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Master thesis

Lévy Processes and Applications

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Abstract

The aim of the present thesis is to introduce the concepts of Lévy processes and jump-diffusion models. At first, we focus on the theory of Lévy processes. Then, there will be presented and analyzed a Lévy jump diffusion process, which is the simplest Lévy process and offers significant insight into the distributional and path structure of a Lévy process. Afterwards, we will state some important results, such as infinite divisibility, Lévy-Khintchine formula and the Lévy-Itô decomposition. Therefore, the Lévy measure and path properties, as well as the elements from martingale theory are presented for better understanding of the features of Lévy processes. Later, we will present some examples as well as applications of Lévy processes in different fields of science, such as financial mathematics and actuarial science. In particular, we will show how important the Lévy processes are in financial modeling, option pricing, construction of optimal hedging portfolios and risk management e.g. computation of risk measures for portfolios in presence of jumps in asset prices. Finally, we will implement Monte Carlo simulation in R for jump-diffusion models with correlational companies and we will also show some numerical results from option pricing.

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Chapter 1

1. Introduction

In this chapter we give the general Definition of Lévy process and study some examples of Lévy processes. By doing so, we will use stochastic analysis to formulate some theoretical results as well as their applications. Specifically we will study the jump processes and the role that certain subtle behavior concerning their fluctuations play in explaining different types of Phenomena appearing in a number of classical models of applied probability.

2. Stochastic Process

Let $(\Omega, \mathcal{F}, F, \mathbb{P})$ denote a stochastic basis, or filtered probability space, i.e. a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ endowed with a filtration $F = (\mathcal{F}_t)_{t\in[0.T]}$.The stochastic basis satisfies the usual conditions if it is right-continuous i.e. $\mathcal{F}_t = \mathcal{F}_{t_+}$, where $\mathcal{F}_{t_+} = \bigcap_{s>t} \mathcal{F}_s$, and is complete, i.e. the σ-algebra $\mathcal{F}$ is $\mathbb{P}$-complete and every $\mathcal{F}_t$ contains all $\mathbb{P}$-null sets of $\mathcal{F}$. Let $T \in [0, \infty]$ denote the time horizon which, in general, can be infinite.

Definition 2.1.(*Stochastic Process*)

Suppose that $(\Omega, \mathcal{F}, P)$ is a probability space, and that $I \subset \mathbb{R}$ is of infinite cardinality. Suppose further that for each $t \in \mathbb{R}$, there is a random variable $X_t: \Omega \to \mathbb{R}$ defined on $(\Omega, \mathcal{F}, P)$ .The function $X: I \times \Omega \to \mathbb{R}$ defined by $X(t, \omega) = X_t(\omega)$ is called a *stochastic process* with indexing set I , and is written $X = \{X_t, t \in I\}$.

Remark 2.2.

A stochastic process $X = \{X(t): t \in T\}$ is a family of random variables which are defined in the same probability space $(\Omega, \mathcal{F}, P)$. We will always assume that the cardinality of I is infinite, either countable or uncountable. If $I = \mathbb{Z}^+$ then, we call X a discrete time Stochastic process, and if $I = [0, \infty)$ then X is said to be a continuous time Stochastic process.

3. Lévy processes

The term “Lévy process” honours the work of the French mathematician Paul Lévy who, although not alone in his contribution, played an instrumental role in bringing together an understanding and characterization of processes with stationary independent increments. In earlier literature, Lévy processes can be found under a number of different names. In the 1940s, Lévy himself referred to them as a sub-class of *processus additif* (additive processes), that is processes with independent increments. For the most part however, research literature through the 1960s and 1970s refers to Lévy processes simply as *processes with stationary independent increments*. One sees a change in language through the 1980s and by the 1990s the use of the term “Lévy process” had become standard. For a detailed discussion on Lévy processes, see e.g. the manuscript [1].

Definition 3.1.(*Lévy Process*)

A càdlàg, adapted, R^d- valued stochastic process $X = (X_t)_{0 \le t \le T}$ with $X_0 = 0$ a.s. is called a *Lévy process* if the following conditions are satisfied:

(L1) X has *independent increments,* i.e. $X_t - X_s$ is independent of $\mathcal{F}_s$ for any $0 \le s < t \le T$

(L2)X has *stationary increments,* i.e. for any $0 \le s, t \le T$ the distribution of $X_{t+s} - X_t$ does not depend on t.

(L3) X is *stochastically continuous,* i.e. for any $0 \le t \le T$ and $\varepsilon > 0$:

$\lim_{s \to t} P(|X_t - X_s| > \varepsilon) = 0$.

Examples of Lévy process

- The linear drift is the simplest Lévy process, a deterministic process.
- The Brownian motion is the only non-deterministic Lévy process with continuous sample paths.
- The Poisson, the Compound Poisson and the Compensated Poisson processes are also examples of Lévy processes.

The sum of a linear drift, a Brownian motion and a Poisson process is again a Lévy process. It’s also called a “jump-diffusion” process and we can see it in figure1. We shall call it *a Lévy jump-diffusion process.*

Definition3.2. (*Brownian Motion*)

A real-valued process $B = \{B_t \ : t \ \geq 0\}$defined on a probability space $(\Omega, \mathcal{F}, P)$is said to be a Brownian motion if the following hold:

i. The paths of B are P-almost surely continuous.

ii. $\mathbb{P}(\boldsymbol{B_0} = \mathbf{0}) = \mathbf{1}$

iii. For $\mathbf{0} \leq \boldsymbol{s} \leq \boldsymbol{t}$, $\boldsymbol{B_t} - \boldsymbol{B_s}$ is equal in distribution to $\boldsymbol{B_{t-s}}$

iv. For $\mathbf{0} \leq \boldsymbol{s} \leq \boldsymbol{t}$, $\boldsymbol{B_t} - \boldsymbol{B_s}$ is independent of $\{\boldsymbol{B_u} : \boldsymbol{u} \leq \boldsymbol{s}\}$.

v. For each $\boldsymbol{t} > \mathbf{0}$, $\boldsymbol{B_t}$ is equal in distribution to a normal random variable with variance t.

Definition3.3. (*Poisson Process*)

A process valued on the non-negative integers $N = \{N_t : t \geq 0\}$, defined on a probability space $(\Omega, \mathcal{F}, P)$, is said to be a Poisson process with intensity λ>0 if the following hold:

i. The paths of N are P-almost surely right continuous with left limits.

ii. $\mathbb{P}(\boldsymbol{N_0} = \mathbf{0}) = \mathbf{1}$

iii. For $\mathbf{0} \leq \boldsymbol{s} \leq \boldsymbol{t}$, $\boldsymbol{N_t} - \boldsymbol{N_s}$ is equal in distribution to $\boldsymbol{N_{t-s}}$.

iv. For $\mathbf{0} \leq \boldsymbol{s} \leq \boldsymbol{t}$, $\boldsymbol{N_t} - \boldsymbol{N_s}$ is independent of $\{\boldsymbol{N_u} : \boldsymbol{u} \leq \boldsymbol{s}\}$.

v. For each t>0, $\boldsymbol{N_t}$ is equal in distribution to a Poisson random variable with parameter λt.

4. 'Toy' Example: A Lévy jump-diffusion

Assume that the process $X = (X_t)_{t \geq 0}$ is a Lévy jump-diffusion i.e. a linear deterministic process plus Brownian motion plus a compensated compound Poisson process. The Lévy jump-diffusion process is the simplest Lévy process that contains both a diffusive part and a jump part [3]. The paths of this process are described by

$$X_t \ = \ bt + \sigma B_t + \left(\sum_{k=1}^{N_t} J_k - t\lambda\beta \right)$$

- $\boldsymbol{B} = (\boldsymbol{B_t})_{\boldsymbol{t} \geq \mathbf{0}}$ is a standard Brownian motion
- $\boldsymbol{N} = (\boldsymbol{N_t})_{\boldsymbol{t} \geq \mathbf{0}}$ is a Poisson process with intensity $\boldsymbol{\lambda} \in \mathbb{R}_{\geq \mathbf{0}}$(i.e. $\boldsymbol{E}[\boldsymbol{N_t}]$=λt)

- $\boldsymbol{J} = (\boldsymbol{J_k})_{\boldsymbol{k \geq 1}}$ is an i.i.d sequence of random variables with probability distribution F and $\boldsymbol{E}[\boldsymbol{J_k}] = \boldsymbol{\beta} < \infty$. Here F describes the distribution of the jumps, which arrive according to the Poisson process N. All sources of randomness are assumed mutually independent.

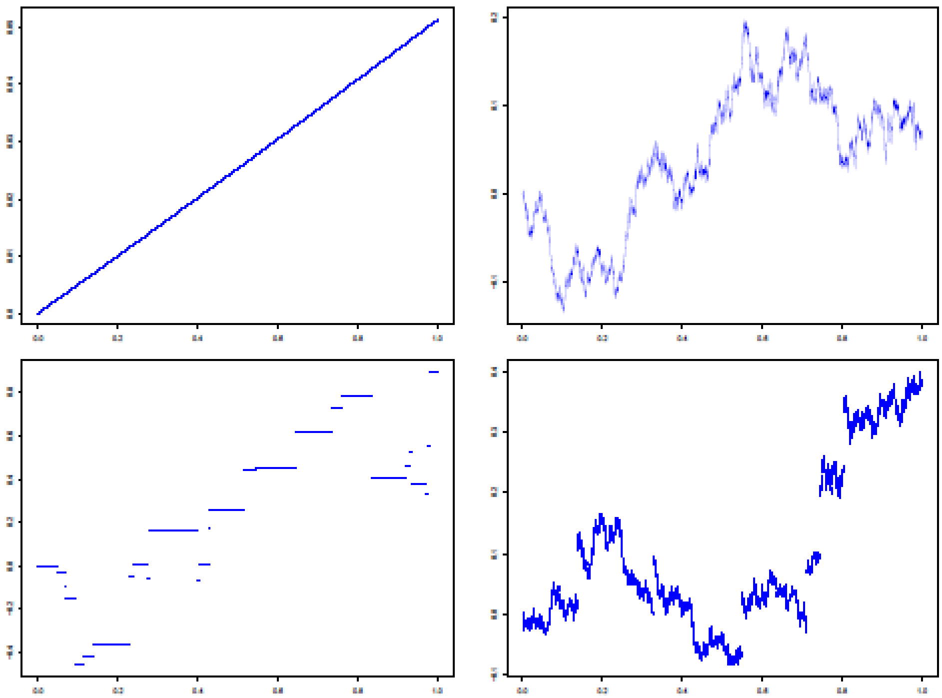

Figure 1. Sample paths of a linear drift process (top-left), a Brownian motion (top-right), a compound Poisson process (bottom-left) and a Lévy-jump diffusion.

Definition 4.1.(*Characteristic function*)

Let $X = (X_1, \dots., X_d)$ be a vector of random variables defined on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and takes values on R^d. The characteristic function

$$\Phi_X(u) = E\left[e^{i\langle u,X\rangle}\right] = E[e^{i\sum_{j=1}^{d} u_j X_j}]$$

for all $u = (u_1, u_2, \dots u_d) \in R^d$, where $\langle u, X\rangle = \sum_{j=1}^{d} u_j X_j$ is the inner product of vectors $u, X \in R^d$. Generally the characteristic function of the probability measure μ (or Fourier transform) defined on space $(R^d, \mathcal{B}(R^d))$ is

$$\varphi(u) = \int_{R^d} e^{i\langle u,X\rangle} \mu(dx) \text{ for all } u \in R^d.$$

The characteristic function defines unambiguously the distribution of the random variable. Therefore, a probability measure μ on the space $(R^d, \mathcal{B}(R^d))$ is determined uniquely by its characteristic function.

We will calculate the characteristic function of the Lévy jump-diffusion[3], since it offers significant insight into the structure of the characteristic function of general Lévy processes. The ***characteristic function of X_t***, taking into account that all sources of randomness are independent , is

$$E\left[e^{iuX_t}\right] = E[\exp\,(iu(bt + \sigma B_t + \sum_{k=1}^{N_t} J_k - t\lambda\beta))]$$

$$= \exp\,[iubt]E[\exp(iu\sigma B_t)]E[\exp\,(iu\sum_{k=1}^{N_t} J_k - iut\lambda\beta)]$$

recalling that the characteristic functions of the normal and the Compound Poisson distributions are

$$E\left[e^{iu\sigma B_t}\right] = e^{-\frac{\sigma^2 u^2 t}{2}}, B_t \sim \mathbb{N}(0, t)$$

$$E\left[e^{iu\sum_{k=1}^{N_t} J_k}\right] = e^{\lambda t(E[e^{iuJ_k - 1}])}, N_t \sim Poi(\lambda t)$$

$$= \exp[iubt]\exp\left[-\frac{u^2\sigma^2}{2}t\right]\exp\,[\lambda t(E\left[e^{iuJ_k} - 1\right] - iuE[J_k])]$$

$$= \exp[iubt]\exp\left[-\frac{u^2\sigma^2}{2}t\right]\exp\,[\lambda t(E\left[e^{iuJ_k} - 1 - iuJ_k\right])]$$

And since the distribution of J_k is F we have

$$= \exp[iubt]\exp\left[-\frac{u^2\sigma^2}{2}t\right]\exp\left[\lambda t\int_{\mathbb{R}} (e^{iux} - 1 - iux)F(dx)\right]$$

Finally, since t is a common factor, we can rewrite the above equation as

$$E\left[e^{iuX_t}\right] = \exp\left[t\left(iub - \frac{u^2\sigma^2}{2} + \int_{\mathbb{R}} (e^{iux} - 1 - iux)\lambda F(dx)\right)\right] \quad (1.1)$$

We can make the following observations based on the structure of the characteristic function of the random variable X_t from the Lévy jump-diffusion:

a) Time and space factorize;

b) The drift, the diffusion and the jump parts are seperated;

c) The jump part decomposes to $\boldsymbol{\lambda \times F}$, where λ is the expected number of jumps and F is the distribution of jump size.

