n) < M$ eventually, where M has been defined in Eq. (4.2) (otherwise $V(\phi(t_n))$ would be bounded and then, from Eq. (4.9), $y^2(t_n)$ would be). Now, if $\phi(t) < M$ is eventually satisfied for all t sufficiently large (not only on the t_n's, namely) then $V(\phi(t)) < V(M) < 0$ eventually, and therefore the hypotheses in Lemma 3 would be satisfied, that would mean that $H(t)$ is unbounded. If, on the other side, there exists an increasing sequence s_n such that $s_n < t_n < s_{n+1}$, $\phi(s_n) = M$ and $\phi(t) < M$ in (s_n, t_n), then it must be, by Eq. (4.9),

$$\frac{1}{2}y^2(s_n) \leq -V(M) + 3K^2,$$

and therefore, using also the growth Assumption 3 made on V and Eqs. (3.12) and (4.8),

$$\begin{aligned}
|y(t_n)| \leq & |y(s_n)| + \left| \int_{s_n}^{t_n} \dot{y}(t)\, dt \right| \\
\leq & \sqrt{2(-V(M) + 3K^2)} + 3 \int_{s_n}^{t_n} |H(t)y(t)|\, dt \\
& + \int_{s_n}^{t_n} V'(\phi(t))\, dt + A \int_{s_n}^{t_n} \rho(t)\, dt \\
\leq & \sqrt{2(-V(M) + 3K^2)} + \frac{3}{2} \int_{s_n}^{t_n} H^2(t)\, dt \\
& + \frac{3}{2} \int_{s_n}^{t_n} y^2(t)\, dt + \lambda \int_{s_n}^{t_n} \left(-V(\phi(t))\right)\, dt + AK \\
\leq & \sqrt{2(-V(M) + 3K^2)} + \frac{1}{2} \int_{s_n}^{t_n} 3H^2(t)\, dt \\
& + \lambda \int_{s_n}^{t_n} \left(-V(\phi(t))\right)\, dt + (3 + A)\, K \\
\leq & \sqrt{2(-V(M) + 3K^2)} + (3 + A)K + (1 + \lambda) \int_{s_n}^{t_n} \left(3H^2(t) - V(\phi(t))\right)\, dt \\
= & \sqrt{2(-V(M) + 3K^2)} + (3 + A)K + (1 + \lambda) \int_{s_n}^{t_n} \left(\frac{1}{2}y^2(t) + \rho(t)\right)\, dt \\
\leq & \sqrt{2\left(-V(M) + 3K^2\right)} + K\left(A + 5 + 2\lambda\right),
\end{aligned}$$

that is a contradiction since $|y(t_n)|$ positively diverges. This means that $H(t)$ cannot be bounded and therefore the result follows, as said in the very first part of this argument, from Lemma 3. ■

4.1.2 Potentials falling into Class B

Potentials falling into class B allow for a similar treatment as potentials of class A. In fact potentials of class B do not have the complication of potentials in class A, since they diverge to $-\infty$ on both directions. Hence, in this section we assume that $V(\phi)$ is a potential satisfying the following assumption.

Assumption 4 *Let $V(\phi) \in \mathcal{C}^2$ such that*

1. $\lim_{\phi \to \pm\infty} V(\phi) = -\infty$,
2. *V has a unique nondegenerate critical point (that has to be, in view of* (1)*, the global maximum),*
3. *There exist $\lambda > 0$ and $M > 0$ such that, $|V'(\phi)| \leq -\lambda V(\phi)$, for all $\phi\,:\,|\phi| > M$.*

Theorem 5 *Let $V(\phi) \in \mathcal{C}^2$ satisfying the Assumption* (4)*. Then $H(t)$ generically negatively diverges in a finite time, i.e. the property*

$$\exists t_* > 0 \textit{ such that } \lim_{t \to t_*^-} H(t) = -\infty,$$

holds.

Proof. The argument follows the same line of the proof used for left unbounded potentials of class A. In the present case, potentials of class B do not have the complication of potentials B, diverging to $-\infty$ on both directions, and therefore, regardless of the behaviour of the scalar field, and recalling Lemma 2, it can be proved with exactly the same argument that, the solution recollapses almost always and the Hubble function negatively diverges in a finite time. ■

4.1.3 Potentials falling into Classes C-E

In the following we incorporate cases C–E into a large class of potentials $V(\phi) \in \mathcal{C}^2$ satisfying some further assumptions.

Assumption 5 *We assume that $V(\phi) \in \mathcal{C}^2$ is such that*

1. $\lim_{\phi\to-\infty} V(\phi) = +\infty$,
2. *There exists a unique $\phi_0 \in \mathbb{R} : V(\phi_0) = 0$. Moreover, V is strictly decreasing for all $\phi \leq \phi_0$,*
3. $\lim_{\phi\to\infty} \frac{V'(\phi)}{V(\phi)} = -\alpha \in \mathbb{R}$, *and* $\lim_{\phi\to-\infty} \frac{V'(\phi)}{V(\phi)} = -\beta \in \mathbb{R}$
4. $\lim_{\phi\to+\infty} V(\phi) = V_\infty \leq 0$ *(possibly $V_\infty = -\infty$).*
5. *There exists a $\mathcal{C}^2$–diffeomorphism, $f : (-\infty, \phi_0] \to [0, s_0)$, such that*
   (a) *The limit* $\lim_{\phi\to-\infty} f'(\phi)$, *exists and is equal to zero,*
   (b) $\lim_{\phi\to-\infty} f(\phi) = 0$,
   (c) $\lim_{\phi\to-\infty} \left[\frac{1}{f'(\phi)}\left(\frac{V''(\phi)}{V(\phi)} - \left(\frac{V'(\phi)}{V(\phi)}\right)^2\right)\right] = 0$,
   (d) $\lim_{\phi\to-\infty} \frac{f''(\phi)}{f'(\phi)} \in \mathbb{R}$.
6. *If $V_\infty = 0$, then there exists a $\phi_M > 0$, such that V is strictly increasing for $\phi \geq \phi_M$. Moreover, we make a similar hypothesis to* (5) *above for $\phi \to +\infty$, assuming the existence of a $\mathcal{C}^2$–diffeomorphism, $g(\phi) : [\phi_M, +\infty) \to (0, s_0]$ such that requests* (5a)–(5d) *hold for $g(\phi)$, as $\phi \to +\infty$.*

Assumptions (5)–(6) are required for situations where the scalar field possibly diverges. In those cases, the above diffeomorphisms are needed to bring a neighbourhood of infinity to a neighbourhood of the origin, [115,122]. It is easy to verify that cases C, D and E of negative potentials mentioned in the Introduction satisfy Assumption 5. In the next section we will examine the possible $\omega-$limit sets of the system, essentially depending on the asymptotic behavior of the scalar field $\phi(t)$. We will see that, except one case described in Proposition 4, solutions to (3.8)–(3.12) always recollapse to a singularity in a finite amount of time. For what follows in this section we suppose that $V(\phi)$ satisfies the Assumption 5. We also define $\mathbb{I}$ to be the maximal interval of definition of a solution to the system (3.8)–(3.12) and

$$\phi_\infty := \lim_{t\to\sup\mathbb{I}} \phi(t),$$

if it exists.

We are going to examine different situations depending on ϕ_∞.

Case $\phi_\infty = -\infty$

We firstly analyse the case E when the scalar field negatively diverges in such a way that $V(\phi(t)) \to +\infty$, see for example [106]. Note that we do not need to assume an a priori estimate on α, β.

Proposition 1 *If* $\phi_\infty = -\infty$ *then the property*

$$\exists t_* > 0 : \lim_{t\to t_*^-} H(t) = -\infty, \tag{4.10}$$

generically holds.

Proof. Since $V(\phi(t)) \to +\infty$, then $H^2(t) \to +\infty$. Eq. (3.10) implies that $\rho = 0$ is an invariant set. Then it follows that $\rho > 0$, if it was initially positive. Then, from Eq.(3.11), $H(t)$ is decreasing. Therefore we conclude that $H(t)$ negatively diverges. It is left to prove that this happens in a finite amount of time. Without loss of genericity suppose $H(0) < 0$. We divide both sides of the constraint (3.12) by H^2,

$$\frac{1}{2}\left(\frac{y}{H}\right)^2 + \frac{V}{H^2} + \frac{\rho}{H^2} = 3. \tag{4.11}$$