Question: Are these observations true for any Lévy process?

The answer for a) and b) is yes, because Lévy processes have stationary and independent increments. The answer for c) is no, because there exist Lévy processes with infinitely many jumps (on any compact time interval) thus their expected number of jumps is also infinite [3].

The basic connections. The next section will be devoted for establishing the connection between the following mathematical objects:

- Lévy processes $\boldsymbol{X} = (\boldsymbol{X_t})_{\boldsymbol{t \geq 0}}$
- Infinitely divisible distributions $\boldsymbol{\rho} = \boldsymbol{\mathcal{L}}(\boldsymbol{X_1})$
- Lévy triplets $(\boldsymbol{b}, \boldsymbol{c}, \boldsymbol{v})$.

The following diagram displays how these connections can be proved, where $\boldsymbol{LK}$stands for the Lévy-Khintchine formula, $\boldsymbol{LI}$ for the Lévy-Itô decomposition, $\boldsymbol{CFE}$ for the Cauchy functional equation and $\boldsymbol{SII}$ for stationary and independent increments[3].

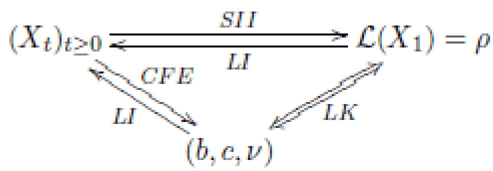

Figure2. Basic connections between Lévy processes, infinitely divisible

distributions and Lévy triplets.

- Show that the law of $\boldsymbol{X_t}$ is infinitely divisible using the stationarity and independence of the increments. (Lemma 5.5)
- Show that for every Lévy triplet $(\boldsymbol{b}, \boldsymbol{c}, \boldsymbol{v})$ that satisfies (6.1) the measure ρ is infinitely divisible.
- Use Kolmogorov's extension theorem to show that for every infinitely divisible distribution ρ, there exists a Lévy process $\boldsymbol{X} = (\boldsymbol{X_t})_{\boldsymbol{t \geq 0}}$ such that $\boldsymbol{P_{X_1}} = \boldsymbol{\rho}$;
- Prove the following version of the Lévy-Itô decomposition: every Lévy process admits the path decomposition(7.1) . (see analytically [2]).

Theorem 4.2.(Lévy continuity Theorem)

Let $(\rho_n)_{n\in\mathbb{N}}$ be a probability measure on $\mathbb{R}^d$ whose characteristic functions $\hat{\rho}_n(u)$ converges to some function $\hat{\rho}(u)$, for all u, where $\hat{\rho}$ is continuous at 0. Then $\hat{\rho}$ is the characteristic function of a probability distribution ρ and $\rho_n \overset{d}{\to} \rho$.

Definition4.3. (Markov property)

Let X be a Lévy process and $t \geq 0$ a fixed time, then the pre-t process $(X_r)_{r\leq t}$ is independent of the post-t process $(X_{t+s} - X_t)_{s\geq 0}$,and the post-t process has the same distribution as X[5].

Theorem 4.4.(*Kac's theorem*)

The random variables $X_1, \dots X_n$ are independent if and only if $\Phi_{X_1,\dots X_n}(u_1, \dots u_n) = E(\exp\,[i\sum_{j=1}^{n}(u_j, X_j)]) = \Phi_{X_1} \cdots \Phi_{X_n}$ for all $u \in R^d$.

5. Lévy processes and Infinite Divisibility

In this chapter we will attempt to give some indication of how rich a class of processes the Lévy processes form. To illustrate the variety of processes captured within the definition of a Lévy process, we explore briefly the relationship of Lévy processes with the infinite divisible distributions. De Finetti (1929) introduced the notion of an *infinitely divisible* distribution and showed that they have an intimate relationship with Lévy processes. This relationship gives a reasonably good impression of how varied the class of Lévy processes really is.

Definition 5.1. (*Infinite Divisibility*)

A random variable X is *infinitely divisible* if for all $n \in \mathbb{N}$, there exist i.i.d random variables $X_1^{(n),}, \dots, X_n^{(n)}$ such that $X = {X_1}^{(n)} + {X_2}^{(n)} \dots + {X_n}^{(n)}$. (5.1)

Definition 5.2.A probability measure ρ is *infinitely divisible* if, for all $n \in \mathbb{N}$, there exists another probability measure ρ_n such that

$$\rho = \underbrace{\rho_n * \rho_n \dots * \rho_n}_{n-times}$$

Proposition 5.3. A probability measure ρ *is infinitely divisible* if and only if, for all ∈ ℕ , there exists another another probability measure ρ_n such that

$$\hat{\rho}(u) = (\hat{\rho}_n(u))^n.$$

Lemma 5.4. If $(\rho_k)_{k\geq 0}$ is a sequence of infinitely divisible distributions and $\rho_k \overset{w}{\rightarrow} \rho$, then ρ is also infinitely divisible.

Examples of infinitely divisible distributions are the Normal, Poisson, Exponential, Geometric, the Negative Binomial, the Cauchy and the strictly stable distributions. Counter examples are the uniform and the binomial distributions.

Lévy processes have Infinitely divisible laws

Lemma 5.5. Let $X = (X_t)_{t\geq 0}$ be a Lévy process. The random variables X_t, $t \geq 0$, are infinitely divisible.

Proof. Let $X = (X_t)_{t\geq 0}$ be a Lévy process; for any $n \in \mathbb{N}$ and any $t > 0$ we trivially have that

$$X_t = X_{\frac{t}{n}} + \left(X_{\frac{2t}{n}} - X_{\frac{t}{n}}\right) + \cdots + \left(X_t - X_{\frac{(n-1)t}{n}}\right) \qquad (5.2)$$

the stationarity of the increments of the Lévy process yields that

$$X_{\frac{tk}{n}} - X_{\frac{t(k-1)}{n}} \triangleq X_{\frac{t}{n}} \text{ for any } k \geq 1,$$

where ≜ is equality in distribution, while the independence of the increments yields that the random variables

$$X_{\frac{tk}{n}} - X_{\frac{t(k-1)}{n}}, k \geq 1$$

are independent of each of other.

Thus, $\left(X_{\frac{tk}{n}} - X_{\frac{t(k-1)}{n}}\right)_{k\geq 1}$ is i.i.d sequence of random variables and, from Definition 5.1. we conclude that the random variable X_t is infinitely divisible.

We showed that for any Lévy process $X = (X_t)_{t\geq 0}$ the random variables $X_t$are infinitely divisible. Next, we would like to compute the characteristic function of X_t. Since X_t is infinitely divisible for any $t \geq 0$ we know that $X_1$is infinitely divisible and has the Lévy-Khinchine representation in terms of some triplet (b, c, v).

6.The Lévy-Khintchine representation

The next result provides a complete characterization of infinitely divisible distributions in terms of their characteristic functions. This is the celebrated Lévy-Khintchine formula. B.de Finetti and A.Kolmogorov were the first to prove versions of this representation under certain assumptions. P.Lévy and A.Khintchine independently proved it in general case, the former by analyzing the sample paths of the process and the latter by a direct analytic methods.

Definition 6.1. We will call (b, c, v) the ***Lévy*** or ***characteristic triplet*** of the infinitely divisible measure **ρ**. We call **b** the ***drift*** term, **c** the ***Gaussian*** or ***diffusion*** coefficient and **v** the ***Lévy measure*** (see definition 9.1).

Theorem 6.2. (*Lévy-Khintchine*)

A measure ρ is infinitely divisible if and only if there exists a triplet (b, c, v) with $b \in R^d$, c a symmetric, non-negative definite, dxd matrix, and v a Lévy measure, such that

$$\widehat{\rho}(u) = \exp\Big(\mathrm{i}\langle \mathrm{u}, \mathrm{b}\rangle - \frac{\langle \mathrm{u}, \mathrm{cu}\rangle}{2} + \int_{\mathrm{R^d}} (\mathrm{e}^{\mathrm{i}\langle \mathrm{u},\mathrm{x}\rangle} - 1 - \mathrm{i}\langle \mathrm{u}, \mathrm{x}\rangle\, 1_{\mathrm{D}} \Big) \mathrm{v}(\mathrm{dx}) \quad (6.1)$$

where D=closed ball in R^d, i.e $D \coloneqq \{|x \le 1|\}$.

Truncation function and Uniqueness

Definition 6.3. A Truncation function is a bounded function $h: R^d \to R^d$ that satisfies h(x)=x in a neighborhood of zero.

Definition 6.4. A truncation function $h': \mathbb{R}^d \to \mathbb{R}$ is a bounded and measurable function, satisfying

$$h'(x) = 1 + o(|x|), as\ |x| \to 0$$

$$h'(x) = O(1/|x|),\ as\ |x| \to \infty$$

Remark 6.5. The two definitions are related via $h(x) = x \cdot h'(x)$.

Example 6.6. The following are some well known examples of truncation functions:

I. $\boldsymbol{h(x) = x1_D(x)}$, typically called the canonical truncation function

II. $\boldsymbol{h(x) \equiv 0\ and\ h(x) \equiv x}$, are also commonly used truncation functions.

III. $\boldsymbol{h(x) = \frac{x}{1+|x|^2}}$, a continuous truncation function.

The Lévy-Khintchine representation of $\hat{\rho}$ depends of the choice of the truncation function. Indeed, if we use another truncation function h instead of the canonical one, then (6.1) can be rewritten as

$$\hat{\rho}(u) = \exp\,(i\langle u, b_h\rangle - \frac{\langle u, cu\rangle}{2} + \int_{\mathbb{R}^d} \big(e^{i\langle u,x\rangle} - 1 - i\langle u, h(x)\rangle\big) v(dx))\ \ with\ \ b_h$$

defined as follows:

$$b_h = b + \int_{\mathbb{R}^d} \big(h(x) - x1_D(x)\big) v(dx)$$

If we want to stress the dependence of the Lévy triplet on the truncation function, we will denote it by

$$(b_{h,}c, v) \text{ } or \text{ } (b, c, v)_h$$

Note that the diffusion characteristic c and the Lévy measure v are invariant with respect to the choice of the truncation function(see figure3).

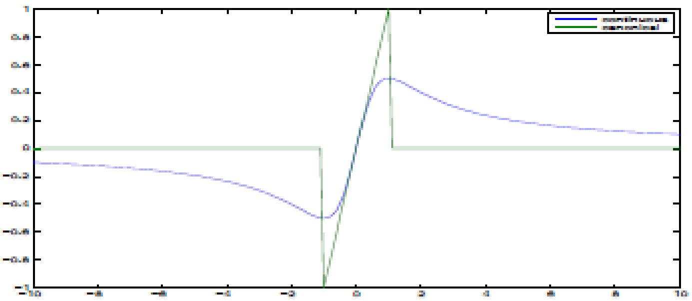

Figure 3.Illustration of the canonical and the continuous truncation functions

Proposition 6.7. The representation of $\hat{\rho}$ by (b, c, v) in (6.1) is unique.

Remark 6.8. There is no rule about which truncation function to use, among the permissible ones. One has to be consistent with ones choice of a truncation function. That is the same choice should be made for the Lévy-Khintchine representation of the characteristic function, the Lévy triplet and the path decomposition of the Lévy process.

Example 6.9. Let us revisit the Lévy jump-diffusion process. In this example since we have assumed that the Lévy measure is finite and we have assumed that $E[J_k] < \infty$, all truncation functions are permissible. The distribution of the random variable X_1 from the Lévy jump-diffusion is infinitely divisible and have Lévy triplet with respect to the canonical truncation function is $(b - \int_{D^c} x\lambda F(dx), \sigma^2, \lambda \times F)$. The triplets with respect to the zero and the linear truncation functions are

$$(b - \int_{\mathbb{R}} x\lambda \, F(dx), \sigma^2, \lambda \times F)_0 \quad \text{and} \quad (b, \sigma^2, \lambda \times F)_{id}.$$

The Lévy Exponent

One way to establish whether a given random variable has an infinitely divisible distribution is via its characteristic exponent. Suppose that Θ has characteristic exponent $\Psi(u) := -\log\ \mathbb{E}(e^{iu\Theta})$ for all $u \in \mathbb{R}$. Then Θ has an infinitely divisible distribution if for all $n \geq 1$, there exists a characteristic exponent of a probability distribution, say Ψ_n , such that $\Psi(u) = n\Psi_n(u)$ for all $u \in \mathbb{R}$ [1].The full extent to which we may characterize infinitely divisible distributions is described by the characteristic exponent Ψ and an expression known as the Lévy-Khintchine formula.

Definition 6.10. (*The Lévy Exponent*)

We define the Lévy exponent ψ of X by

$$\psi(u) = i\langle u, b\rangle - \frac{\langle u, cu\rangle}{2} + \int_R (e^{i\langle u,x\rangle} - 1 - i\langle u, x\rangle 1_D (x)v(dx) \quad (6.2)$$

where $\left[e^{i\langle u,X_1\rangle}\right] = e^{\psi(u)}$.

A special case of Lévy-Khintchine formula was established by Kolmogorov (1932) for infinitely divisible distributions with second moments. However it was Lévy (1934) who gave a complete characterization of infinitely divisible distributions and in doing so he also characterized the general class of processes with stationary independent increments. Let us now discuss in further detail the relationship between infinitely divisible distributions and processes with stationary independent increments.

From the definition of a Lévy process we see that for any t>0, X_t is a random variable belonging to the class of infinitely divisible distributions. According to [1], this follows from the fact that for any n=1,2,…

$$X_t = X_{t/n} + (X_{2t/n} - X_{t/n}) + \cdots + (X_t - X_{(n-1)t/n})$$

Together with the fact that X has stationary independent increments. Suppose now that we define for all $\theta \in \mathbb{R}, t \geq 0$,

$$\Psi_t(\theta) = -\log\ E(e^{i\theta X_t})$$

And hence for rational t>0, $\Psi_t\ (\theta) = t\Psi_1(\theta)$.

In conclusion, any Lévy process has the property that for all $t \geq 0$

$$E\left[e^{i\theta X_t}\right] = e^{-t\Psi(\theta)}\ ,$$

Where $\Psi(\theta) := \Psi_1(\theta)$ is the characteristic exponent of X_1, which has an infinitely divisible distribution.

Theorem 6.11.

Let $X = (X\ _t)_{t\geq 0}$ be a Lévy process, then $E\left[e^{i\langle u,X_t\rangle}\right] = e^{t\Psi(u)}$ where ψ is the Lévy exponent of X.

Proof. Define the function $\varphi_u(t) = E\left[e^{i\langle u,X_t\rangle}\right]$. Using the independence and stationarity of the increments we have that

$$\varphi_u(t+s) = E\left[e^{i\langle u,X_{t+s}\rangle}\right] = E\left[e^{i\langle u,X_{t+s}-X_s\rangle}e^{i\langle u,X_s\rangle}\right] =$$

$$E\left[e^{i\langle u,X_{t+s}-X_s\rangle}\right]E\left[e^{\langle iu,X_s\rangle}\right] = \varphi_u(t)\varphi_u(s). \quad (6.3)$$

Moreover, $\varphi_u(0) = E\left[e^{i\langle u,X_0\rangle}\right] = 1$ by definition. Since X is stochastically continuous, we can show that $\varphi \vdash> \varphi_u(t)$ is continuous. Note that (6.3) is Cauchy's second functional equation and the unique continuous solution to this equation has the form$\varphi_u(t) = e^{t\theta(u)}$, where θ: $\mathbb{R}^d \to \mathbb{C}$.Now the result follows since X_1 is infinitely divisible, which yields $\varphi_u(1) = E\left[e^{i\langle u,X_1\rangle}\right] = e^{\psi(u)}$.

7. The Lévy-Itô Decomposition

In the previous section, we showed that for any Lévy process $X = (X_t)_{t\geq 0}$ the random variables $X_t, t \geq 0$ have an infinitely divisible distribution and determined this distribution using the Lévy-Khintchine representation. The aim of this section if to prove an «inverse» result: starting from an infinitely divisible distribution ρ, or equivalently from a Lévy triplet (b, c, v), we want to construct a Lévy process $X = (X_t)_{t\geq 0}$ such that $P_{X_1} = \rho$.

Theorem 7.1. (*Lévy-Itô Decomposition*)

Let ρ be an infinitely divisible distribution with Lévy triplet (b, c, v), where $b \in R^d$, $c \in S^d{}_{\geq 0}$ and v is a Lévy measure. Then there exists a probability space $(\Omega, \mathcal{F}, P)$on which four independent Lévy processes exist, $X^{(1)}, X^{(2)}, X^{(3)}, X^{(4)}$, where : $X^{(1)}$ is a constant drift , $X^{(2)}$is a BM, $X^{(3)}$ is a compound Poisson process and $X^{(4)}$ is a square integrable, pure jump martingale with a.s countable number of jumps of magnitude less than 1 in each finite time interval. Setting $X = X^{(1)}+X^{(2)}+X^{(3)} + X^{(4)}$, we have that there exists a probability space on which a Lévy process $X = (X_t)_{t\geq 0}$ is defined, with Lévy exponent $\psi(u) = i\langle u,b\rangle - \frac{\langle u,cu\rangle}{2} + \int_{R^d}(e^{i\langle u,x\rangle} - 1 - i\langle u,x\rangle\, 1_D(x))v(dx)$, for all $u \in R^d$, and path, or Lévy –Itô Decomposition

$$X_t = bt + \sqrt{c}B_t + \int_0^t \int_{D^c} x\mu^X(ds,dx) + \int_0^t \int_D x(\mu^x - v^x)(ds,dx) \quad (7.1)$$

where $v^x = Leb \otimes v$.

As we see the Lévy-Itô decomposition describes the structure of a general Lévy process in terms of three independent auxialary Lévy processes, each of which with different types path behavior. Understanding the Lévy- Itô decomposition will allow to distinguish a number of important general subclasses of Lévy processes according to their path type. In doing so it will be necessary to digress temporarily into the theory of Poisson random measures and associated square integrable martingales.

8. Analysis of jumps and Poisson random measures

The jump process $\Delta X = (\Delta X_t)_{0\leq t\leq T}$ associated to the Lévy Process X is defined for each $0 \leq t \leq T$ via

$$\Delta X_t = X_t - X_{t-}$$

where $X_{t-} = \lim_{s\to t} X_s$. The condition of stochastic continuity of Lévy process yields immediately that for any Lévy process X and fixed t>0 then $\Delta X_t = 0$ a.s; hence a Lévy process has *no fixed times of discontinuity*. In general, the sum of the jumps of a Lévy process does not converge, in other words it is possible that

$$\sum_{s\leq t}|\Delta X_s| = \infty \text{ a.s}$$

but we always have that

$$\sum_{s\leq t}|\Delta X_s|^2 < \infty \text{ a.s}$$

which allows us to handle Lévy processes by martingale techniques.A convenient tool for analyzing the jumps of a Lévy Process is the *random measure of jumps* of the process. Consider a set $A \in \mathcal{B}(\mathbb{R}\backslash\{0\})$ such that $0 \notin \bar{A}$ and let $0 \leq t \leq T$; define the random measure of the jumps of the Lévy process X by

$$\mu^L(\omega; t, A) = \#\{0 \leq s \leq t; \Delta X_s(\omega) \in A\} = \sum_{s\leq t} 1_A(\Delta X_s(\omega)) ,$$

hence the measure $\mu^L(\omega; t, A)$ *counts the jumps of the process X of size in A up to the time t.*

Definition 8.1. (Poisson random measure)

Let $(E, \mathcal{E}, v)$ be a σ-finite measure space. Consider a mapping $\mu: \mathcal{E} \to \mathbb{N} \cup \{\infty\}$ such that $\{\mu(A): A \in \mathcal{E}\}$ is a family of random variables defined on some probability space $(\Omega, F, \mathbb{P})$. Then μ is called *a Poisson random measure with intensity v* if

(1) μ is $\mathbb{P}$-a.s.a measure on $(E, \mathcal{E})$;
(2) for each $A \in \mathcal{E}$, $\mu(A)$ is Poisson distributed with parameter $v(A)$, where $v(A) \in [0, \infty]$;
(3) for mutually disjoint sets $A_1, \dots A_n$ in $\mathcal{E}$, the random variables $\mu(A_1), \dots \mu(A_n)$ are independent.

Hence, $\mu^X(\cdot, A)$is *a Poisson process* and μ^X is a *Poisson random measure*. The *intensity* of this Poisson process is $v(A) = E[\mu^X(1, A)]$.

Definition8.2. The measure v defined by $v(A) = E[\mu^L(1, A)] = E[\sum_{s\leq 1} 1_A(\Delta X_s(\omega))]$ is the Lévy measure of the Lévy process X.

Remark 8.3. The process $(\int_{|x|<1} \tilde{N}(t, dx), t \geq 0)$ describes the 'small jumps' that happen to the Lévy process, and the process $(\int_{|x|\geq 1} xN(t, dx), t \geq 0)$ describes the 'big jumps' and is called Compound Poisson process.

Poisson random measures and Stochastic processes

In this sequel, we want to make the connection between Poisson random measures and Stochastic processes.

We will work in the following σ-finite space $(E, \mathcal{E}^X, v^X) = (\mathbb{R}_{\geq 0} \times \mathbb{R}^d, (\mathcal{B}(\mathbb{R}_{\geq 0}) \times \mathcal{B}(\mathbb{R}^d), Leb \otimes v)$ where v is a Lévy measure. We will denote the Poisson random measure on this space by μ^X.

If we consider a time space interval of the form $[s, t] \times A, s \leq t$ where $A \subset \mathbb{R}^d$ such that $0 \notin \bar{A}$ then the integral with respect to μ^X, denoted by

$$\int_{[s,t]} \int_A x\mu^X(ds, dx) =: X$$

is a Compound Poisson random variable with intensity $(t - s)v(A)$.

Let us consider the collection of random variables $(\int_0^t \int_A x\mu^X(ds, dx))_{t\geq 0}$. This is a Compound Poisson *stochastic process*.

9. The Lévy measure and path properties

The Lévy measure is the most interesting part of a Lévy process and is responsible for the richness of the class of these processes (and carries useful information about the structure of the process). The behavior of the sample paths of a Lévy process, as well as many properties, e.g. existence of moments, smoothness of densities, etc, can be completely characterized based on the Lévy measure and the presence or absence of a Brownian component.

Definition 9.1.(*Lévy measure*)

Let v be a Borel measure on R^d.We say that v is a Lévy measure if it satisfies

$$v(\{0\}) = 0 \quad \text{and} \quad \int_{R^d} (|x|^2 \wedge 1) v(dx) < \infty \, . \quad (9.1)$$

We can deduce that the Lévy measure satisfies $\mathbb{E}[\mu^X([0,1] \times A)] = \upsilon(A)$ for every set $A \in \mathcal{B}(\mathbb{R}^d \setminus \{0\})$.

In other words the Lévy measure describes *the expected number of jumps of certain height in a time interval of length one.* The relation between Poisson random measures and Lévy measures allows us to draw the following conclusion about the sample paths of Lévy processes based on their Lévy measure: the Lévy measure has no mass at the origin while singularities (i.e. infinitely many jumps) can occur around the origin (i.e. small jumps), thus a Lévy process can have an infinite number of small jumps-"small" here means bounded by one in absolute value, although we can consider any ε>0 instead of one. Moreover, the mass away from the origin is bounded, hence only a finite number of big jumps can occur again, "big" here means greater than one in absolute value.

In the figures bellow we see the distribution functions as well as the density functions of the Levy measure of some distributions of particular interest.

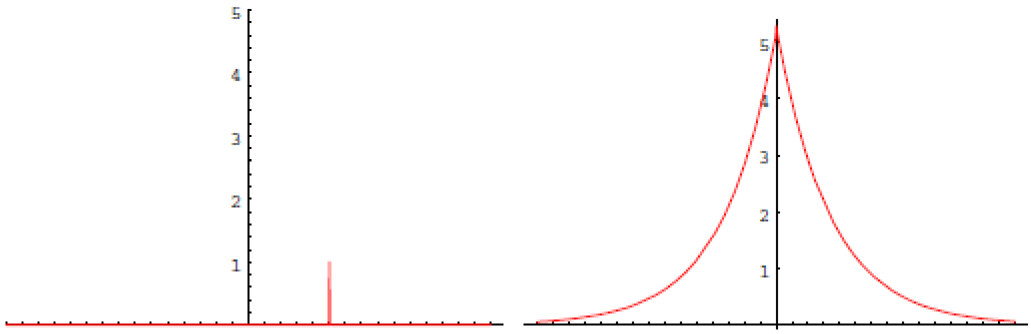

Figure 4. The distribution function of the Lévy measure of the standard Poisson process (left) and the density of the Lévy measure of a compound Poisson process with double-exponentially distributed jumps.

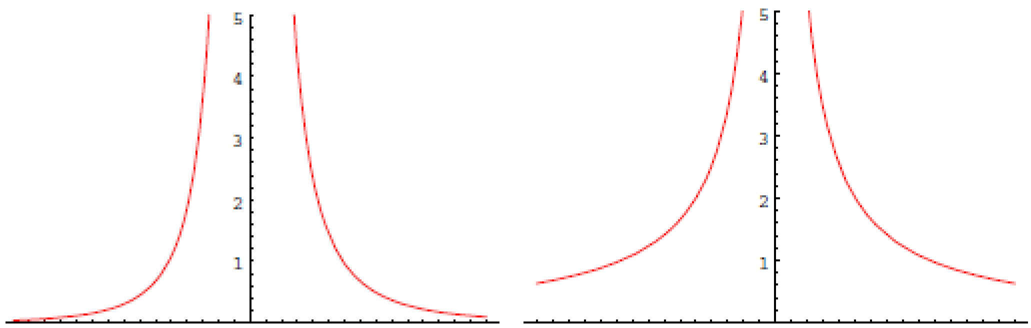

Figure 5. The density of the Lévy measure of a normal inverse Gaussian (NIG, left) and a α-stable process.

Recall the example of the Lévy jump-diffusion; the Lévy measure is

$$v(dx) = \lambda \times F(dx)$$

from that we can deduce that the expected number of jumps is λ and the jump size is distributed according to F .More generally, if v is a finite measure, i.e.

$$\lambda \coloneqq v(\mathbb{R}) = \int_{\mathbb{R}} v(dx) < \infty$$

then we can define

$$F(dx) \coloneqq \frac{v(dx)}{v}$$

which is a probability measure. If $v(\mathbb{R}) = \infty$, then an infinite number of (small) jumps is expected.

Path properties

We would like to discuss some finer properties of the paths of a Lévy process, in particular, when are paths continuous or piecewise constant and when they have finite or infinite variation. Throughout in this section we assume that $X = (X_t)_{t\geq 0}$ is a Lévy process with triplet (b, c, v).

Proposition 9.2. The paths of $X = (X_t)_{t\geq 0}$ are a.s. continuous if and only if $v \equiv 0$.

Proposition 9.3. The paths of $(X_t)_{t\geq 0}$ are a.s. piecewise constant if and only if X is a compound Poisson process without drift.

Definition 9.4.A Lévy process X has an *infinite activity* if the sample paths of X have an a.s. countably infinite number of jumps on every compact time interval [0,T]. Otherwise, X has *finite activity*.

Proposition 9.5.

(1) If $v(\mathbb{R}^d) = \infty$ then X has infinite activity

(2) If $\boldsymbol{v}(\mathbb{R}^{\boldsymbol{d}}) < \infty$ then X has finite activity.