Similarly to the previous chapter, we introduce expansion-normalised variables,

$$\phi, \qquad x = \frac{1}{H}, \qquad w = \frac{y}{H}, \qquad z = \frac{\sqrt{\rho}}{H}, \tag{4.12}$$

and a new time coordinate τ, defined by

$$\tau = -\ln a. \tag{4.13}$$

Substituting to the constraint (4.11), we get,

$$V(\phi)x^2 + \frac{1}{2}w^2 + z^2 = 3. \tag{4.14}$$

By differentiating with respect to the new time variable τ, we obtain,

$$\frac{d\phi}{d\tau} = \frac{d\phi}{dt}\frac{dt}{d\tau} = -\frac{1}{H}\dot{\phi} = -w, \tag{4.15}$$

$$\begin{aligned} \frac{dx}{d\tau} &= \frac{dx}{dt}\frac{dt}{d\tau} = -\frac{\dot{H}}{H^2}\left(-\frac{1}{H}\right) = -\frac{1}{H}\left(\frac{1}{2}\frac{y^2}{H^2} + \frac{\gamma}{2}\frac{\rho}{H^2}\right) \\ &= -x\left(\frac{1}{2}w^2 + \frac{\gamma}{2}z^2\right), \end{aligned} \tag{4.16}$$

$$\begin{aligned} \frac{dw}{d\tau} &= \frac{dw}{dt}\frac{dt}{d\tau} \\ &= -\frac{1}{H}\left(-3y - \frac{V'(\phi)}{H} + \frac{4-3\gamma}{2}Q\frac{\rho}{H} + \frac{1}{2}\frac{y^3}{H^2} + \frac{\gamma}{2}\frac{\rho y}{H^2}\right) \\ &= 3w + \frac{V'(\phi)}{V(\phi)}V(\phi)x^2 - \frac{4-3\gamma}{2}Qz^2 - \frac{1}{2}w^3 - \frac{\gamma}{2}z^2 w, \end{aligned} \tag{4.17}$$

$$
\begin{aligned}
\frac{dz}{d\tau} &= \frac{dz}{dt}\frac{dt}{d\tau} = -\frac{1}{H}\left(\frac{\dot{\rho}}{2H\sqrt{\rho}} - \frac{\sqrt{\rho}\dot{H}}{H^2}\right) \\
&= -\frac{1}{H}\left(\frac{-\sqrt{\rho}\left(3\gamma H + \frac{4-3\gamma}{2}Qy\right)}{2H} - \frac{\sqrt{\rho}\dot{H}}{H^2}\right) \\
&= -\frac{\sqrt{\rho}}{2H}\left(-3\gamma - \frac{4-3\gamma}{2}Q\frac{y}{H} + \frac{y^2}{H^2} + \gamma\frac{\rho}{H^2}\right) \\
&= -\frac{1}{2}z\left(-3\gamma - \frac{4-3\gamma}{2}Qw + w^2 + \gamma z^2\right), \qquad (4.18)
\end{aligned}
$$

where we remind that an overdot denotes differentiation with respect to time t. We use the constraint (4.14), to eliminate $x(\tau)$ from the system (4.15)–(4.18), thus we come to the following system for the triple $(\phi(\tau), w(\tau), z(\tau))$:

$$
\frac{d\phi}{d\tau} = -w, \qquad (4.19)
$$

$$
\frac{dw}{d\tau} = -\left(\frac{1}{2}w^2 - 3\right)\left(w + \frac{V'(\phi)}{V(\phi)}\right) - z^2\left(\frac{\gamma}{2}w + \frac{4-3\gamma}{2}Q + \frac{V'(\phi)}{V(\phi)}\right), \qquad (4.20)
$$

$$
\frac{dz}{d\tau} = -\frac{1}{2}z\left(w^2 - \frac{4-3\gamma}{2}Qw + \gamma(z^2 - 3)\right). \qquad (4.21)
$$

We recall that our assumption is, $\phi_\infty = -\infty$, hence we are interested in the dynamics near the critical point "at infinity", $\phi \to -\infty$. Therefore, we introduce the variable $s = f(\phi)$, where f is defined in Assumption 5 and its derivative with respect to time τ, is

$$
\begin{aligned}
\frac{ds}{d\tau} &= \frac{df}{d\phi}\frac{d\phi}{dt}\frac{dt}{d\tau} = -\frac{1}{H}f'(\phi)\dot{\phi} = -\frac{y}{H}f'\left(f^{-1}(s)\right) \\
&= -wf'\left(f^{-1}(s)\right). \qquad (4.22)
\end{aligned}
$$

In this way we obtain a system in the variables $(w(\tau), z(\tau), s(\tau))$, ruled by equations (4.20), (4.21) and (4.22).

Since $V(\phi(t))$ is eventually positive, the system (4.20)–(4.22) is subject to the constraint,

$$\frac{1}{2}w^2 + z^2 < 3. \tag{4.23}$$

We consider critical points of (4.20)–(4.22) such that $s = 0$, which are candidates to be ω–limit points for the solutions we are interested in. There are up to seven critical points which are further restricted to critical points with $w \geq 0$, since we expect both y and H to be eventually negative and $z \leq 0$. We are left with four admissible critical points and their (w, z)–coordinates are then,

$$\begin{aligned}
\mathcal{A} &= \left(\sqrt{6}, 0\right), \\
\mathcal{B} &= (\beta, 0), \\
\mathcal{C} &= \left(\frac{2\frac{4-3\gamma}{2}Q_-}{2-\gamma}, -\frac{\sqrt{-2\left(\frac{4-3\gamma}{2}Q_-\right)^2 + 3(2-\gamma)^2}}{2-\gamma}\right), \\
\mathcal{D} &= \left(-\frac{3\gamma}{\frac{4-3\gamma}{2}Q_- - \beta}, \frac{\sqrt{3(-3\gamma - \frac{4-3\gamma}{2}Q_-\beta + \beta^2)}}{\frac{4-3\gamma}{2}Q_- - \beta}\right).
\end{aligned}$$

It is easy to check that all these points, except possibly $\mathcal{B}$, do not coincide with the origin $(0, 0)$. In the particular case when $\lambda = 0$, then $\mathcal{B} = (0, 0)$, but the eigenvalues of the linearised system associated with this critical point are $\{0, 3, \frac{3}{2}\gamma\}$ and so this point is definitely an unstable equilibrium.

The generical situation therefore is that there exists a $\delta > 0 \,:\, \frac{1}{2}w^2 + z^2 \geq \delta$ eventually. Then, recalling (3.11),

$$\frac{1}{H(t)} - \frac{1}{H(t_0)} = \int_{t_0}^{t} -\frac{\dot{H}(\sigma)}{H(\sigma)^2}\, d\sigma = \int_{t_0}^{t} \frac{1}{2}(w^2 + \gamma z^2)\, d\sigma \geq \delta(t - t_0).$$

Since $H(t_0) < 0$, we conclude that $H(t)$ diverges in a finite amount of time.

■

Remark 2 *The same dynamics described in the above proposition applies* *to the more general case* $\liminf_{t\to\sup\mathbb{I}} V(\phi(t)) = -\infty$. *Indeed, this situation implies again that* $H(t) \to -\infty$, *and in the above proposition we have proved that the dynamics near the point "at infinity"* $\phi \to -\infty$, *give necessarily rise to solutions that completely recollapse in a finite time.*

Case $\phi_\infty \in \mathbb{R}$.

We briefly examine what happens if the scalar field converges to a positive value.

Proposition 2 *If* $\phi_\infty \in \mathbb{R}$ *then the property*

$$\exists t_* > 0 : \lim_{t\to t_*^-} H(t) = -\infty, \tag{4.24}$$

generically holds.

Proof. If $\phi_\infty \in \mathbb{R}$ then $\phi(t)$ is bounded. Then, if $H(t)$ was bounded too, by (3.12) also $y(t), \rho(t)$ would be bounded so the solution would be bounded which by Remark 1 is a non generic situation. Then $H(t)$ is unbounded and since it is decreasing by (3.11), it must diverge to $-\infty$. At this point, Lemma 3 applies to give the result. ■

Case $\phi_\infty = +\infty$

In this situation we must split the argument into two subcases, depending on the value of V_∞. We start by considering shortly the case when this limit value is strictly negative (case E), possibly $-\infty$ (case A).

Proposition 3 *If* $V_\infty < 0$ *and* $\phi_\infty = +\infty$ *then the property*

$$\exists t_* > 0 : \lim_{t\to t_*^-} H(t) = -\infty, \tag{4.25}$$

generically holds.

Proof. If $\phi_\infty = +\infty$ then, since $V_\infty < 0$, there exists a $\bar{V} < 0$, such as, $V(\phi(t)) \leq \bar{V} < 0$ eventually and then Lemma 3 applies to give the result. ■

A more subtle case happens when $V_\infty = 0$, as is the case of the ekpyrotic potentials. In this situation the critical point "at infinity" corresponding to $\phi \to +\infty$ must be studied carefully, since it may give rise to ever expanding cosmologies. Before we state the precise theorem, the following preliminary result is needed.

Lemma 4 *If $V_\infty = 0$ and $\phi_\infty = +\infty$, then $H_\infty = \lim_{t\to\sup\mathbb{I}} H(t) \leq 0$.*

Proof. Suppose by contradiction, that $H_\infty > 0$. Then,

$$\frac{1}{2}y^2 + \rho \to 3H_\infty^2$$

and therefore

$$\sup\mathbb{I} \in \mathbb{R},$$

otherwise it would be

$$H(t) - H(0) = -\frac{1}{2}\int_{t_0}^{t} \left(y(s)^2 + \gamma\rho(s)\right)\, ds \to -\infty,$$

as $t \to \infty$, a contradiction. Using the Cauchy-Schwarz inequality we obtain,

$$\begin{aligned}
(\phi(t) - \phi(t_0))^2 &= \left(\int_{t_0}^{t} \dot{\phi}(s)\, ds\right)^2 \\
&= \left(\int_{t_0}^{t} y(s)\, ds\right)^2 = \left(\int_{t_0}^{t} 1\cdot y(s)\, ds\right)^2 \\
&\leq \left(\int_{t_0}^{t} 1^2(s)\, ds\right)\left(\int_{t_0}^{t} y^2(s)\, ds\right) \\
&\leq (t - t_0)\int_{t_0}^{t} y^2(s)\, ds \leq (t - t_0)\int_{t_0}^{t} -2\dot{H}(s)\, ds \\
&= 2(t - t_0)\left(H(t_0) - H(t)\right),
\end{aligned}$$

that converges to the finite value $2(\sup \mathbb{I} - t_0)(H(t_0) - H_\infty) \in \mathbb{R}$, hence $\phi(t)$ is bounded, which is a contradiction. Therefore, $H_\infty \leq 0$. ■

Proposition 4 *Suppose that $\phi_\infty = +\infty$ and $V_\infty = 0$. If*

$$0 < \gamma < \frac{4}{3}, \quad 0 < Q_+ < \sqrt{6}\frac{2-\gamma}{4-3\gamma}, \quad \alpha > \frac{4-3\gamma}{2}Q_+ + \frac{3(2-\gamma)\gamma}{(4-3\gamma)Q_+} \tag{4.26}$$

does not hold, then the property

$$\exists t_* > 0 : \lim_{t \to t_*^-} H(t) = -\infty, \tag{4.27}$$

generically holds. Otherwise, i.e. if (4.26) *holds, either* $\exists t_* > 0 : \lim_{t \to t_*^-} H(t) = -\infty$*, generically holds or the solution expands forever, with $\phi(t) \to +\infty$ and $y(t), \rho(t)$ and $H(t)$ infinitesimal as $t \to +\infty$.*

Proof. By Lemma 4, $H_\infty \leq 0$. If H_∞ is strictly negative then the results follows from Lemma 3. Suppose now it is zero; this means that the solution expands forever and a normalised variables scheme can be used to study the critical point "at infinity". We use variables (ϕ, x, w, z) as in Proposition 1, which are functions of a new time τ coordinate defined by $d\tau/dt = H$. Note that unlike the case treated in Proposition 1, now $H > 0$. We use the function g, defined in Assumption 5 and the same arguments as in the proof of Proposition 1, that is,

$$\phi, \qquad x = \frac{1}{H}, \qquad w = \frac{y}{H}, \qquad z = \frac{\sqrt{\rho}}{H},$$

are the expansion-normalised variables, and the constraint in the new variables is,

$$V(\phi)x^2 + \frac{1}{2}w^2 + z^2 = 3.$$

Differentiation with respect to the new time variable τ, yields,

$$\frac{dw}{d\tau} = \left(\frac{1}{2}w^2 - 3\right)\left(w + \frac{V'(g^{-1}(s))}{V(g^{-1}(s))}\right) + z^2\left(\frac{\gamma}{2}w + \frac{4-3\gamma}{2}Q + \frac{V'(g^{-1}(s))}{V(g^{-1}(s))}\right), \tag{4.28}$$

$$\frac{dz}{d\tau} = \frac{1}{2}z\left[w^2 - \frac{4-3\gamma}{2}Qw + \gamma(z^2-3)\right], \tag{4.29}$$

$$\frac{ds}{d\tau} = wg'(g^{-1}(s)), \tag{4.30}$$

where we used the constraint, to eliminate $x(\tau)$ from the system. Note that Q is a function depending on ϕ, or equivalently, $Q(\phi) = Q\left(g^{-1}(s)\right)$.

We are interested in solutions such that $s \to 0$ and $\frac{1}{2}w^2 + z^2 > 3$, with $w, z \geq 0$ eventually. Therefore, the (w,z)–coordinates of the critical points that are admissible candidates to be ω–limit points are,

$$\mathcal{A} = \left(\sqrt{6}, 0\right),$$

$$\mathcal{B} = (\alpha, 0),$$

$$\mathcal{C} = \left(\frac{2\frac{4-3\gamma}{2}Q_+}{2-\gamma}, \frac{\sqrt{-2\left(\frac{4-3\gamma}{2}Q_+\right)^2 + 3(2-\gamma)^2}}{2-\gamma}\right),$$

$$\mathcal{D} = \left(-\frac{3\gamma}{\frac{4-3\gamma}{2}Q_+ - \alpha}, -\frac{\sqrt{3(-3\gamma - \frac{4-3\gamma}{2}Q_+\alpha + \alpha^2)}}{\frac{4-3\gamma}{2}Q_+ - \alpha}\right).$$

The analysis of these critical points reveals that the only sink can be $\mathcal{C}$, and this happens precisely when (4.26) holds. In this case, we obtain ever expanding solutions such that $H_\infty = 0$, and consequently, both y and ρ tend to zero; since the solution is defined for $\tau \to +\infty$, and recalling that t is an increasing function of τ, we get $\sup \mathbb{I} = +\infty$, i.e. also the corresponding solution to (3.8)–(3.12) is defined for $t \to +\infty$.

If the solution does not start into the basin of attraction of $\mathcal{C}$, it is unbounded, thus, $\frac{1}{2}w^2 + z^2 \to +\infty$. Suppose by contradiction that they also correspond to ever expanding cosmologies with $H_\infty = 0$. Then $\sup \mathbb{I} = +\infty$.

Set

$$\widetilde{x} = \frac{\sqrt{|V|}}{H},$$

and observe that

$$\frac{d\widetilde{x}}{d\tau} = \frac{1}{2}\widetilde{x}\left(\frac{V'(g^{-1}(s))}{V(g^{-1}(s))}w + w^2 + \gamma z^2\right) \approx \frac{1}{2}\widetilde{x}(w^2 + \gamma z^2),$$

where the symbol $\approx$ is used to denote the dominant terms. Now,

$$\widetilde{x}^2 = \frac{1}{2}w^2 + z^2 - 3 \approx \frac{1}{2}w^2 + z^2 \approx K(w^2 + \gamma z^2),$$

for some constant $K > 0$, so $d\widetilde{x}/d\tau \approx A\widetilde{x}^3$ for some $A > 0$, which implies $\widetilde{x}(\tau) \approx (a - b\tau)^{-1/2}$ for suitable $a, b > 0$.

Then,

$$\frac{1}{2}(w^2 + \gamma z^2) \approx \frac{d\widetilde{x}}{d\tau}\frac{1}{\widetilde{x}} \approx \frac{b}{2}(a - b\tau)^{-1},$$

hence,

$$\frac{dH}{d\tau} = -H\frac{1}{2}(w^2 + \gamma z^2) \approx -H(\tau)\frac{b}{2}(a - b\tau)^{-1},$$

from which $H(\tau) \approx H_0\sqrt{a - b\tau}$. This implies that

$$t = \int_{\tau_0}^{\tau} \frac{1}{H(\sigma)}d\sigma \approx \int_{\tau_0}^{\tau} \frac{1}{H_0\sqrt{a - b\sigma}}\, d\sigma,$$

which converges as $\tau \to a/b$. This means that $\sup \mathbb{I} \in \mathbb{R}$, that is a contradiction. Therefore, $H(t) < 0$ eventually. Since $V(\phi(t)) < 0$ eventually, the conclusion follows from Lemma 3. ■

To illustrate the situation depicted in Proposition 4, let us consider as an example the double exponential potential (3.15), where we choose $\alpha = 4, \beta = 5$ and $V_2 = -V_1 = 1$. For the case of dust, $\gamma = 1$, and a constant coupling $Q = 1$, both expansion and recollapse may take place, depending on the initial conditions.

With initial conditions for instance, $H(0) = 1$, $\phi(0) = 2$ and $y(0) = -1$ (the initial value $\rho(0)$ is not arbitrary, but is determined by the constraint (3.16)), the scalar field positively diverges in an infinite time and the Hubble function remains always positive, tending asymptotically to zero; therefore the Universe expands forever.

Simply changing the initial conditions, for instance $y(0) = -2$, then the scalar positively diverges again, but now in a finite amount of time. Indeed, $H(t)$ changes sign and once it becomes negative, the solution is forced to recollapse and develop a singularity.

Case ϕ_∞ does not exist

In this subsection we study the case when $\phi(t)$ neither converges nor diverges.

Proposition 5 *If ϕ_∞ does not exist, then the property*

$$\exists t_* > 0 : \lim_{t \to t_*^-} H(t) = -\infty,$$

generically holds.

Proof. First we claim that

$$H_\infty = \lim_{t \to \sup \mathbb{I}} H(t) = -\infty, \tag{4.31}$$

generically holds, by considering the following subcases.

1. Suppose $\liminf_{t \to \sup \mathbb{I}} V(\phi(t)) \geq 0$. If by contradiction, $H(t)$ was bounded, then from (3.12) we could conclude that $y(t), \rho(t)$ were bounded too. Then, for the solution to be generic (recall again Lemma 2), $\phi(t)$ should be unbounded. But since $V(\phi(t))$ must be eventually non negative, this would imply that $\limsup_{t \to \sup \mathbb{I}} V(\phi(t)) = +\infty$. Then a sequence $t_n \to \sup \mathbb{I}$ exists, such that $H(t_n)^2 \to +\infty$, which

means that $H(t_n) \to -\infty$, which is a contradiction. Thus, $H(t)$ cannot be bounded and therefore (4.31) must hold.

2. Suppose $\liminf_{t\to\sup\mathbb{I}} V(\phi(t)) < 0$. In this case there exist sequences $\{t_n\}, \{s_n\}$, such that

$$t_n, s_n \to \sup\mathbb{I}, \quad t_n < s_n < t_{n+1},$$

with $V(\phi(t_n)), V(\phi(s_n)) < 0$ and $\phi(t)$ lies between $\phi(t_n)$ and $\phi(s_n)$, $\forall t \in [t_n, s_n]$. Using Cauchy-Schwarz inequality as in the proof of Lemma 4 we get

$$(\phi(t_n)-\phi(s_n))^2 \leq (s_n-t_n)\int_{t_n}^{s_n} -2\dot{H}(s)\,\mathrm{d}s = 2(s_n-t_n)(H(t_n)-H(s_n)),$$

and therefore

$$s_n - t_n \geq \frac{(\phi(t_n)-\phi(s_n))^2}{2(H(t_n)-H(s_n))}. \tag{4.32}$$

Now, if by contradiction $H_\infty \in \mathbb{R}$ then (4.32) would imply that $s_n - t_n \to +\infty$ and as a consequence $\sup\mathbb{I} = +\infty$. Moreover comparison theorems in ODE would say that $H(t) \leq z(t)$ in $[t_n, s_n]$, where $z(t)$ solves the Cauchy problem

$$\dot{z}(t) = \frac{\gamma}{2}(-3z(t)^2 + \bar{V}), \qquad z(t_n) = H(t_n),$$

and $\bar{V}$ is a (negative) constant such that $V(\phi) < \bar{V}$, for every ϕ between ϕ_t and ϕ_s. Now, observe that the solution $z(t)$ to the Cauchy problem above negatively diverges for some $t_n + \delta_n$, where δ_n is uniformly bounded with respect to n, whereas $s_n - t_n \to +\infty$, and this is a contradiction. Hence $H_\infty = -\infty$, i.e. (4.31) holds.