Intuitively speaking, a Lévy process with infinite activity will jump more often than a process with finite activity.

Proposition 9.6.

Let X be a Lévy process with triplet (b, c, v)

(1)If c=0 and $\int_{|x|\leq 1} |x| v(dx) < \infty$, then almost all paths of X have finite variation.

(2)If $c \neq 0$ and $\int_{|x|\leq 1} |x|\, v(dx) = \infty$, then almost all paths of X have infinite variation.

Bellow are graphically exhibited the different functions a Lévy measure has to integrate in order to have finite activity or variation (see figure6). The compound Poisson process has finite measure, hence it has finite variation as well; on the contrary, the NIG Lévy process has an infinite measure and has infinite variation. In addition, the CGMY Lévy process has infinite activity, but the paths have finite variation.

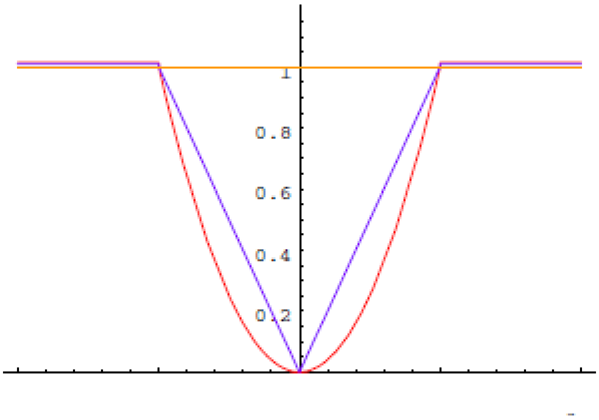

Figure 6. The Lévy measure must integrate $|x|^2 \wedge 1$ (red line); It has finite variation if it integrates $|x| \wedge 1$ (blue line); it is finite if it integrates 1 (orange line).

The Lévy measure also carries information about the finiteness of the moments of a Lévy process. This is particularly useful information in mathematical finance, related to the existence of a martingale measure. The finiteness of the moments of a Lévy process is related to the finiteness of an integral over the Lévy measure (more precisely, the restriction of the Lévy measure to jumps larger than 1 in absolute value, i.e. big jumps).

Proposition 9.7.

Let X be a Lévy process with triplet (b, c, v). Then

(1) $\boldsymbol{X_t}$ has finite p-th moment for $\boldsymbol{p} \in \mathbb{R}_{\geq \mathbf{0}}$ $(\mathbb{E}|\boldsymbol{X_t}|^{\boldsymbol{p}} < \infty)$ if and only if $\int_{|x|\geq 1} |\boldsymbol{x}|^{\boldsymbol{p}}\, \boldsymbol{v}(\boldsymbol{dx}) < \infty$.

(2) $\boldsymbol{X_t}$ has finite p-th exponential moment for $\boldsymbol{p} \in \mathbb{R}$ $(\mathbb{E}[\boldsymbol{e^{pX_t}}] < \infty$ if and only if $\int_{|x|\geq 1} \boldsymbol{e^{px}} \boldsymbol{v}(\boldsymbol{dx}) < \infty$.

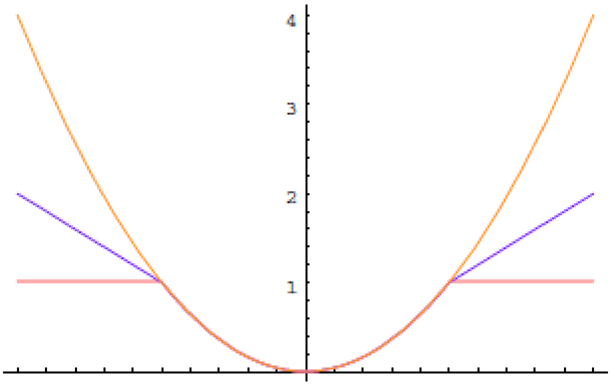

Figure 7. A Lévy process has first moment if the Lévy measure integrates $|x|$ for $|x| \geq 1$ (blue line) and second moment if it integrates x^2 for $|x| \geq 1$ (orange line).

In the figure8 are presented the simulated sample paths of a continuous Lévy process with infinite variation (i.e. Brownian motion) and a purely discontinuous one (i.e. NIG process). We can observe that, locally, the pure-jump infinite variation process behaves like a Brownian motion, as it proceeds by infinitesimally small movements. However, these small jumps are interlaced with, less frequent, big jumps.

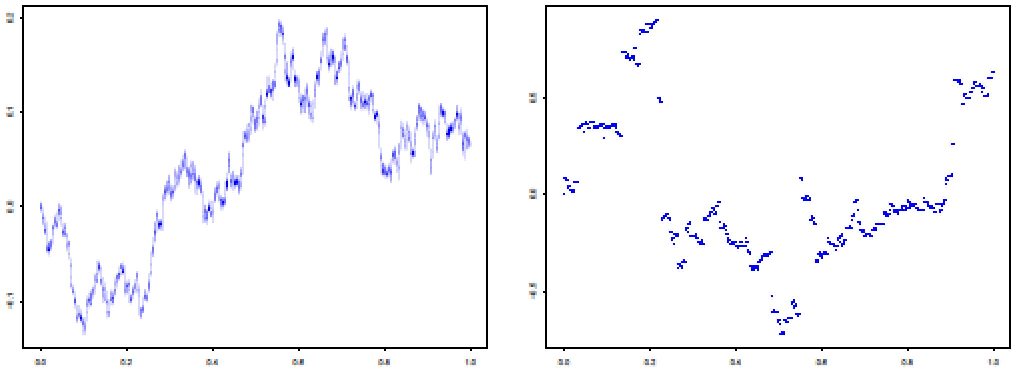

Figure 8. Simulated paths of two infinite variation Lévy processes: Brownian motion (left) and NIG process.

Remark 9.8. As can be observed from the propositions (9.5),(9.6) and (9.7), the *variation* of a Lévy process depends on the *small jumps* (and the Brownian motion), the *moment* properties depend on the *big jumps*, while the *activity* of a Lévy process depends on *all* the jumps of the process.

Remark 9.9.

Assume that the *jump part* of the Lévy process X has finite variation, i.e. it holds that

$$\int_{|x|\leq 1} |x|\, v(dx) < \infty.$$

Then the Lévy-Itô decomposition of X takes the form

$$X_t = b_0 t + \sqrt{c} W_t + \int_0^t \int_{\mathbb{R}^d} x \mu^x (ds, dx)$$

and the Lévy-Khintchine formula can be written as

$$E[e^{i\langle u, X_1\rangle}] = \exp\left(i\langle u, b_0\rangle - \frac{\langle u, cu\rangle}{2} + \int_{\mathbb{R}^d} (e^{i\langle u,x\rangle} - 1) v(dx)\right).$$

In other words, we can use the truncation function h(x)=0 and the drift term relative to this truncation function(denoted by b_o) is related to the drift term b in (6.1) via

$$b_o = b - \int_{|x|\leq 1} x v(dx)$$

Note that, this process is not necessarily a compound Poisson process as the activity of the process might be infinite. (i.e.$v(D) = \infty$)[3].

Remark 9.10.

Consider a Lévy process X with triplet such that the following properties holds:

$$b \geq 0,\ \ c \geq 0\ ,\ \ v\big((-\infty, 0]\big) = 0 \qquad \text{and} \qquad \int_{(0,1]} |x|\, v(dx) = \infty$$

This process has the Lévy-Itô decomposition

$$X_t = bt + \int_0^t \int_{\mathbb{R}^+} x(\mu^x - v^x)(ds, dx)$$

Its paths are fluctuating but are *not* increasing- the paths have infinite variation –and this process is not a subordinator(an almost surely increasing Lévy process). The intuitive explanation for this behavior is that the jump part will converge only if we add an “infinitely strong” deterministic term in the negative direction to compensate for the divergent sum of jumps. This term cannot be negated, however large we choose b[3].

Variation of the paths

Next we will analyze the variation of the paths of a Lévy process. We will consider a real-valued Lévy process for simplicity, although the main result is also valid for $\mathbb{R}^d$ valued Lévy processes.

Definition 9.11.

Consider a function $f: [a, b] \to \mathbb{R}$. The total variation of f over [a,b] is

$$TV = sup_{\pi} \sum_{i=1}^{n} |f(t_i) - f(t_{i-1})|$$

where $\pi = \{a = t_0 < t_1 < \cdots < t_n$=b} is a partition of the interval [a,b].

Quadratic Variation

A commonly used measure of continuous time processes in financial economics, financial econometrics, derivative pricing and stochastic analysis is the quadratic variation (QV) process. This has two steps. First, time is split into small intervals

$$t_0^r = 0 < t_1^r < \cdots < t_{m^r}^r = t$$

Then the QV process is

$$[Y](t) = P - \lim_{r\to\infty} \sum \{Y(t_i^r + 1) - \ Y\ (t_i^r)\}^2$$

where $\sup\{t_{i+1}^r - t_i^r\} \to 0 \ for \ r \to \infty$.

It's sometimes helpful to work with an alternative and equivalent definition of QV which is written in terms of a Stochastic Integral. It's that

$$[Y]_t = Y_t^2 - 2\int_0^t Y_{u\_} dY_u.$$

This series looks at the partial sum of squared increments over tiny intervals of time. In general the QV process of a Lévy process is a (different) Lévy process whose the increments are independent and stationary (because QV is just sums the squares of independent and stationary increments). Further, it can be regarded as a subordinator for the increments are non-negative.

10. Martingales and Lévy processes

Proposition 10.1.

Let $X = (X_t)_{0\le t\le T}$ be a Lévy process with Lévy triplet (b, c, v) and assume that $E[|X_t|] < \infty$. X is a martingale if and only if b=0. Similarly, X is a submartingale if b>0 and a supermartingale if b<0.

Proposition 10.2.

Let X be an $\mathbb{R}^d$-valued, Lévy process with Lévy triplet (b, c, v) characteristic exponent ψ and cumulant generating function ϕ([3]).

(1) If $\int_{|x|>1} |\boldsymbol{x}|\, \boldsymbol{v}(\boldsymbol{dx}) < \infty$, then X is a martingale if and only if $\boldsymbol{b} + \int_{|x|>1} \boldsymbol{x}\boldsymbol{v}(\boldsymbol{dx}) = \boldsymbol{0}$

(2) If $\int_{|x|>1} |\boldsymbol{x}|\, \boldsymbol{v}(\boldsymbol{dx}) < \infty$, then $(\boldsymbol{X_t} - \mathbb{E}[\boldsymbol{X_t}])_{t\ge 0}$ is a martingale

(3) If $\int_{|x|>1} \boldsymbol{e}^{\langle u,x\rangle}\, \boldsymbol{v}(\boldsymbol{dx}) < \infty$ for some $\boldsymbol{u} \in \mathbb{R}^{\boldsymbol{d}}$, then $\boldsymbol{M} = (\boldsymbol{M_t})_{t\ge 0}$ is a martingale, where $\boldsymbol{M_t} = \frac{e^{\langle u,X_t\rangle}}{e^{t\varphi(u)}}$.

(4) The process $\boldsymbol{N} = (\boldsymbol{N_t})_{t\ge 0}$ is a complex-valued martingale, where $\boldsymbol{N_t} = \frac{e^{i\langle u,X_t\rangle}}{e^{t\psi(u)}}$.

Definition 10.3.

A random variable X is square integrable if $E(X^2) < \infty$. A process X(t) on the time interval [0,T],where T can be infinite, is square integrable if $sup_{t\in[0,T]} EX^2 < \infty$ i.e. second moments are bounded.

Examples

1. Brownian Motion B(t) on a finite time interval $0 \le t \le T$ is a square integrable martingale, since $EB^2(t) = t < T < \infty$. Similarly,$B^2(t) - t$ is a square integrable martingale. They are not square integrable when $T = \infty$.
2. If f(x) is a bounded and continuous function on $\mathbb{R}$, then Itô integrals $\int_0^t f(B(s))dB(s)$ and $\int_0^t f(s)dB(s)$ are square integrable martingales on any finite time interval $0 \le t \le T$. Since, $|f(x)| \le K$
$$E\left(\int_0^t f(B(s))dB(s)\right)^2 = E\left(\int_0^t f^2(B(s))ds\right) \le K^2 t \le K^2 T < \infty$$
If moreover, $\int_0^t f^2(s)ds < \infty$ then $\int_0^t f(s)dB(s)$ is a square integrable martingale on $[0, \infty)$.

Lemma 10.4.

Assume that $\int_{\mathbb{A}} |x|v(dx) < \infty$. Then, $M_t = \int_0^t x\mu^X(ds, dx) - t\int_{\mathbb{A}} xv(dx), t \geq 0$ is a $\mathbb{P}$-martingale relative to the filtration generated by the Poisson random random measure μ^X $\mathcal{F}_t \coloneqq \sigma(\mu^X(\mathcal{G}): \mathcal{G} \in \mathcal{B}([0,t]) \times \mathcal{B}(\mathbb{R}^d))$, $t \geq 0$. If in addition, $\int_{\mathbb{A}} |x|^2\, v(dx) < \infty$ then M is a square-integrable martingale.

11. Some classes of particular interest

Subordinator

A Subordinator is an a.s. increasing (in t) Lévy process[2]. Equivalently, for X to be a subordinator, the triplet must satisfy

$$v(-\infty, 0) = 0,\ \ c = 0,\ \ \int_{(0,1)} xv(dx) < \infty \ and\ \gamma = b - \int_{(0,1)} xv(dx) > 0.$$

The Lévy –Itô decomposition of a subordinator is

$$X_t = \gamma t + \int_0^t \int_{(0,\infty)} x\mu^L(ds, dx) \quad (11.1)$$

And the Lévy-Khintchine formula takes the form

$$E\left[e^{iuX_t}\right] = \exp\left[t\left(iu\gamma + \int_{(0,\infty)}(e^{iux} - 1\right)v(dx))\right].$$

Two examples of subordinators are the Poisson and the inverse Gaussian processes.