In both cases (2a) and (2b) we have shown that (4.31) holds. Let us prove that this happens in a finite amount of time. If $\liminf_{t\to\sup\mathbb{I}} \phi(t) \in \mathbb{R}$

then there exists a $\bar{V} \in \mathbb{R}$, such that $V(\phi(t)) \leq \bar{V}$ eventually, and the result follows from Lemma 3.

If $\liminf_{t\to\sup\mathbb{I}} \phi(t) = -\infty$, i.e., $\limsup_{t\to\sup\mathbb{I}} V(\phi(t)) = +\infty$, then we can consider the same system in normalised variables used in case (1c) before, see Remark 2 after Proposition 1. ■

The results proved in this section may be collected in the following main theorem.

Theorem 6 *Let $V(\phi)$ satisfy Assumption 5. Then, if either $V_\infty < 0$, or condition,*

$$0 < \gamma < \frac{4}{3}, \quad 0 < Q_+ < \sqrt{6}\frac{2-\gamma}{4-3\gamma}, \quad \alpha > \frac{4-3\gamma}{2}Q_+ + \frac{3(2-\gamma)\gamma}{(4-3\gamma)Q_+} \tag{4.33}$$

*does **not** hold, then a solution to* (3.8)–(3.12)*, up to a zero–measured set of initial data, recollapses to a singularity in a finite amount of time, i.e.,*

$$\exists t_* > 0 : \lim_{t\to t_*^-} H(t) = -\infty. \tag{4.34}$$

Otherwise, if $V_\infty = 0$ and (4.33) *does hold, a solution to* (3.8)–(3.12) *either generically recollapses to a singularity in a finite time or expands forever, with $\phi(t) \to +\infty$ and $y(t), \rho(t)$ and $H(t)$ infinitesimal as $t \to +\infty$.*

4.2 The Double Exponential Potential revisited

To illustrate the results depicted in this chapter, we consider the general form of the potential

$$V(\phi) = V_1 e^{-\alpha\phi} + V_2 e^{-\beta\phi},$$

where α, β, V_1, V_2 are nonzero real constants of arbitrary sign.

Without loss of generality, we assume $\alpha \neq \beta$. For $\alpha = \beta$ the case reduces to a single exponential potential. Different assumptions on the parameters α, β, V_1, V_2, yield to different forms of the potentials, as shown in Table 4.1. The cases $0 < \alpha < \beta$ and $0 < \beta < \alpha$ have already been analysed in Chapter 3.

Potentials shown in Fig. 4.6(a), where $\alpha < 0 < \beta$ and $V_1, V_2 > 0$, have a strictly positive minimum, say $V_{\min}$, and the de Sitter solution with $H = \sqrt{V_{\min}/3}$, is the future attractor for the system, [79]. This follows directly either from the original equations (3.17)-(3.19), or from the system (3.27) written in the new variables. Moreover, it is easy to see that a matter era represented by a saddle equilibrium $\mathcal{B}$, precedes the final accelerated epoch. Potentials in Fig. 4.6(b) belong to the Class E for the "twin" cases $\beta < 0 < \alpha$, $\alpha < 0 < \beta$, potentials in Fig. 4.6(c) also belong to the Class E, while potentials in Fig. 4.6(d) have a strictly negative maximum, therefore this is not a physically interesting case.

The case $\beta < 0 < \alpha$ is a mere renaming of the parameters and some equilibria and yields to the same conclusions. The different forms of the potentials with respect to the different signs of V_1, V_2 are shown in Fig. 4.6.

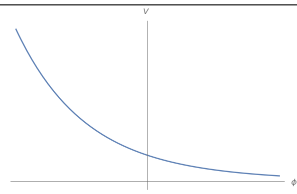
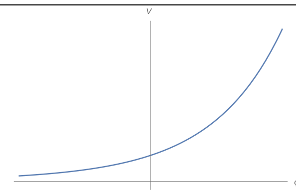
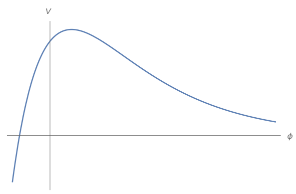
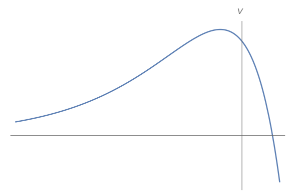
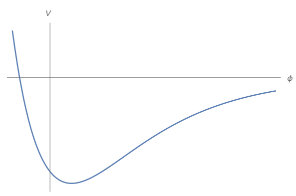
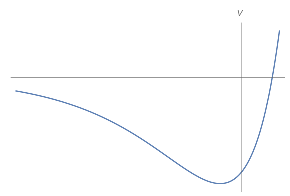
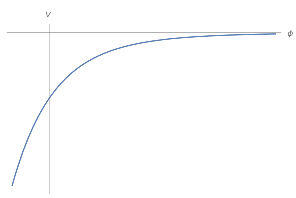
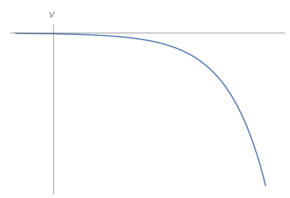
4.3 Conclusion

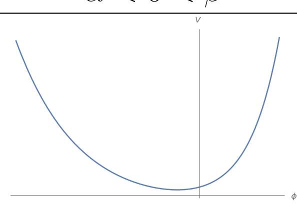
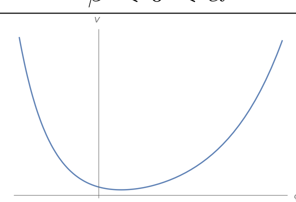
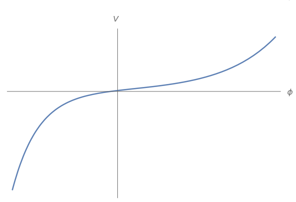
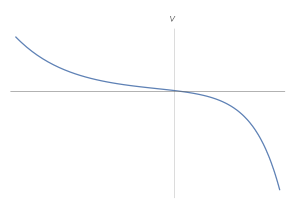
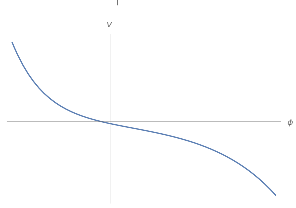
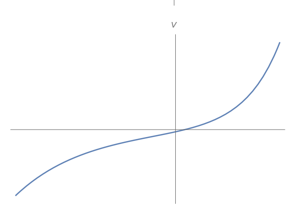
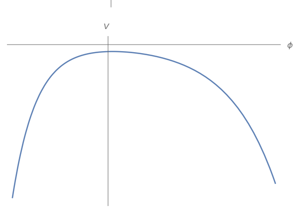
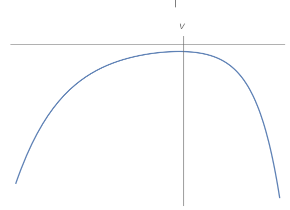
In this chapter we completed the analysis of the main classes of negative potentials encountered in the literature. We proved that a solution to (3.8)–(3.12) generically recollapses in a finite time, that is

$$\exists t_* > 0 \text{ such that } \lim_{t \to t_*^-} H(t) = -\infty.$$

We have investigated the qualitative behaviour of the Hubble function, examining all possible cases for the asymptotic behaviour of the scalar field.

Table 4.1: Forms of the potentials $V(\phi) = V_1 e^{-\alpha\phi} + V_2 e^{-\beta\phi}$.

Parameters V_1, V_2	$0 < \alpha < \beta$	$\beta < \alpha < 0$
$V_1, V_2 > 0$		
$V_1 > 0, V_2 < 0$		
$V_1 < 0, V_2 > 0$		
$V_1, V_2 < 0$		

Parameters V_1, V_2	$\alpha < 0 < \beta$	$\beta < 0 < \alpha$
$V_1, V_2 > 0$		
$V_1 > 0, V_2 < 0$		
$V_1 < 0, V_2 > 0$		
$V_1, V_2 < 0$		

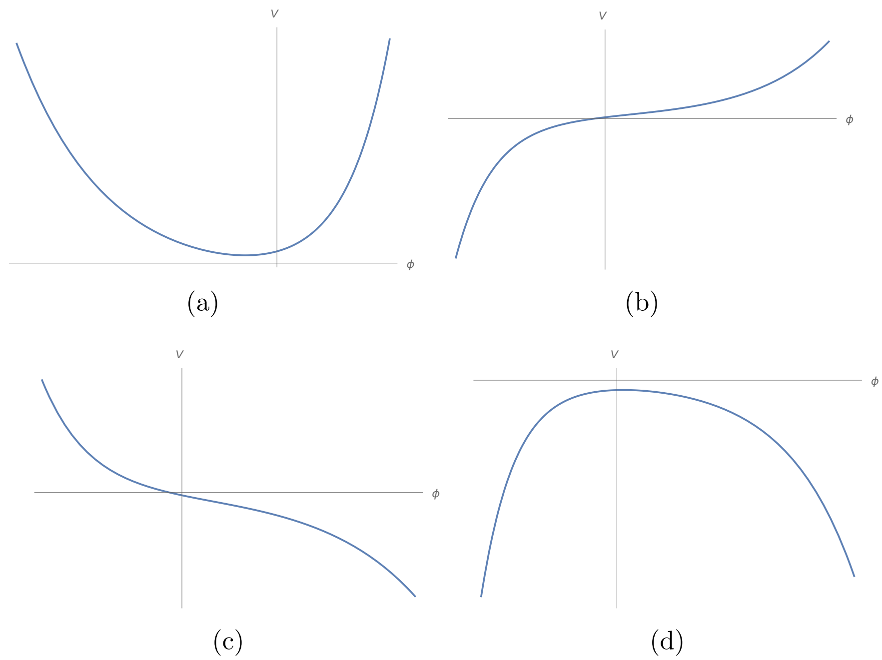


Figure 4.6: Potentials with $\alpha < 0 < \beta$. (a) $V_1, V_2 > 0$, (b) $V_1 > 0, V_2 < 0$, (c) $V_1 < 0, V_2 > 0$, (d) $V_1, V_2 < 0$

We have found that the recollapse and the formation of a future singularity always take place in a generic way, i.e. stable with respect to perturbations of the initial data of the system. Moreover, recollapse is the only generical situation allowed, except in case the potential goes to zero from below as $\phi \to +\infty$ and (4.33) holds; in this case there also exists generical choices of initial data that do not lead to recollapse, producing an ever–expanding cosmology where the scalar field positively diverges. Our conclusions are valid for scalar fields coupled to matter, as well as for uncoupled models studied so far in the literature.

Cosmology with negative potentials is the basis of the cyclic Universes in the context of the ekpyrotic scenario. Our results may be helpful in building solid models of cyclic cosmologies and therefore avoid the fragility of this scenario with respect to the unknown physics at the singularity.

Chapter 5

Conclusions and Future Work

In this thesis we have focused on a general treatment of a scalar field with a potential function, non-minimally coupled to matter.

In Chapter 3, we treated the case where the potential is the sum of two exponentials. This form arises as the asymptotic form of other potentials. It is therefore of great interest to see if these models are cosmologically viable. To study the system we have used the expansion-normalised variables techniques. The analysis of the critical points of the dynamical system was complicated, it revealed though that the model predicts a late accelerated phase of the Universe for a wide range of the parameters, α, β, γ and Q. We found that there is a solution of the resulted dynamical system which may represent a viable cosmological history. In fact, there exists a saddle matter point with the appropriate time dependence of scale factor, which allows for the construction of matter, followed by a stable accelerated point. However, in most cases the scale factor near transient matter points evolves as $a\left(t\right) \sim t^{q(Q)}$, where the exponent q is in general different from the usual 2/3. This "wrong" matter epoch is associated with the strong coupling of order $Q \sim 1$; in this case it was found that the coupling constant has to be almost to zero. This result, indicates that cosmological evolution imposes strict constraints on the choice of the correct Lagrangian of a gravity theory.

In Chapter 3 we assumed a constant coupling Q. It would be interesting to study the case where Q depends on ϕ and see if the coupling constant has to vanish for an acceptable cosmological history. Such a result could lead to a generalisation of the attractor mechanism of scalar-tensor theories towards General Relativity.

In Chapter 4, we treated rigorously potentials that take negative values. We provided a list that includes the main classes of negative potentials encountered in the literature. We proved that in most cases, initially expanding Universes collapse in a finite time. Our results may be helpful in building mathematical rigorous models in the cyclic scenario. It would be interesting to extend our study on negative potentials beyond the case of flat Universes.

Another path of future research consists of the study of theories of gravity that include many coupled scalar fields with arbitrary couplings to the curvature. They belong to a large class of theories based on the concept of wave maps. Preliminary results indicate that under certain conditions the action is conformally equivalent to General Relativity with a minimally coupled scalar field.

Appendix A

Conformal Transformations

Conformal transformation techniques are used widely in theories of gravity [10,66,67,137]. Conformal transformations are obtained by multiplying the metric by a non-vanishing spacetime-depended function ω,

$$\tilde{g}_{\mu\nu} = \omega^2 g_{\mu\nu}. \tag{A.1}$$

A conformal transformation (A.1) keeps the sign of the line element and the angle between two vectors unchanged. The original spacetime is the so-called Jordan frame, while the conformal spacetime is called the Einstein frame. The following transformation formulas hold:

$$\begin{aligned}
\tilde{g}^{\mu\nu} =& \omega^{-2} g^{\mu\nu}, \qquad \tilde{g} = \omega^{2n} g, \\
\tilde{R}_{\mu\nu} =& R_{\mu\nu} - \left[(n-2)\,\delta^{\alpha}_{\mu}\delta^{\beta}_{\nu} + g_{\mu\nu}g^{\alpha\beta}\right]\omega^{-1}\left(\nabla_\alpha\nabla_\beta\omega\right) \\
&+ \left[2\,(n-2)\,\delta^{\alpha}_{\mu}\delta^{\beta}_{\nu} - (n-3)\,g_{\mu\nu}g^{\alpha\beta}\right]\omega^{-2}\left(\nabla_\alpha\omega\nabla_\beta\omega\right), \\
\tilde{R} =& \omega^{-2}R - 2\,(n-1)\,g^{\alpha\beta}\omega^{-3}\left(\nabla_\alpha\nabla_\beta\omega\right) \\
&- (n-1)\,(n-4)\,g^{\alpha\beta}\omega^{-4}\left(\nabla_\alpha\omega\right)\left(\nabla_\beta\omega\right), \\
\tilde{\Box}\phi =& \omega^{-2}\Box\phi + (n-2)\,g^{\alpha\beta}\omega^{-3}\left(\nabla_\alpha\omega\right)\left(\nabla_\beta\phi\right).
\end{aligned}$$

If $g_{\mu\nu}$ in (A.1) is the FLRW metric, the corresponding non vanishing

Christoffel symbols are

$$
\begin{aligned}
&\tilde{\Gamma}^0_{00} = \frac{\dot{\omega}}{\omega}, \\
&\tilde{\Gamma}^0_{11} = \left(\frac{\dot{\omega}}{\omega} + \frac{\dot{a}}{a}\right) a^2, \quad \tilde{\Gamma}^0_{22} = \left(\frac{\dot{\omega}}{\omega} + \frac{\dot{a}}{a}\right) a^2 r^2, \qquad \tilde{\Gamma}^0_{33} = \left(\frac{\dot{\omega}}{\omega} + \frac{\dot{a}}{a}\right) a^2 r^2 \sin^2\theta, \\
&\tilde{\Gamma}^1_{01} = \frac{\dot{\omega}}{\omega} + \frac{\dot{a}}{a}, \qquad\qquad\qquad\qquad \tilde{\Gamma}^1_{22} = -r, \qquad \tilde{\Gamma}^1_{33} = -r\sin^2\theta, \\
&\tilde{\Gamma}^2_{02} = \frac{\dot{\omega}}{\omega} + \frac{\dot{a}}{a}, \qquad\qquad\qquad\qquad \tilde{\Gamma}^2_{12} = \frac{1}{r}, \qquad \tilde{\Gamma}^2_{33} = -\sin\theta\cos\theta, \\
&\tilde{\Gamma}^3_{03} = \frac{\dot{\omega}}{\omega} + \frac{\dot{a}}{a}, \qquad\qquad\qquad\qquad \tilde{\Gamma}^3_{13} = \frac{1}{r}, \qquad \tilde{\Gamma}^3_{23} = \cot\theta.
\end{aligned}
$$

We also compute

$$\nabla_\alpha \omega = \partial_\alpha \omega = \dot{\omega}\delta^0_\alpha := \omega_\alpha,$$

$$
\begin{aligned}
\nabla_\alpha \nabla_\beta \omega &= \nabla_\alpha \omega_\beta = \partial_\alpha \omega_\beta - \Gamma^\lambda_{\alpha\beta}\omega_\lambda \\
&= \delta^0_\alpha \delta^0_\beta \ddot{\omega} - \Gamma^\lambda_{\alpha\beta}\delta^0_\lambda \dot{\omega} = \delta^0_\alpha \delta^0_\beta \ddot{\omega} - \Gamma^0_{\alpha\beta}\dot{\omega},
\end{aligned}
$$

hence,

$$
\begin{aligned}
&\nabla_0 \nabla_0 \omega = \ddot{\omega}, \\
&\nabla_i \nabla_i \omega = -\Gamma^0_{ii}\dot{\omega} = \begin{cases} -a\dot{a}\dot{\omega}, & \text{if } i = 1, \\ -a\dot{a}r^2\dot{\omega}, & \text{if } i = 2, \\ -a\dot{a}r^2\sin^2\theta\dot{\omega}, & \text{if } i = 3. \end{cases}
\end{aligned}
$$

The $\tilde{R}_{\mu\nu}$ are given by

$$\begin{aligned}
\tilde{R}_{00} &= -3\frac{\ddot{a}}{a} - 3\frac{\ddot{\omega}}{\omega} - 3\frac{\dot{\omega}}{\omega}\frac{\dot{a}}{a} + 3\frac{\dot{\omega}^2}{\omega^2}, \\
\tilde{R}_{11} &= a^2\left(\frac{\ddot{a}}{a} + 2\frac{\dot{a}^2}{a^2} + \frac{\ddot{\omega}}{\omega} + 5\frac{\dot{\omega}}{\omega}\frac{\dot{a}}{a} + \frac{\dot{\omega}^2}{\omega^2}\right), \\
\tilde{R}_{22} &= a^2r^2\left(\frac{\ddot{a}}{a} + 2\frac{\dot{a}^2}{a^2} + \frac{\ddot{\omega}}{\omega} + 5\frac{\dot{\omega}}{\omega}\frac{\dot{a}}{a} + \frac{\dot{\omega}^2}{\omega^2}\right), \\
\tilde{R}_{33} &= a^2r^2\sin^2\theta\left(\frac{\ddot{a}}{a} + 2\frac{\dot{a}^2}{a^2} + \frac{\ddot{\omega}}{\omega} + 5\frac{\dot{\omega}}{\omega}\frac{\dot{a}}{a} + \frac{\dot{\omega}^2}{\omega^2}\right).
\end{aligned}$$