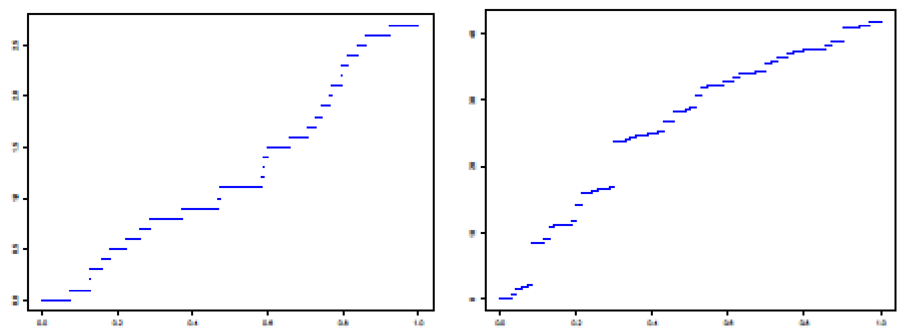

Figure 9. Simulated paths of a finite activity (left) and an infinite activity subordinator.

Jumps of finite variation

A Lévy process has jumps of finite variation if and only if $\int_{|x|\leq 1}|x|v(dx) < \infty$. In this case, the Lévy –Itô decomposition of X resumes the form

$$X_t = \gamma t + \sqrt{c}W_t + \int_0^t \int_{\mathbb{R}} x\mu^X(ds, dx) \quad (11.2)$$

and the Lévy-Khintchine formula takes the form

$$E\left[e^{iuX}\right] = \exp\left[t(iu\gamma - \frac{u^2 c}{2} + \int_{\mathbb{R}} \left(e^{iux} - 1\right)v(dx))\right]$$

where γ is defined as in the previous section in the definition of the subordinator. Moreover, if $v([-1,1]) < \infty$, which means that $v(\mathbb{R}) < \infty$, then the jumps of X correspond to a compound Poisson process.

Spectrally one sided

A Lévy process is called *spectrally negative* if it has only negative jumps. The Lévy-Itô decomposition of a spectrally negative Lévy process has the form

$$X_t = bt + \sqrt{c}W_t + \int_0^t \int_{x<-1} x\mu^X(ds, dx) + \int_0^t \int_{-1<x<0} x(\mu^X - v^X)(ds, dx) \quad (11.3)$$

and the Lévy-Khintchine formula takes the form

$$E\left[e^{iuX_t}\right] = \exp\left[t(iub - \frac{u^2 c}{2} + \int_{(-\infty,0)} e^{iux} - 1 - iu1_{\{x>-1\}}v(dx)\right]$$

Similarly, a Lévy process is called *spectrally positive* if –X is spectrally negative.

Finite first moment

As we have seen already, a Lévy process has a *finite first moment* if and only if $\int_{|x|\geq 1}|x|v(dx) < \infty$. Therefore, we can also compensate the big jumps to form a martingale hence the Lévy-Itô decomposition of X resumes the form

$$X_t = b't + \sqrt{c}W_t + \int_0^t \int_{\mathbb{R}} x(\mu^X - v^X)(ds, dx) \quad (11.4)$$

hence the Lévy-Khintchine formula takes the form

$$E\left[e^{iuX_t}\right] = \exp\left[t(iub' - \frac{u^2 c}{2} + \int_{\mathbb{R}} \left(e^{iux} - 1 - iux\right)v(dx))\right]$$

where $b' = b + \int_{|x|\geq 1} xv(dx)$.

Examples of Lévy Processes

1. Poisson Processes

For each λ>0 consider a probability distribution μ_λ which is concentrated on $k = 0,1,2 \ldots$ such that $\mu_\lambda = (\{\kappa\}) = {e^{-\lambda}\lambda^\kappa}/{\kappa!}$. That is to say the Poisson distribution. An easy calculation reveals that

$$\sum_{\kappa\geq 0} e^{i\theta\kappa}\,\mu_\lambda(\{\kappa\}) = e^{-\lambda(1-e^{i\theta})} = \left[e^{-\frac{\lambda\left(1-e^{i\theta}\right)}{n}}\right]^n.$$

The right hand side is the characteristic function of the sum of n independent Poisson processes, each of which with parameter λ/n. In the Lévy-Khintchine decomposition we see that $b = c = 0$ and $v = \lambda\delta_1$, the Dirac measure supported on {1}.

Recall that a Poisson process$\{N_t : t \geq 0\}$, is a Lévy Process with distribution at time t>0, which is Poisson with parameter λt. From the above calculations we have $E\left(e^{i\theta N_t}\right) = e^{-\lambda t(1-e^{i\theta})}$ and hence its characteristic exponent is given by $\Psi(\theta) = \lambda(1 - e^{\iota\theta})$ for $\theta \in \mathbb{R}$.(see manuscript [1])

2. Compound Poisson Processes

Suppose that N is a Poisson random variable with parameter λ>0 and that $\{\xi_i : i \geq 1\}$ is an i.i.d. sequence of random variables(independent of N) with common law F having no atom at zero. By first conditioning on N, we have for ∈ $\mathbb{R}$,

$$E\left(e^{i\theta\sum_{i=1}^{N}\xi_\iota}\right) = \sum_{n\geq 0} E(e^{i\theta\sum_{i=1}^{N}\xi_i})e^{-\lambda}\frac{\lambda^n}{n!} = \sum_{n\geq 0}\left(\int_{\mathbb{R}} e^{i\theta x}\,F(dx)\right)^n e^{-\lambda}\frac{\lambda^n}{n!}$$
$$= e^{-\lambda\int_{\mathbb{R}}(1-e^{i\theta x})F(dx)}.$$

We see that the distributions of the form $\sum_{i=1}^{N}\xi_i$ are infinitely divisible with triplet $b = -\lambda\int_{0<|x|<1} xF(dx),\ \sigma = 0\ and\ v(dx) = \lambda F(dx)$. When F has an atom of unit mass at 1 then we have simply a Poisson distribution.

Suppose now that $\{N_t : t \geq 0\}$ is a Poisson process with parameter λ and consider a compound Poisson process $\{X_t : t \geq 0\}$ defined by

$$X_t = \sum_{i=0}^{N_i}\xi_i\,, t \geq 0. \quad (11.5)$$

Using the fact that N has stationary independent increments together with the mutual independence of random variables $\{\xi_\iota : i \geq 1\}$, for $0 \leq s < t < \infty$, by writing

$$X_t = X_s + \sum_{i=N_{s+1}}^{N_t} \xi_i \quad (11.6)$$

it is clear that X_t is the sum of X_s and an independent copy of X_{t-s}. Right continuity and left limits of the process N also ensure right continuity and left limits of X. Thus compound Poisson processes are Lévy Processes. From the calculations in the previous paragraph, for each $t \geq 0$ we may substitute N_t for the variable N to discover that the Lévy-Khintchine formula for a compound Poisson process takes the form $\Psi(\theta) = \lambda\left(1 - e^{i\theta x}\right)F(dx)$. *Note in particular that the Lévy measure of a compound Poisson process is always finite with total mass equal to the rate λ of the underlying process N.*

Compound Poisson processes provide direct link between Lévy processes and random walks; that is discrete time processes of the form $S = \{S_n : n \geq 0\}$

Where $S_0 = 0 \; and \; S_n = \sum_{i=1}^{n} \xi_i \; for \; n \geq 1.$

Indeed a compound Poisson process is nothing more than a random walk whose jumps have been spaced out with independent and exponentially distributed periods[1].

3.Linear Brownian Motion

Take the probability law

$$\mu_{s,\gamma} := \frac{1}{\sqrt{2\pi s^2}} e^{\frac{-(x-\gamma)^2}{2s^2}} dx$$

supported on $\mathbb{R}$ where $\gamma \in \mathbb{R}$ and s>0; the well-known Gaussian distribution with mean γ and variance s^2 .It is known that

$$\int_{\mathbb{R}} e^{i\theta x} \mu_{s,\gamma}(dx) = e^{-\frac{s^2\theta^2 + i\theta\gamma}{2}} = \left[e^{-\frac{1}{2}\left(\frac{s}{\sqrt{n}}\right)^2 \theta^2 + i\theta\frac{\gamma}{n}} \right]^n$$

showing again that it is an infinitely divisible distribution, this time with $b = -\gamma, \sigma = s$ and $v = 0$.

We immediately recognize the characteristic exponent $\Psi(\theta) = \frac{s^2\theta^2}{2} - i\theta\gamma$ is also that of a scale Brownian motion with linear drift, $X_t := sB_t + \gamma t, t \geq 0$, where $B = \{B_t : t \geq 0\}$ is a standard Brownian; that is to say a linear Brownian motion with parameters $\sigma = 1$ and $\gamma = 0$. It is trivial exercise to verify that X has stationary independent increments with continuous paths as a consequence of the fact that B does[1].

4. Stable processes

Stable processes are the class of Lévy processes whose characteristic exponent corresponds to those of stable distributions. Stable distributions were introduced by Lévy (1924,1925) as a third example of infinitely divisible distributions after Gaussian and Poisson distributions. A random variable, Y, is said to have a stable distribution if for all $n \geq 1$ it observes the distributional equality

$$Y_1 + \cdots + Y_n \triangleq a_n Y + b_n \,, \quad (11.7)$$

where $Y_1, \ldots, Y_n$ are independent copies of Y, $a_n > 0$ and $b_n \in \mathbb{R}$. By subtracting $b_n/_n$ from each part of the terms on the left-hand side of (11.7) one sees in particular that this definition implies that any stable random variable is infinitely divisible. It turns out that necessarily $a_n = n^{1/a}$ for $a \in (0,2]$; In that case we refer to the parameter α as the *index.* A smaller class of distributions are *the strictly stable* distributions. A random variable Y is said to have strictly stable distribution if it observes (11.7) but with $b_n = 0$.

In this case, we necessarily have

$$Y_1 + \cdots + Y_n \triangleq n^{1/a} Y.$$

The case α=2 corresponds to zero mean Gaussian random variables.

Stable random variables observing the relation (11.7) for $\alpha \in (0,1) \cup (1,2)$ have characteristic exponents of the form

$$\Psi(\theta) = c|\theta|^{\alpha}\left(1 - i\beta tan\frac{\pi\alpha}{2} sgn\theta\right) + i\theta\eta \quad (11.8)$$

where $\beta \in [-1,1],\ \eta \in \mathbb{R}\ and\ c > 0.$

Stable random variables observing the relation 11.7 for $\alpha = 1$, have characteristic exponents of the form

$$\Psi(\theta) = c|\theta|\left(1 + i\beta\frac{2}{\pi} sgn\theta log|\theta|\right) + i\theta\eta \quad (11.9)$$

where $\beta \in [-1,1],\ \eta \in \mathbb{R}\ and\ c > 0.$

Here we work with the definition of the sign function $sgn\theta = 1_{(\theta>0)} - 1_{(\theta<0)}$.

To make the connection with the Lévy-Khintchine formula, one needs σ=0 and

$$v(dx) = \begin{cases} c_1 x^{-1-a} dx\ for\ x \in (0,\infty) \\ c_2 |x|^{-1-a} dx\ for\ x \in (-\infty, 0) \end{cases}$$

where $c = c_1 + c_2$, $c_1, c_2 \geq 0$ and $\beta = \frac{(c_1 - c_2)}{(c_1 + c_2)}$ *if* $a \in (0,1) \cup (1,2)$ and $c_1 = c_2$ if *a=1.*

Unlike the previous examples, the distributions that lie behind these characteristic exponents are heavy tailed in the sense that the tails of their distributions decay slowly enough to zero so they only have moments strictly less than α. The value of the parameter β gives a measure of asymmetry in the Lévy measure and likewise for the distributional asymmetry. The densities of stable processes are known explicitly in the form of convergent power series.

Suppose that $S(c,\alpha,\beta,\eta)$ is the distribution of a stable random variable with parameters c, α, β and η. For each choice of c>0, $\alpha \in (0,2), \beta \in [-1,1]\ and\ \eta \in \mathbb{R}$ there exists a Lévy process, with characteristic exponent given by 11.8 or 11.9 according to the choice of parameters. Further, form the definition of its characteristic exponent it's clear that at each fixed time the α-stable process will have distribution $S(ct,\alpha,\beta,\eta)$.

We sometimes refer to an α-stable process to mean a Lévy process based on a strictly stable distribution. Necessarily this means that the associated characteristic exponent takes the form

$$\Psi(\theta) = \begin{cases} c|\theta|^{\alpha}(1 - i\beta tan\frac{\pi\alpha}{2}sgn\theta & ,for\ \alpha \in (0,1) \cup (1,2) \\ c|\theta| + i\eta & for\ \alpha = 1 \end{cases}$$

where the parameters ranges for c and β as above. The reason for the restriction to the strictly stable distribution is essentially that we shall want to use the following fact. If $\{X_t : t \geq 0\}$ is an X α-stable process, then from its characteristic exponent (or equivalently the scaling properties of strictly stable random variables) we see that for all λ>0 $\{X_{\lambda t} : t \geq 0\}$ has the same law as $\{\lambda^{\frac{1}{\alpha}} X_t : t \geq 0\}$.

Chapter 2

Applications

Lévy processes play a central role in several fields of science, such as *physics*, in the study of turbulence, laser cooling and in quantum field theory; in *engineering*, for the study of networks, queues and dams; in economics, for continuous time- series models; in the *actuarial science*, for the calculation of insurance and re-insurance risk; and of course, in *mathematical finance* and *biology*.