The Ricci scalar is given by

$$\tilde{R} = \frac{6}{\omega^2}\left(\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2} + \frac{\ddot{\omega}}{\omega} + 3\frac{\dot{\omega}}{\omega}\frac{\dot{a}}{a}\right).$$

The components of the Einstein tensor $\tilde{G}_{\mu\nu} = \tilde{R}_{\mu\nu} - \frac{1}{2}\tilde{R}\tilde{g}_{\mu\nu}$, are

$$\begin{aligned}
\tilde{G}_{00} &= 3\left(\frac{\dot{a}}{a} + \frac{\dot{\omega}}{\omega}\right)^2, \\
\tilde{G}_{11} &= a^2\left(-2\frac{\ddot{a}}{a} - \frac{\dot{a}^2}{a^2} - 2\frac{\ddot{\omega}}{\omega} - 4\frac{\dot{\omega}}{\omega}\frac{\dot{a}}{a} + \frac{\dot{\omega}^2}{\omega^2}\right), \\
\tilde{G}_{22} &= a^2r^2\left(-2\frac{\ddot{a}}{a} - \frac{\dot{a}^2}{a^2} - 2\frac{\ddot{\omega}}{\omega} - 4\frac{\dot{\omega}}{\omega}\frac{\dot{a}}{a} + \frac{\dot{\omega}^2}{\omega^2}\right), \\
\tilde{G}_{33} &= a^2r^2\sin^2\theta\left(-2\frac{\ddot{a}}{a} - \frac{\dot{a}^2}{a^2} - 2\frac{\ddot{\omega}}{\omega} - 4\frac{\dot{\omega}}{\omega}\frac{\dot{a}}{a} + \frac{\dot{\omega}^2}{\omega^2}\right),
\end{aligned}$$

where the ii components have the same time-dependence for $i = 1, 2, 3$,

$$g_{ii}\left(-2\frac{\ddot{a}}{a} - \frac{\dot{a}^2}{a^2} - 2\frac{\ddot{\omega}}{\omega} - 4\frac{\dot{\omega}}{\omega}\frac{\dot{a}}{a} + \frac{\dot{\omega}^2}{\omega^2}\right),$$

as expected due to isotropy.

Finally, the D' Alembertian is,

$$\tilde{\Box}\phi = -\frac{1}{\omega^2}\left(\ddot{\phi} + 3H\dot{\phi} + 2\frac{\dot{\omega}}{\omega}\dot{\phi}\right).$$

For $\omega = 1$, we obtain the usual formulas for the Ricci and the Einstein tensors for homogeneous and isotropic spacetimes.

Appendix B

Field Equations of $f(R)$ Theories and Conformal Equivalence

B.1 Derivation of the Field Equations

The action of an $f(R)$ theory is given by

$$S = \int d^4x \frac{1}{2}\sqrt{-g}f(R) + S_{\mathrm{m}}(g_{\mu\nu}, \Psi),$$

where Ψ denotes all matter fields collectively.

By definition of the metric tensor $g_{\mu\nu}$, it holds

$$g_{\mu\nu}g^{\nu\rho} = \delta^{\rho}_{\mu}, \tag{B.1}$$

where δ^{ρ}_{μ} is the Kronecker delta. From (B.1)

$$(\delta g_{\mu\nu})\, g^{\nu\rho} + (\delta g^{\nu\rho})\, g_{\mu\nu} = \delta(\delta^{\rho}_{\mu}) = 0,$$

since δ^ρ_μ is constant. Contracting by $g_{\rho\sigma}$, we get

$$(\delta g_{\mu\nu})\,\delta^\nu_\sigma + (\delta g^{\nu\rho})\,g_{\mu\nu}g_{\rho\sigma} = 0,$$

and by renaming some of the indices

$$\delta g_{\mu\nu} = -g_{\mu\rho}g_{\nu\sigma}\delta g^{\rho\sigma}.$$

For a metric tensor $g_{\mu\nu}$ also holds,

$$g_{\mu\nu}g^{\mu\nu} = n, \tag{B.2}$$

where n is the dimension of the Riemannian space. Since n is a constant, varying (B.2), we get

$$g_{\mu\nu}\delta g^{\mu\nu} = -g^{\mu\nu}\delta g_{\mu\nu}.$$

For the determinant of the above metric, $g := \det g_{\mu\nu}$, we apply the Jacobi's formula $g = \mathrm{trace}\,(\ln g_{\mu\nu})$. Therefore, varying the above identity, we get

$$\frac{1}{\det g_{\mu\nu}}\delta(\det g_{\mu\nu}) = \mathrm{trace}((g_{\mu\nu})^{-1}\delta(g_{\mu\nu})),$$

that is,

$$\frac{1}{g}\delta g = g^{\mu\nu}\delta g_{\mu\nu},$$

or

$$\delta g = -g g_{\mu\nu}\delta g^{\mu\nu}.$$

Now varying the square root of the determinant, g, we have

$$\begin{aligned}\delta\left(\sqrt{g}\right) &= \frac{1}{2\sqrt{g}}\delta g \\ &= -\frac{1}{2\sqrt{g}} g g_{\mu\nu}\delta g^{\mu\nu} \\ &= -\frac{1}{2}\sqrt{g} g_{\mu\nu}\delta g^{\mu\nu},\end{aligned}$$

and

$$\delta\left(\sqrt{-g}\right) = -\frac{1}{2}\sqrt{-g} g_{\mu\nu}\delta g^{\mu\nu}. \tag{B.3}$$

We now compute the variation of the Ricci scalar.

$$\begin{aligned}\delta R =& \delta(g^{\mu\nu}R_{\mu\nu}) \\ =& R_{\mu\nu}\delta g^{\mu\nu} + g^{\mu\nu}\delta R_{\mu\nu} \\ =& R_{\mu\nu}\delta g^{\mu\nu} + g^{\mu\nu}\delta R^{\sigma}_{\mu\sigma\nu} \\ =& R_{\mu\nu}\delta g^{\mu\nu} \\ &+ g^{\mu\nu}\left(\partial_\sigma \delta\Gamma^{\sigma}_{\nu\mu} - \partial_\nu \delta\Gamma^{\sigma}_{\sigma\mu} + \delta\Gamma^{\sigma}_{\sigma\lambda}\Gamma^{\lambda}_{\nu\mu} + \Gamma^{\sigma}_{\sigma\lambda}\delta\Gamma^{\lambda}_{\nu\mu} - \delta\Gamma^{\sigma}_{\nu\lambda}\Gamma^{\lambda}_{\sigma\mu} - \Gamma^{\sigma}_{\nu\lambda}\delta\Gamma^{\lambda}_{\sigma\mu}\right) \\ =& R_{\mu\nu}\delta g^{\mu\nu} + g^{\mu\nu}\left(\nabla_\sigma(\delta\Gamma^{\sigma}_{\nu\mu}) - \nabla_\nu(\delta\Gamma^{\sigma}_{\sigma\mu})\right) \\ =& R_{\mu\nu}\delta g^{\mu\nu} + \nabla_\rho(g^{\mu\nu}\delta\Gamma^{\rho}_{\nu\mu} - g^{\mu\rho}\delta\Gamma^{\sigma}_{\sigma\mu}).\end{aligned}$$

For the second term and using Stoke's theorem, we have

$$\int d^4x\sqrt{-g} g^{\mu\nu}\nabla_\rho(g^{\mu\nu}\delta\Gamma^{\rho}_{\nu\mu} - g^{\mu\rho}\delta\Gamma^{\sigma}_{\sigma\mu}) = 0,$$

thus

$$\delta R = R_{\mu\nu}\delta g^{\mu\nu}.$$

Varying the action with respect to the metric tensor, we get

$$
\begin{aligned}
0 =& \delta S = \delta\left(S_{\text{HOG}} + S_{\text{M}}\right) \\
=& \int d^4x \left[\frac{1}{2}\delta\left(\sqrt{-g}f(R)\right) + \delta\left(\sqrt{-g}\mathcal{L}_{\text{M}}\right)\right] \\
=& \int d^4x \left[\frac{1}{2}\left(\delta f(R)\sqrt{-g} + f(R)\delta\sqrt{-g}\right) + \delta\left(\sqrt{-g}\mathcal{L}_{\text{M}}\right)\right] \\
=& \int d^4x \left[\frac{1}{2}\left(f'(R)\delta R\sqrt{-g} - \frac{1}{2}\sqrt{-g}g_{\mu\nu}\delta g^{\mu\nu} f(R)\right) + \delta\left(\sqrt{-g}\mathcal{L}_{\text{M}}\right)\right] \\
=& \int d^4x \frac{1}{2}\sqrt{-g}\left[f'(R)\left(R_{\mu\nu}\delta g^{\mu\nu} + g_{\mu\nu}\Box\delta g^{\mu\nu} - \nabla_\mu\nabla_\nu\delta g^{\mu\nu}\right) - \frac{1}{2}g_{\mu\nu}\delta g^{\mu\nu} f(R)\right] \\
&+ \int d^4x\delta\left(\sqrt{-g}\mathcal{L}_{\text{M}}\right) \\
=& \int d^4x\sqrt{-g}\delta g^{\mu\nu}\frac{1}{2}\left(f'(R)R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}f(R) + \left(g_{\mu\nu}\Box - \nabla_\mu\nabla_\nu\right)f'(R) + 2\frac{\delta(\sqrt{-g}\mathcal{L}_{\text{M}})}{\sqrt{-g}\delta g^{\mu\nu}}\right)
\end{aligned}
$$

Since the variation $\delta g^{\mu\nu}$ is arbitrary, we have

$$
f'(R)R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}f(R) + \left(g_{\mu\nu}\Box - \nabla_\mu\nabla_\nu\right)f'(R) + 2\frac{\delta(\sqrt{-g}\mathcal{L}_{\text{M}})}{\sqrt{-g}\delta g^{\mu\nu}} = 0
$$

that is,

$$
f'(R)R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}f(R) + \left(g_{\mu\nu}\Box - \nabla_\mu\nabla_\nu\right)f'(R) = -2\frac{\delta(\sqrt{-g}\mathcal{L}_{\text{M}})}{\sqrt{-g}\delta g^{\mu\nu}}. \quad \text{(B.4)}
$$

We define the right hand side of (B.4) as the energy-momentum tensor, $T_{\mu\nu}$,

$$
T_{\mu\nu} := -2\frac{1}{\sqrt{-g}}\frac{\delta\left(\sqrt{-g}\mathcal{L}_{\text{M}}\right)}{\delta g^{\mu\nu}}.
$$

The tensor $T_{\mu\nu}$ describes the distribution of energy, momentum and stress associated to any force field. Using the above results into (B.4), we derive the Einstein field equations

$$
f'(R)R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}f(R) + \left(g_{\mu\nu}\Box - \nabla_\mu\nabla_\nu\right)f'(R) = T_{\mu\nu}. \quad \text{(B.5)}
$$

B.2 Conformal Equivalence of $f(R)$ Theories

It was proved in [34, 138] that under the conformal transformation,

$$\tilde{g}_{\mu\nu} = f'(R) g_{\mu\nu},$$

the field equations, under the assumption, $f'(R) > 0$, reduce to the Einstein field equations with a scalar field as a matter source

$$\tilde{G}_{\mu\nu} = T_{\mu\nu}(\tilde{g}, \phi),$$

where the energy momentum tensor is

$$T_{\mu\nu}(\tilde{g}, \phi) = \partial_\mu \phi \partial_\nu \phi - \frac{1}{2}\tilde{g}_{\mu\nu}\left((\partial\phi)^2 - 2V(\phi)\right),$$

and

$$\phi = \sqrt{\frac{3}{2}} \ln f'(R). \tag{B.6}$$

Under our assumption on $f'(R)$, Eq. (B.6) can be solved for R, to obtain a function $R(\phi)$. The corresponding potential is given by

$$V(\phi) = \frac{1}{2(f')^2}\left(Rf' - f\right).$$

In case of matter fields, the tensor $T^{(\mathrm{m})}_{\mu\nu}$ is conserved,

$$\begin{aligned}
\nabla^\mu T^{(\mathrm{m})}_{\mu\nu} &= -\frac{1}{2}\nabla_\nu f(R) + \nabla^\mu (R_{\mu\nu} f'(R)) + \nabla^\mu (g_{\mu\nu}\Box - \nabla_\mu \nabla_\nu) f'(R) \\
&= -\frac{1}{2}\nabla_\nu R f'(R) + \nabla^\mu R_{\mu\nu} f'(R) + R_{\mu\nu}\nabla^\mu f'(R) - R_{\mu\nu}\nabla^\mu f'(R) \\
&= -\frac{1}{2}\nabla_\nu R f'(R) + \nabla^\mu R_{\mu\nu} f'(R) \\
&= -\frac{1}{2}\nabla_\nu R f'(R) + \frac{1}{2}\nabla_\nu R f'(R) = 0,
\end{aligned}$$

and the Einstein tensor $G_{\mu\nu}$ is also conserved,

$$\nabla^{\mu} G_{\mu\nu} = 0.$$

One can define an effective energy momentum tensor $T^{(\text{eff})}_{\mu\nu}$ as

$$T^{(\text{eff})}_{\mu\nu} := \frac{1}{2}(f(R) - R)g_{\mu\nu} + (1 - f'(R))\, R_{\mu\nu} - (g_{\mu\nu}\Box - \nabla_{\mu}\nabla_{\nu})f'(R),$$

which is also conserved since

$$0 = \nabla^{\mu} G_{\mu\nu} = \nabla^{\mu}(T^{(\text{eff})}_{\mu\nu} + T^{(\text{m})}_{\mu\nu}).$$

Appendix C

A glossary to Dynamical Systems

For the convenience of the reader we present some definitions and theorems of the theory of the dynamical systems we have used in this thesis. Standard textbooks in Dynamical Systems are [83, 139–144].

C.1 Dynamical systems, trajectories and critical points

A *dynamical system* is described by a system of n ordinary differential equations of the form

$$\frac{d\mathbf{x}}{dt} \equiv \dot{\mathbf{x}} = \mathbf{f}\left(\mathbf{x}\right), \tag{C.1}$$

where $\mathbf{x} = \mathbf{x}\left(t\right) \in \mathbb{R}^n$, is a vector-valued function of the independent variable $t \in I \subseteq \mathbb{R}$, $\mathbf{f}$ is at least of class C^1 and is defined on an open subset E of $\mathbb{R}^n$, i.e., $\mathbf{f} : E \to \mathbb{R}^n$ defines a *vector field*. The set E of the dependent variables of (C.1) is called the *phase space* of (C.1). If an initial condition

$$\mathbf{x}(t_0) = \mathbf{x}_0 \in E, \tag{C.2}$$

is given, then the system of the form

$$\begin{aligned} \dot{\mathbf{x}} &= \mathbf{f}\left(\mathbf{x}\right), \\ \mathbf{x}\left(t_0\right) &= \mathbf{x}_0, \end{aligned} \tag{C.3}$$

is called an *initial value problem.* A *solution* or *trajectory* of (C.3) starting at $\mathbf{x}_0$, is a function $\phi : I \to \mathbb{R}^n$, satisfying (C.1) for all $t \in \mathbb{R}$, i.e.,

$$\dot{\phi}(t, \mathbf{x}_0) = \mathbf{f}(\phi(t, \mathbf{x}_0)), \quad \forall t \in I. \tag{C.4}$$

The *phase portrait* of a dynamical system is the set of all solutions of the system. Without loss of generality, solutions based on time t_0 can always be translated to $t_0 = 0$, due to vector field's invariance to translations in time. The following theorem is of great importance in the theory of dynamical systems, [139].

Theorem 7 (The fundamental existence-uniqueness theorem) *Suppose that* $\mathbf{f} \in C^1(E)$ *where* E *is an open subset of* $\mathbb{R}^n$ *containing* $\mathbf{x}_0$*. Then there exists an* $\varepsilon > 0$ *such that the initial value problem*

$$\begin{aligned} \dot{\mathbf{x}} &= \mathbf{f}(\mathbf{x}), \\ \mathbf{x}(0) &= \mathbf{x}_0, \end{aligned} \tag{C.5}$$

has a unique solution $\mathbf{x}(t)$ *on the interval* $[-\varepsilon, \varepsilon]$*.*

The fundamental existence-uniqueness theorem ensures that the trajectories of the system (C.1) do not cross. Furthermore, it is shown that the solution can be extended on a *maximal interval of existence*, $I_{\max}$. In fact, the following theorem is proved.

Theorem 8 *Under the hypotheses of Theorem 7, then for each point* $\mathbf{x}_0 \in E$*, there is a maximal open interval* $I_{\max}$ *on which the initial value problem* (C.5) *has a unique solution.*

The image of the solution on $\mathbb{R}^n$ is called the *orbit* of the system that passes through $\mathbf{x}_0$. Systems of the form (C.1) are called *autonomous* because the vector field $\mathbf{f}$ depends only on the variable $\mathbf{x}$, that is, does not contain time explicitly. The *flow* ϕ_t, generated by the vector field $\mathbf{f}$, is a smooth function

$$\begin{aligned} \phi_t &: E \to \mathbb{R}^n, \\ &\mathbf{x} \mapsto \phi_t(\mathbf{x}) \equiv \phi(\mathbf{x}, t), \end{aligned} \tag{C.6}$$

satisfying,

$$\dot{\phi}_t(\mathbf{x}, t_0) = \mathbf{f}\left(\phi_t(\mathbf{x}, t_0)\right), \quad \forall \mathbf{x}_0 \in E \;\; \text{and} \;\; t_0 \in I. \tag{C.7}$$

C.2 Linearization

A starting point to study the dynamical system (C.1), is to determine the critical points of (C.1) and the behaviour of the system near those points. A *critical point (or equilibrium point or fixed point)* of (C.1) is a point $\mathbf{x}_0$ such that

$$\mathbf{f}\left(\mathbf{x}_0\right) = \mathbf{0}. \tag{C.8}$$

By Taylor's theorem,

$$\mathbf{f}(\mathbf{x}) = \mathbf{f}(\mathbf{x}_0) + D\mathbf{f}(\mathbf{x}_0)(\mathbf{x} - \mathbf{x}_0) + \mathcal{O}(|\mathbf{x} - \mathbf{x}_0|)^2, \tag{C.9}$$

where

$$D\mathbf{f}(\mathbf{x}_0) = \left(\frac{\partial f_i}{\partial x_j}\right)_{\mathbf{x}=\mathbf{x}_0}$$

is the Jacobi matrix and $\mathcal{O}(|\mathbf{x}-\mathbf{x}_0|)^2$ stands for higher order terms. By the definition of the critical point, $\mathbf{f}(\mathbf{x}_0) = \mathbf{0}$, thus a good first approximation to the nonlinear vector field $\mathbf{f}$ near $\mathbf{x}_0$ is the linear function $D\mathbf{f}(\mathbf{x}_0)(\mathbf{x}-\mathbf{x}_0)$, and the local behaviour of the original system (C.1) near $\mathbf{x}_0$ is qualitatively determined by the behaviour of the linear system

$$\dot{\mathbf{x}} = D\mathbf{f}(\mathbf{x}_0)(\mathbf{x} - \mathbf{x}_0). \tag{C.10}$$

A critical point $\mathbf{x}_0$ is said to be *hyperbolic* if none of the eigenvalues of $D\mathbf{f}(\mathbf{x}_0)$ have zero real part, otherwise it is called *non-hyperbolic*. If a critical point is non-hyperbolic, linear stability techniques cannot be applied. For simplicity we assume the critical point has been translated to the origin, i.e., $\mathbf{x}_0 = \mathbf{0}$ and $A := D\mathbf{f}(\mathbf{0})$.