In mathematical finance, Lévy processes are becoming extremely fashionable because they can describe the observed reality of financial markets in a more accurate way than models based on Brownian motion. In the ''real'' world we observe that asset price processes have jumps (e.g. big price changes) or spikes, and risk managers have to take them into consideration. Moreover, the empirical distribution of asset returns exhibits fat tails and skewness, behavior that deviates from normality. Hence, models that accurately fit return distributions are essential for the estimation of profit and loss (P& L) distributions.

Similarly in the ''risk-neutral'' world, we observe that implied volatilities are constant neither across strike nor across maturities. Therefore, traders need models that can capture the behavior of the implied volatility smiles more accurately, in order to handle the risk of traders. Lévy processes provide us with the appropriate tools to adequately and consistently describe all these observations, both in the ''real'' and in the ''risk-neutral'' world.

We describe the possible approaches in modeling the price process of a financial asset using Lévy processes under ''real'' and ''risk-neutral'' world, and give a brief account of market incompleteness which links the two worlds. Then we present a primer of Lévy models in the mathematical finance literature. Furthermore, there will be implemented the Monte Carlo simulation for various stochastic processes as well as for stock and option pricing.

We review applications which emphasize the importance of jumps in stock price modeling, namely construction of optimal hedging portfolios and computation of risk measures for dynamically insured portfolios in presence of jumps in asset prices. These examples show how Lévy-based models provide a better understanding of risk. In the last years, the research departments of major banks started to accept jump-diffusions and Lévy processes as a valuable tool in their modeling. The increasing interest to jump models in finance is mainly due to the following reasons. First, in a model with continuous paths like a diffusion model, the price process behaves locally like a Brownian motion and the probability that the stock moves by a large amount over a short period of time is very small. In such models the prices of options should be much lower than it is in real markets. On the other hand, if stock prices are allowed to jump, even the time to maturity is very short, there is a non-negligible probability that after a sudden change in the stock price the option will move in the money.

Second, from the point of view of hedging, continuous models of stock price behavior generally lead to a complete market or to a market, which can be completed. In such a market every terminal payoff can be replicated and the very existence of traded options is a problem. This can be solved by using discontinuous models. In real markets, due to the presence of jumps in the prices, perfect hedging is impossible and the options enable to hedge risks that cannot be hedged using underlying only.

Last but not least, from a risk management perspective, jumps allow to quantify and take into account the risk of strong price movements over short time intervals, which appears non-existant in the diffusion model.

12. Lévy Processes and some applied Probability models

In this section we introduce some classical probability models, which are structured around basic examples of Lévy processes. This section provides a particular motivation for the study of fluctuation theory that follows.

Insurance ruin

A compound Poisson process $(Z_t)_{t\leq 0}$ with positive jump sizes can be interpreted as a claim process recording the total claim amount incurred before time t. If there is linear premium income at rate c>0, then also the gain process $ct - Z_t, t \geq 0$, is a Lévy process. For an initial reserve of x >0, the reserve process $x + ct - Z_t$ is a shifted Lévy process starting from a non-zero initial value x.

Cramer-Lundberg Risk Process

Consider the following model of the revenue of an insurance company as a process in time proposed by Lundberg(1903).The insurance company collects premiums at a fixed rate c>0 from its customers. At times of a Poisson process, a customer will make a claim causing the revenue to jump downwards. The size of claims is independent and identically distributed. If we call X_t the capital of the company at time t, then the latter description amounts to

$$X_t = x + ct - \sum_{i=1}^{N_t} \xi_\iota \ , t \geq 0$$

where $x > 0$ is the initial capital of the company, $N = \{N_t : t \geq 0\}$ is a Poisson process with rate λ>0, and $\{\xi_\iota : i \geq 1\}$ is a sequence of positive, independent and identically distributed random variables also independent of N. The process $X = \{X_t : t \geq 0\}$ is nothing more than a compound Poisson process with drift of rate c, initiated from $x > 0$.

Financial ruin in this model (or just ruin for short) will occur if the revenue of the insurance company drops below zero. Since this will happen with probability one if

$\mathbb{p}(\lim_{t\to\infty} inf X_t = -\infty) = 1$, an additional assumption imposed on the model is that $\lim_{t\to\infty} X_t = \infty$.

A sufficient condition to guarantee the latter is that the distribution of ξ has finite mean, say μ>0, and that

$\frac{\lambda\mu}{c} < 1$, the so-called *net profit condition*.

To see why this presents a sufficient condition, note that the Strong Law of Large Numbers and the obvious fact that $\lim_{t\to\infty} N_t = \infty$ imply that

$$\operatorname{Lim}_{t\to\infty} \frac{X_t}{t} = \lim_{t\to\infty} \left(\frac{x}{t} + c + \frac{N_t}{t} \frac{\sum_{i=1}^{N_t} \xi_i}{N_t} \right) = c - \lambda\mu > 0,$$

under the net profit condition it follows that ruin will occur only with probability less than one. Fundamental quantities of interest in this model thus become the distribution of the time to ruin and the deficit at ruin; otherwise identified as

$$r_0^- := \inf\{t > 0: X_t < 0\} \, and \; X_{r_0^-} \; on \; \{r_0^- < \infty\}$$

when the process X drifts to infinity. The following classic result links the probability of ruin to the conditional distribution

$$\eta(x) = \mathbb{P}(-X_{r_0^-} \leq x | r_0^- < \infty).$$

The M/G/1 queue

Let us recall the definition of M/G/1 queue. Customers arrive at a service desk according to a Poisson process and join a queue. Customers have service times that are independent and identically distributed. Once served, they leave the queue.

The workload W_t, at each time $t \geq 0$, is defined to be the time it will take a customer who joins the back of the queue at that moment to reach the service desk, that is to say the amount of processing time remaining in the queue at time t. Suppose that at an arbitrary moment, which we shall call time zero, the server is not idle and the workload is equal to w>0 . On the event that t is before the first time the queue becomes empty, we have that W_t is equal to

$$w + \sum_{i=1}^{N_t} \xi_i - t$$

where, as with the Cramer-Lundberg risk process, $N = \{N_t : t \geq 0\}$ is a Poisson process with intensity λ>0, and $\{\xi_i : i \geq 0\}$ are positive variables that are independent and identically distributed with common distribution F and mean $\mu < \infty$. The process N represents the arrival of new customers and $\{\xi_i : i \geq 0\}$ are understood as their respective service times that are added to the workload. The negative unit drift simply corresponds to the decrease in time as the server deals with jobs. Thanks to the lack of memory property, once the queue becomes empty, the queue remains empty for an exponentially distributed period of time with the parameter λ after which a new arrival incurs a jump in W, which has distribution F. The process proceeds as the Compound Poisson process described above until the queue next empties and so on.

The workload is clearly not a Lévy process as it is impossible for $W_t : t \geq 0$ to decrease in value from the state zero whereas it can decrease in value from any other state x>0. However, it turns out that it is quite easy to link the workload to a familiar functional of a Lévy process, which is also a Markov process.

13. Popular models

In this section we review some popular models in the mathematical finance literature from the point of view of Lévy processes. (see manuscript [2]).

Black-Scholes

The most famous asset price model based on a Lévy process is that of Samuelson (1965), Black-Scholes (1973) and Merton (1973). The log-returns are normally distributed with mean μ and variance σ^2, i.e. $X_t \sim Normal(\mu, \sigma^2)$ and the density is $f_{X_1}(x) = \frac{1}{\sigma\sqrt{2\pi}} exp\left[-\frac{(x-\mu)^2}{2\sigma^2}\right]$. The canonical decomposition of the X is

$X_t = \mu t + \sigma W_t$ and the Lévy triplet is $(\mu, \sigma^2, 0)$.

Merton

Merton was one of the first to use a discontinuous price process to model asset returns. The canonical decomposition of the driving process is

$X_t = \mu t + \sigma W_t + \sum_{k=1}^{N_t} J_k$ where $J_k \sim Normal(\mu_J, \sigma_J^2), k = 1 \ldots$ hence the distribution of the jump size has density $f_J(x) = \frac{1}{\sigma_{J\sqrt{2\pi}}} \exp\left[-\frac{(x-\mu_J)^2}{2\sigma_J^2}\right]$ and Lévy triplet $(\mu, \sigma^2, \lambda \times f_J)$.

Normal Inverse Gaussian

The NIG distribution is a special case of the GH (Generalized Hyperbolic) distribution for λ=-1/2. It was introduced to finance in Barndoff-Nielsen (1997). The canonical decomposition is

$$X_t = t\mathbb{E}[X_1] + \int_0^t \int_{\mathbb{R}} x(\mu^x - v^{NIG})(ds, dx)$$

where the Lévy measure has the simplified form

$$v^{NIG}(dx) = e^{\beta x} \frac{\delta\alpha}{\pi|x|} K_1(a|x|)dx$$

where K_1 is the modified Bessel function of the second kind.

CGMY

The CGMY Lévy process was introduced by Carr, Geman, Madan and Yor(2002); another name for this process is (generalized) tempered stable process.

The Lévy measure of this process admits the representation

$$v^{CGMY}(dx) = c\frac{e^{-Mx}}{x^{1+Y}} 1_{\{x<0\}} dx + c\frac{e^{Gx}}{|x|^{1+Y}} 1_{\{x<0\}} dx,$$

where C>0,G>0,M>0 and Y<2. The CGMY process is a pure jump Lévy process with canonical decomposition

$$X_t = t\mathbb{E}[X_1] + \int_0^t \int_{\mathbb{R}} x(\mu^X - v^{CGMY})(ds, dx)$$

and Lévy triplet $(\mathbb{E}[X_1], 0, v^{CGMY})$

14. Lévy Processes in Asset pricing

Financial Stock prices

Brownian Motion $(B_t)_{t\geq 0}$ or linear Brownian Motion $(\sigma B_t + \mu t, t \geq 0)$ was the first model of stock prices, introduced by Bachelier in 1900. Black, Scholes and Merton studied Geometric Brownian Motion $\exp(\sigma B_t + \mu t)$ in 1973, which is not itself a

Lévy Process but can be studied with similar methods. The Economics Nobel price in 1997 was awarded to them for their work. Several deficiencies of Black-Scholes model have been identified, e.g. the Gaussian density decreases quickly, no variation of the volatility σ over time, no macroscopic jumps in the price processes. The deficiencies can be addressed by models based on Lévy processes.

Asset price model – real world measure

Exponential Lévy models

To ensure positivity as well as the independence and stationarity of log-returns, *stock prices* are usually modeled as exponentials of Lévy processes[6]. Under the real-world measure, we model the asset price process as the exponential of a Lévy process, that is

$$S_t = S_0 exp X_t\ ,\ \ 0 \leq t \leq T \quad (14.1)$$

where, X is the Lévy process whose infinitely divisible distribution has been estimated from the data set available for the particular asset. Hence, the log-returns of the model have independent and stationary increments, which are distributed-along time intervals of specific length, e.g. 1-according to an infinitely divisible distribution. Naturally, the path properties of the process X carry over to S; This fact allows us to capture, up to certain extent, the microstructure of the price fluctuations, even on an intraday time scale.

In the jump-diffusion case this gives

$$S_t = S_o \exp\left(\mu t + \sigma B_t + \sum_{i=1}^{N_t} Y_i\right) \quad (14.2)$$

Between the jumps the process evolves like a geometric Brownian motion, and after each jump, the value of S_t is multiplied by e^{Y_i} . This model is a generalization of the Black-Scholes model:

$$\frac{dS_t}{S_{t_-}} = \bar{\mu} dt + \sigma dB_t + dJ_t \quad (14.3)$$

This is a Stochastic differential equation that we use to represent the GBM. Here, J_t is a Compound Poisson process such that the i-th jump of J is equal to $e^{Y_i} - 1$. For instance, if Y_i has Gaussian distribution, S will have log-normally distributed jumps. The notation S_{t-}means that whenever there is a jump, the value of the process just before the jump is used on the left-hand side of the formula[9].

Definition 14.1.

The market is said to be *complete* if each claim is either attainable or asymptotically attainable.

Theorem 14.2.(The second Fundamental Theorem of Asset Pricing)

Assuming that the model is *arbitrage free*, it is complete *iff* $\mathcal{P}(\mathrm{P})$ contains *exactly one* measure[4].

We denote by $\mathcal{P}(P)$ the class of all probability measures on $(\Omega, \mathcal{F})$ which are *equivalent* to P and under which the discounted process $\bar{S}$ is a martingale.

Remark 14.3. The fact that the price process is driven by a Lévy process makes the market in general incomplete[2]. The only exceptions are the markets driven by the normal (Black-Scholes model) and Poisson distributions. Therefore, there exists a large set of equivalent martingale measures, i.e. candidate measures for risk-neutral valuation.

Eberlein and Jacod (1997) provide an analysis and characterization of the set of equivalent martingale measures for Lévy-driven models. Moreover, they prove that the range of option process for a convex payoff function, e.g. a call option, under all possible equivalent martingale measures spans the whole no-arbitrage interval, e.g. $[(S_o - Ke^{-rT})^+, S_o]$ for a European call option with strike K. Selivanov (2005) discusses the existence and uniqueness of the martingale measures for exponential Lévy models in finite and infinite time horizon and various specifications of the no-arbitrage condition.

The Lévy market can be completed using particular assets, such as moment derivatives (e.g. variance swaps), and then there exists a unique equivalent martingale measure. For example, if an asset is driven by a Lévy jump-diffusion $X_t = bt + \sqrt{c}B_t + \sum_{k=1}^{N_t} J_k$, then the market can be completed using only variance swaps on this asset;

Risk-Neutral measure

Under the risk-neutral measure $\bar{P}$ we model the asset price process as an exponential Lévy process $S_t = S_0 exp X_t$ where the Lévy process X has the triplet $(\bar{b}, \bar{c}, \bar{v})$.The process X has the canonical decomposition

$$X_t = \bar{b}t + \sqrt{\bar{c}}\bar{W}_t + \int_0^t \int_{\mathbb{R}} x(\mu^X - \bar{v}^X)(ds, dx)$$

where $\bar{W}$ is a $\bar{P}$ -Brownian motion and $\bar{v}^X$ is the $\bar{P}$ –compensator of the jump measure μ^X.

15. Empirical motivation

The main empirical motivation of using Lévy processes in *finance* comes from fitting asset return distributions[6]. For example, consider the *daily returns of S&P 500* *index* (SPX) from June 1, 2015 to May 29, 2020. We plot the histogram of normalized daily log-returns in SPSS along with the standard normal density function.

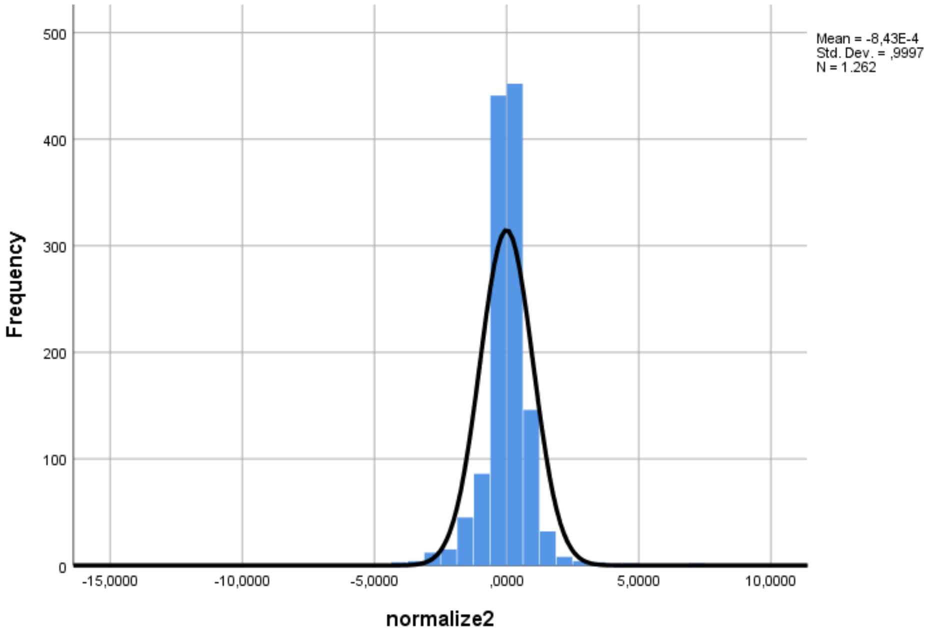

Figure 10. Histogram of normalized S&P 500 index Daily log-returns and the Standard Normal Density

The histogram displays a high peak and two asymmetric heavy tails (leptokurtic feature). This is true for almost all financial asset prices, e.g. *worldwide stock indices, foreign exchange rates, interest rates, individual stocks*. In fact it's so evident that a name "leptokurtic distribution" is given, which means the kurtosis of the distribution is large. The kurtosis and the skewness are defined as

$$K = E\left(\frac{(X-\mu)^4}{\sigma^4}\right), S = E\left(\frac{(X-\mu)^3}{\sigma^3}\right).$$

For standard normal density K=3, and if K>3 then the distribution is called leptokurtic and the distribution will have a higher peak and two heavier tails than those of the normal distribution.

The classical geometric BM model, which models the stock price as

$$S_t = S(0)e^{\mu t+\sigma B_t}$$

with B_t the standard BM, is inconsistent with this feature, because in this model the return, $\ln\left(\frac{S(t)}{S(0)}\right)$ has a normal distribution. Lévy processes, among other processes have been proposed to incorporate the leptokurtic feature.

Lévy processes provide a natural generalization of the sum of independent and identically distributed (i.i.d) random variables. Any Lévy process can be written as a drift term μt, a BM with variance and covariance matrix A, and a possible infinite sum of independent compound Poisson processes which are related to an intensity measure v(dx).

This implies that a Lévy process can be approximated by jump-diffusion processes. This has important numerical applications in finance, as jump-diffusion models are widely used in finance.

Volatility clustering Effect

In addition to the leptokurtic feature, returns distributions also have an interesting dependent structure, called the volatility clustering effect; (see Engle 1995). More precisely, the volatility of returns (which are correlated to the squared returns) are correlated, but asset returns themselves have almost no autocorrelation. In other words, a large movement in asset prices, either upside or downside tend to generate large movements in the future asset prices, although the direction of the movements is unpredictable. In particular any model for stock returns with independent increments (such as Lévy processes) cannot incorporate the volatility clustering effect. However, one can combine Lévy processes with other processes (e.g, Duffie, Pan, Singleton, 2000, Barndoff Nielsen and Shepherd 2001) or consider time changed Brownian motion and Lévy processes to incorporate the volatility clustering effect.

16. Monte Carlo Simulation in R for various Stochastic processes

Brownian Motion, which is non-standard, will have two parameters just like Normal Distribution, known as drift and diffusion[10]. Using B(t) we therefore give a **Stochastic Differential Equation** for any Brownian Motion

$$dX(t) = \mu(t)dt + \sigma(t)dB(t)$$

where μ is a drift component and σ^2 is a diffusion coefficient.

Sample Paths Generations

Solving the SDE presented above we can write the equation in terms of $X(t_i), \mu(s), \sigma(t)$

$$X(t_{i+1}) = X(t_i) + \int_{t_i}^{t_{i+1}} \mu(s)ds + \sqrt{\int_{t_i}^{t_{i+1}} \sigma^2(u)du}\ Z_{i+1}$$

In the R-code we present later we have assumed that μ and σ are constant.

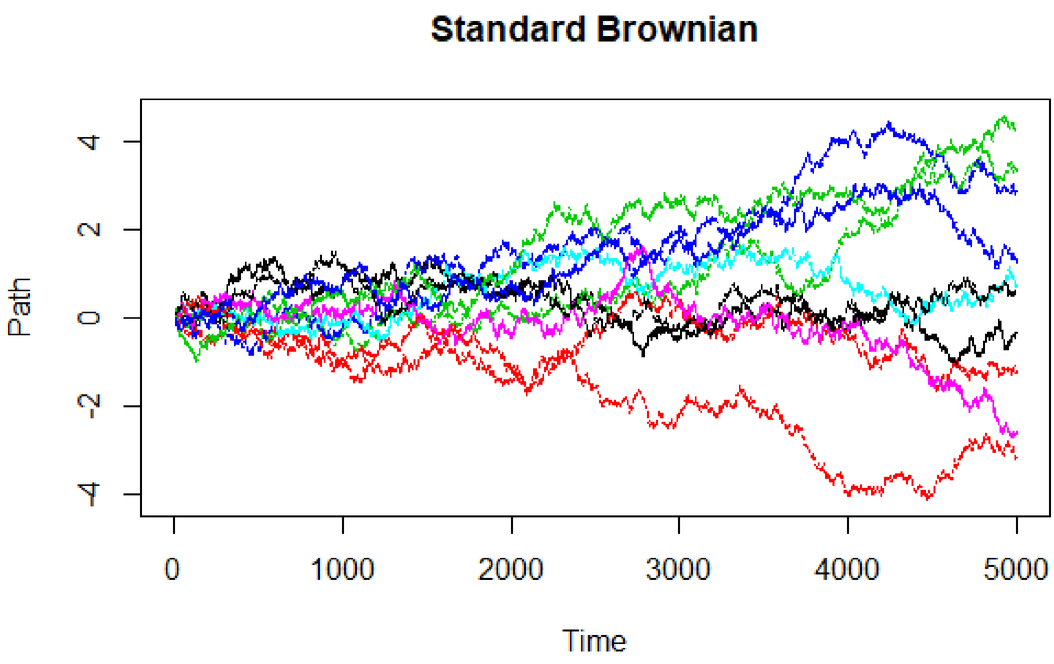

Figure 11. Simulation of Standard Brownian Motion.

The above image (figure 11) represents 10 paths the code generated for the Standard Brownian Motion.

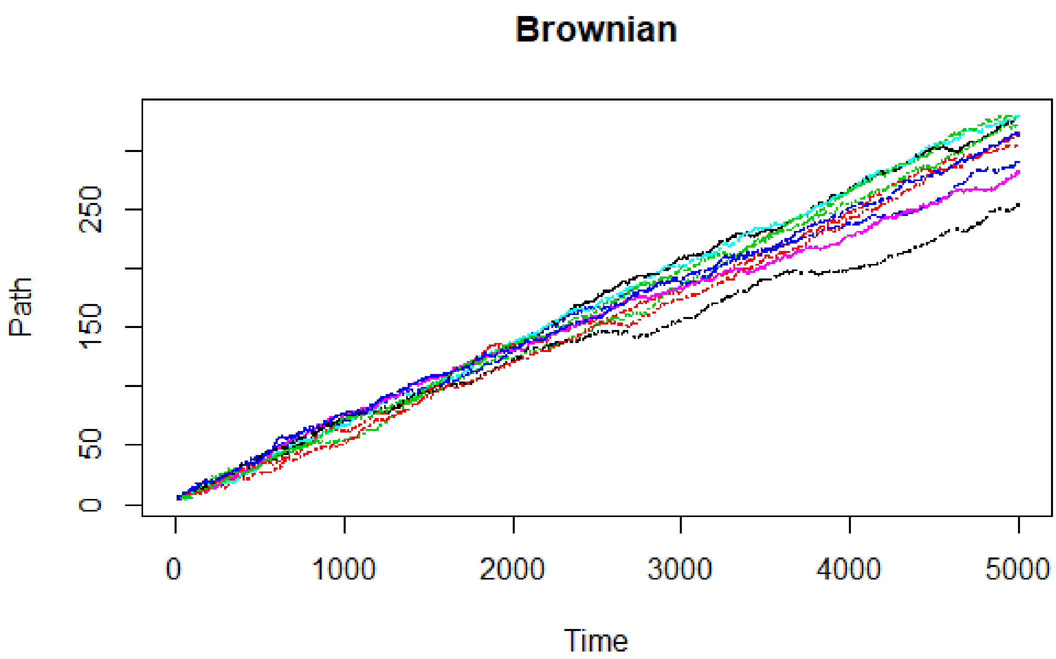

Figure 12. Simulation of Brownian Motion.

The code used the formula defined for Brownian Motion to generate 10 paths for it and also used **rnorm(1)** a function defined in R to generate a Standard Normal Random Variable.

Geometric Brownian Motion properties

Suppose that $S(t)$ is Geometric Brownian Motion with drift parameter μ volatility σ and initial value S_o then

(1) $\boldsymbol{S}$ is a stochastic process with initial value $\boldsymbol{S_o}$

(2) S has independent growth factors: for any sequence on non-overlapping intervals $(t_j, t_j + h_j]$ the growth factors $\frac{S(t_j+h_j)}{S(t_j)}$ are independent.

(3)For all $t \geq 0$ and $h > 0$ the growth factor $\frac{S(t+h)}{S(t)}$ is log-normal $e^{(N(\mu h, \sigma^2 h))}$ with mean $e^{\left(\mu+\frac{1}{2\sigma^2}\right)h}$ and variance $e^{(2\mu+\sigma^2)h}(e^{\sigma^2 h} - 1)$.

The Stochastic Differential Equation that we use now to represent the Geometric Brownian Motion is $\frac{dS(t)}{S(t)} = \mu dt + \sigma dB(t)$.

The solution of S.D.E for Geometric Brownian Motion is

$$S(t) = S(0)exp\left(\left[\mu - \frac{1}{2}\sigma^2\right]t + \sigma B(t)\right),$$

where $B(t)$ is replaced with $\sigma\sqrt{T}Z_i$

and that's *how by using Monte Carlo Simulation we could also simulate the paths of a Stock price or of a Geometric Brownian Motion.*

For such simulation we would again have to discretize the time line into some N points to generate Stock Price at all such points. Let us take initial Stock price to be 100. The plot looks

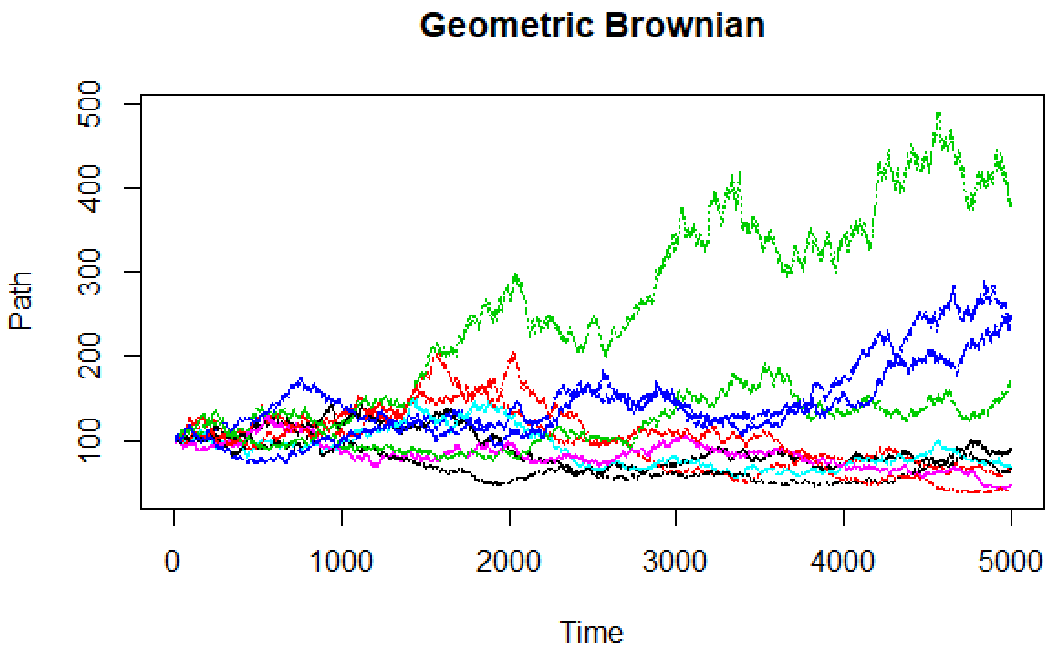

Figure 13. Simulation of Geometric Brownian Motion

Hence we can use all such paths to finally get a S(t) and calculate the premium of European Option and finally give the average estimate of the European Option Price.