Theorem 9 (The Hartman–Grobman Theorem) *Let* $\mathbf{f} \in C^1$ *in an open* $E \subset \mathbb{R}^n$ *containing the origin,* ϕ_t *be the flow of the dynamical system* (C.1) *and* $\mathbf{0}$ *is a hyperbolic critical point of the* (C.1). *Then there exists a homeomorphism* h *of an open set* U *containing the origin onto an open set* V *containing the origin such that for each* $\mathbf{x}_0 \in U$, *there exists an open interval* $I_0 \subset \mathbb{R}$ *containing zero such that for all* $\mathbf{x}_0 \in U$ *and* $t \in I_0$

$$h \circ \phi_t(\mathbf{x}) = e^{At} h(\mathbf{x}), \tag{C.11}$$

i.e., h *maps trajectories of* (C.1) *near the critical point onto trajectories of* (C.10) *near the critical point and preserves the parametrization by time.*

The Hartman–Grobman Theorem is very important since it states that under certain assumptions on the matrix A, we can analyse the local behaviour of the nonlinear system (C.1) near its critical points by studying the corresponding linear system (C.12). Therefore, the analysis of the local behaviour of a nonlinear system (C.1) near a critical point $\mathbf{x}_0$ is reduced to the analysis of the equivalent linear system

$$\dot{\mathbf{x}} = A\mathbf{x}. \tag{C.12}$$

The eigenvalues of the Jacobi matrix computed at the critical point determines the stability of the point in question, provided the eigenvalues have non zero real part. Two autonomous dynamical systems are said to be *topologically equivalent* or to have the same qualitative structure in a neighbourhood of a critical point if there is a homeomorphism h mapping an open set U containing the critical point onto an open set V containing the critical point which maps trajectories of the first dynamical system in U onto trajectories of the second dynamical system in V, preserving their orientation by time.

A critical point $\mathbf{x}_0$ is said to be *stable* if a solution of (C.1) near $\mathbf{x}_0$ remains close to $\mathbf{x}_0$ for all time. A critical point that is not stable is said to be *unstable*. More precisely a critical point $\mathbf{x}_0$ of the system (C.1) is said to be *Lyapunov stable* if for every $\varepsilon > 0$ there exists a $\delta > 0$ such that, for any solution of (C.1) starting at $\mathbf{y}_0$, i.e. $\mathbf{y}(t_0) = \mathbf{y}_0$, with $|\mathbf{x}_0 - \mathbf{y}_0| < \delta$, then $|\mathbf{x}_0 - \mathbf{y}_t| < \varepsilon$, for $t > t_0$. If a critical point $\mathbf{x}_0$ is stable and in addition there exists a $\delta > 0$ such that $|\mathbf{x} - \mathbf{x}_0| < \delta$, implies $\lim_{t\to\infty} \phi_t(\mathbf{x}) = \mathbf{x}_0$, then $\mathbf{x}_0$ is called *asymptotically stable* critical point. A classification regarding the stability of critical points is given in the following definition.

Definition 1 *Assuming a linear system* (C.12)*, then the equilibrium point* $\mathbf{x}_0$*, is said to be:*

- *Sink, if all of the eigenvalues of the matrix* $D\mathbf{f}(\mathbf{x}_0)$ *have negative real part.*

- *Source, if all of the eigenvalues of the matrix* $D\mathbf{f}(\mathbf{x}_0)$ *have positive real part.*

- *Saddle, if it is a hyperbolic critical point and the matrix* $D\mathbf{f}(\mathbf{x}_0)$ *has at least one eigenvalue with negative real part and at least one eigenvalue with positive real part.*

In 2-dimensions the following theorem determines the stability of a critical point of the linear system (C.10) and subsequently of its topologically equivalent nonlinear system, [139].

Theorem 10 *Let $\delta = \det A$ and $\tau = \operatorname{tr} D\mathbf{f}(\mathbf{x}_0)$ the determinant and trace of the matrix A of the linear system $\dot{\mathbf{x}} = A\mathbf{x}$ respectively.*

- *If $\delta < 0$ then (C.10) has a saddle at the origin.*
- *If $\delta > 0$ and $\tau^2 - 4\delta \geq 0$ then (C.10) has a stable node at the origin if $\tau < 0$ or an unstable node at the origin if $\tau > 0$.*
- *If $\delta > 0$ and $\tau^2 - 4\delta < 0$ then (C.10) has a stable focus at the origin if $\tau < 0$ or an unstable focus at the origin if $\tau > 0$.*
- *If $\delta > 0$ and $\tau = 0$ then (C.10) has a center at the origin.*

Definition 2 *The stable subspace E^s, the center subspace E^c, and the unstable subspace E^u are the subspaces of $\mathbb{R}^n$, spanned by the real and imaginary parts of the generalised eigenvectors $\mathbf{w}_i$ of the real matrix $D\mathbf{f}(\mathbf{x}_0)$ corresponding to the eigenvalues $\lambda_i = a_i + b_i$, with negative, zero and positive real parts respectively.*

Theorem 11 (The Stable Manifold Theorem) *Let $\mathbf{f} \in C^1$ be a function defined on an open subset $E \subset \mathbb{R}^n$, $\mathbf{0} \in E$, and ϕ_t be the flow of (C.1). Suppose that $\mathbf{f}(\mathbf{0}) = \mathbf{0}$ and that $D\mathbf{f}(\mathbf{0})$ has k eigenvalues with negative real part and $n - k$ eigenvalues with positive real parts. Then there exists a k-dimensional differentiable manifold S tangent to the stable subspace E^s of the linear system (C.10) at $\mathbf{0}$ such that $\forall t \geq 0,\ \phi_t(S) \subset S$ and $\forall \mathbf{x}_0 \in S$*

$$\lim_{t\to\infty} \phi_t(\mathbf{x}_0) = \mathbf{0};$$

there exists an $n - k$-dimensional differentiable manifold U tangent to the unstable subspace E^u of the linear system (C.10) at $\mathbf{0}$ such that $\forall t \leq$

$0,\ \phi_t(U) \subset U$ *and* $\forall \mathbf{x}_0 \in U$

$$\lim_{t\to -\infty} \phi_t(\mathbf{x}_0) = \mathbf{0};$$

Definition 3 *Let ϕ_t be the flow of the nonlinear system* (C.1)*. The stable manifold of* (C.1) *at $\mathbf{x}_0$ is defined as*

$$W^s(\mathbf{x}_0) = \bigcup_{t\leq 0} \phi_t(S),$$

and the unstable manifold as

$$W^u(\mathbf{x}_0) = \bigcup_{t\geq 0} \phi_t(S).$$

C.3 Asymptotic Behaviour

A set S is said to be *invariant* under the flow ϕ_t if $\phi_t(S) \subset S$ for all $t \in \mathbb{R}$. A set S is said to be *positively invariant* under the flow ϕ_t if $\phi_t(S) \subset S$ for all $t \geq 0$. A point $\mathbf{x}_0$ is called an ω *limit point,* $\omega(\mathbf{x})$*,* of $\mathbf{x} \in \mathbb{R}^n$, if there exists a sequence $\{t_i\}, t_i \to \infty$ as $i \to \infty$, such that

$$\phi(t_i, \mathbf{x}) \to \mathbf{x}_0.$$

Similarly, a point $\mathbf{x}_0$ is called an α *limit point,* $\omega(\mathbf{x})$*,* of $\mathbf{x} \in \mathbb{R}^n$, if there exists a sequence $\{t_i\}, t_i \to -\infty$, such that

$$\phi(t_i, \mathbf{x}) \to \mathbf{x}_0.$$

The set of all ω limit points of a flow is called *the ω limit set*, and the set of all α limit points of a flow is called *the α limit set.* It is proved [141] that ω limit points (similar for α limit points, for the reversed flow), have the following properties:

(i) $\omega(\mathbf{p}) \neq \varnothing$,

(ii) $\omega(\mathbf{p})$ is closed,

(iii) $\omega(\mathbf{p})$ is invariant under the flow,

(iv) $\omega(\mathbf{p})$ is connected,

where $\mathbf{p} \in \mathcal{M}$, $\mathcal{M}$ is a positively invariant set for this flow. An *attracting set* A, is a closed invariant subset of $\mathbb{R}^n$ for which there is some neighbourhood U of A such that for all $t \geq 0$

$$\phi(t, U) \subset U \quad \text{and} \quad \bigcap_{t>0} \phi(t, U) = A.$$

The open set U is referred to as a *trapping region*, and the union $\bigcup_{t\leq 0} \phi(t, U)$ is referred to as the *basin of attraction* of an attracting set A. A closed invariant set A is called an *attractor* if, for any two open sets $U, V \subset A$, for all $t \in \mathbb{R}$

$$\phi(t, U) \cap V \in \mathbb{R} - \{0\}.$$

Finally, we state the following very useful Theorem, see for example [141].

Theorem 12 (LaSalle Invariance Principle) *Let $\mathcal{M}$ be a trapping region, $V(\mathbf{x})$ a Liapunov function on $\mathcal{M}$ and M the union set of all trajectories that start in the set $E := \{x \in \mathcal{M} |\ \dot{V}(x) = 0\}$ and remain in the set E for all $t > 0$. Then, for all $x \in \mathcal{M}$, $\phi_t(\mathbf{x}) \to M$ as $t \to \infty$.*

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