Chapter 3

17. Lévy processes in Option Pricing

Although the Black-Scholes model is one of the most widely used frameworks nowadays, the real prices show properties which contradict the assumptions of this model[7].

The assumptions of the model	The properties of real prices
1. Asset log returns have been modeled in continuous time as diffusion 2. Asset return increments are normally distributed 3. The implied volatility should be constant	1. Return dynamics are devoid of diffusion component 2. the increments are skewed to the left and have a fat tail than those of normal distribution 3. the implied volatility curve resembles a smile /skew meaning it is a convex curve of the strike price

For option pricing, we will explicitly include the interest rate into the definition of the exponential Lévy model:

$$S_t = S_o e^{rt+X_t} \quad (17.1)$$

The model (18.1) admits no arbitrage opportunity if there exists an equivalent probability under which e^{X_t} is a martingale. For Lévy processes it can be shown that this is always the case, namely an exponential Lévy model is arbitrage-free if and only if the trajectories of X are not almost surely increasing nor almost surely decreasing (thus we exclude the cases: constant drift, Poisson process, subordinator)

Pricing Option CGMY model

Nowadays, recently researchers have proposed CGMY process as the most suitable model to catch the assumptions fail. The model name refers to mathematician names: Carr, Geman, Madan and Yor allowing to take into account both phenomenon, indefinite activity (process incorporate frequent small moves and rare large jumps) and finite/infinite variation. CGMY model has been employed to study statistical process needed to assess risk-neutral process to pricing option through the characteristic function of return price.

The CGMY process

Contrary to the Variance Gamma process which can be represented as time changed Brownian motion (i.e. Brownian motion subordinated to a Gamma subordinator), CGMY process is not known through such representation, it is only known by its Lévy measure.

Let $(\Omega, \mathcal{F}_{t\in[0,\infty)}, \mathbb{P})$ be a filtered probability space, we define the CGMY process $X_{CGMY}(t; C, G, M, Y)$ as a Lévy process with Lévy triplet $(A^x = 0, \rho^x, b^x = 0)$ and Lévy measure

$$\rho^x_{CGMY} = \frac{Cexp^{(-G|x|)}}{|x|^{1+Y}} 1_{x<0} + \frac{Cexp^{(-Mx)}}{x^{1+Y}} 1_{x>0} \quad (17.2)$$

where $C > 0, G \geq 0, M \geq 0$ and $Y < 2$. The parameter C controls overall arrival rate of jumps, G and M are the exponential decay rates on the right and on the left of the Lévy measure leading to skewed distribution when they are unequal. When G=M, the Lévy measure is symmetric. For G<M, the left tail of the distribution of X_t is heavier than the right tail. We say that the arrival rate of negative jumps is higher than that of positive jumps. The most interesting parameter is Y, it allows to understand the structure of the process since it describes the behavior of the Lévy measure i.e. whether it is completely monotone, it has finite/infinite activity and finite/infinite variation.

European call option pricing under CGMY model

CGMY asset price process

Let $(\Omega, \mathcal{F}_{t\in[0,T]}, \mathbb{Q})$ be a filtered risk neutral probability space. Asset price dynamics $S_{t\in[0,T]}$ is an exponential Lévy process $X_{t\in[0,T]}$ of the form

$$S_t = S_0 exp X_t$$

and the choice of the Lévy process is the CGMY process plus a drift

$$X_t \equiv \{(r - \beta)t + X_{CGMY}\big(t; C_{\mathbb{Q}}, G_{\mathbb{Q}}, M_{\mathbb{Q}}, Y_{\mathbb{Q}}\big)\}$$

where $r \geq 0$ is the mean rate of return on the asset and β is the convexity correction in CGMY model.

Characteristic function formulation for solution

In this section we are motivated to present closed-form solutions to know more about the process structure and about pricing the option under CGMY model. The risk-neutral log asset price dynamics have the form

$$lnS_t = lnS_0 + (r-\beta)t + X_{CGMY}(t; C_{\mathbb{Q}}G_{\mathbb{Q}}M_{\mathbb{Q}}Y_{\mathbb{Q}})$$

since the density of the process $X_t$is expressed in the equation (18.1) and then

$$X_t = lnS_t - [lnS_0 + (r-\beta)t]$$

Fourier transform inversion

We obtain the CGMY-FT call pricing formula [7]

$$C_{CGMY}(T,K) = \frac{e^{-ak}}{2\pi}\int_{-\infty}^{\infty} e^{-i\omega k}\frac{e^{-rT}\varphi_T(\omega-(\alpha+1)i)}{a^2+a-\omega^2+i(2a+1)\omega}d\omega$$

where $\varphi_T(\omega)$ is the characteristic function of the log asset price, defined as

$$\varphi_T(\omega) = \exp\{i\omega S_0 + (r-\beta)T\} \times \exp\left\{\begin{matrix} TCG^YG(-Y)\left[\left(1+\frac{i\omega}{G}\right)^Y - 1 - \frac{i\omega Y}{G}\right] + \\ TCM^Y\Gamma(-Y)[\left(1-\frac{\iota\omega}{M}\right)^Y - 1 + \frac{\iota\omega Y}{M}] \end{matrix}\right\}$$

Numerical results

The implementation of the CGMY Fourier transform formula with decay rates parameters G=2.0, M=3.5, overall arrival rate C=0.5 and Y=1.34. The common parameters are S0=100.0, r=0.05. While ω_n varies from $\omega_n = 1, \dots N$ and which is assumed to be equal in length. Considering the maturity T=1.0, time grid M=50 and finally, simulation size I=10000. Bellow we see the implementation of the CGMY model in Python. The illustration shows dynamic of stock price in market free-arbitrage.

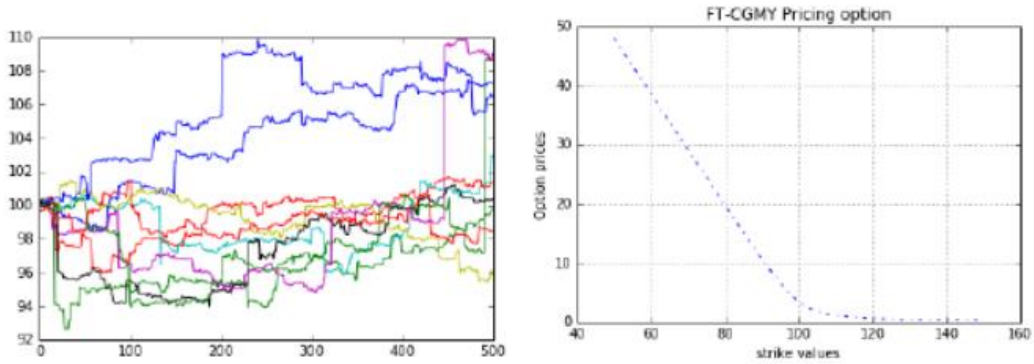

Figure 14. Option pricing with CGMY model In Python.

Here we see the calibration of FT-CGMY option pricing approach with the modified call price. The calibration result suggests that the extra parameters of CGMY model allow the *negative skewness* and the *excess of kurtosis (leptokurtic)*.Moreover the dynamics of implied Lévy density is asymmetric and has an infinite activity. Despite the efficiency of CGMY process, all option pricing models are biased.

Monte Carlo Option pricing for tempered stable (CGMY) processes

Lévy processes are popular models for stock price behavior since they allow to take into account jump risk and reproduce the implied volatility smile. The tempered stable (also known as CGMY) processes form a flexible 6-parameter family of Lévy processes with infinite jump intensity. It is shown that under an appropriate equivalent probability measure a tempered stable process becomes a stable process whose increments can be simulated exactly. This provides a fast Monte Carlo algorithm for computing the expectation of any functional of tempered stable process and this method can be used to price European options[8].

A method of Monte Carlo evaluates the functional of the tempered stable process which avoids direct simulation of the increments of this process. Instead, we construct an equivalent probability measure under which the original tempered stable process becomes a stable process. Since the method for direct simulation of stable random variables is well-known and the measure change is explicit, this provides the desired algorithm.

The following theorem shows that under an appropriate change of measure the tempered stable process becomes a sum of one-sided stable process.

Tempered stable processes

A one dimensional *tempered stable process* is obtained by taking a one-dimensional stable process and multiplying the Lévy measure with a decreasing exponential on each half of the real axis. Thus, a tempered stable process is a Lévy process on $\mathbb{R}$ with no Gaussian component and Lévy density of the form:

$$v(x) = \frac{c_+ e^{-\lambda_+ x}}{x^{1+a}} 1_{x>0} + \frac{c_- e^{-\lambda_- |x|}}{|x|^{1+a}} 1_{x<0}$$

with parameters satisfy $c_- > 0, c_+ > 0, \lambda_- > 0, \lambda_+ > 0$ and $0 < \alpha < 2$.

In particular, the version when $c_+ = c_-$ and $a_+ = a_-$ was studied under the name CGMY process with Lévy measure

$$v_{CGMY}(x) = c\left[\frac{e^{-Mx}}{x^{1+Y}} 1_{x>0} + \frac{e^{-G|x|}}{|x|^{1+Y}} 1_{x<0}\right]$$

Theorem17.1.

Let X_t be a (generalized) tempered stable Lévy process on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with Lévy density $v(x) = \frac{c_+ e^{-\lambda_+ x}}{x^{1+a_+}} 1_{x>0} + \frac{c_- e^{-\lambda_- |x|}}{|x|^{1+a_-}} 1_{x<0}$, and let $(X_t^+)$and (X_t^-) be tempered stable Lévy processes such that $X = X^+ + X^-$ with characteristic triplets $(0, v_+, \gamma^+)$and $(0, v_-, \gamma^-)$ where

$$v_+(x) = \frac{c_+ e^{-\lambda_+ x}}{x^{1+a_+}} 1_{x>0} \text{and } v_-(x) = \frac{c_- e^{-\lambda_- |x|}}{|x|^{1+a_-}} 1_{x<0}$$

Then the following holds:

1. There exists a unique constant c such that $\boldsymbol{e^{U_t}}$ is a P-martingale, where $\boldsymbol{U_t = \lambda_+ X_t^+ - \lambda_- X_t^- + ct}$.
2. One can define a probability measure $\mathbb{Q}$ such that $\mathbb{Q}|_{\boldsymbol{F_t}} \sim \mathbb{P}|_{\boldsymbol{F_t}}$for every t by $\frac{\boldsymbol{d\mathbb{Q}}}{\boldsymbol{d\mathbb{P}}}|_{\boldsymbol{\mathcal{F}_t}} = \boldsymbol{e^{U_t}}$.
3. Under $\mathbb{Q}$, the processes $(\boldsymbol{X_t^+})$ and $(\boldsymbol{X_t^-})$ are stable processes with characteristic triplets $(\boldsymbol{0}, \boldsymbol{\tilde{v}_+}, \boldsymbol{\gamma^+})$ and $(\boldsymbol{0}, \boldsymbol{\tilde{v}_-}, \boldsymbol{\gamma^-})$ where

$\boldsymbol{\tilde{v}_+(x) = \frac{c_+}{x^{1+a_+}} 1_{x>0}}$ and $\boldsymbol{\tilde{v}_-(x) = \frac{c_-}{|x|^{1+a_-}} 1_{x<0}}$

The expectation of any F_t measurable random variable H_T can be evaluated via

$$E^P[H^T] = E^P[H_T e^{-\lambda_+ X_T^+ + \lambda_- X_T^- - cT}]$$

In particular, if $H_T = f(X_T)$ then $E^P[H_T] = E^P[f(X_T)] = E^Q[f(X_T)e^{-\lambda_+ X_T^+ + \lambda_- X_T^- - cT}]$

The **Monte Carlo estimator** $\overline{H}$ of $E[H_T]$ is given by

$$\overline{H} = \frac{1}{N}\sum_{i=1}^{N} H_T^i \exp\left(-\lambda_+ X_T^{+,\iota} + \lambda_- X_T^{-,\iota} - cT\right]$$

where X_T^i for $i = 1, \ldots N$ are independent realizations of X_T under $\mathbb{Q}$ and H_T^i are corresponding realizations of H_T .(see [8])

18. Monte Carlo Option Pricing Algorithms for Jump Diffusion Models with Correlational Companies.

Option is a one of the financial derivatives and its pricing is an important problem in practice. The process of the stock prices are represented as Geometric Brownian motion or jump diffusion processes. In the early 1970's, Black and Scholes proposed a method in order to calculate option price. For option pricing, we solve numerically Black-Scholes equation, that is represented as a differential equation. The methods proposed are either to solve equations directly or by using Monte Carlo methods. In this section algorithms and visualizations are implemented by Monte Carlo method in order to calculate European option price for three equations by Geometric Brownian motion and jump diffusion processes and furthermore a model that presents jumps among companies which affect each other.

Black-Scholes Model

The process of stock prices are basically represented as Geometric Brownian motion.

$$dS(t) = \alpha(t)S(t)dt + \sigma(t)S(t)dB(t) \quad (18.1)$$

where s(t) denotes stock price at time t, α is the drift parameter and σ is the volatility. B(t) is Brownian motion. The solution of the equation is given by

$$S(t) = S(0)\exp\left\{\sigma B(t) + \left(\alpha - \frac{1}{2}\sigma^2\right)t\right\} \quad (18.2)$$

Price of European option

There are many types of options in the stock market. European call option can not execute until the duration T is finished, and its strike price is K. Option prices are calculated under the risk-neutral probability. The *European call option* price is given by

$$Price = \mathbb{E}[\max(S(T) - K, 0)]$$

where E[X] denotes expected value. The *European put option* price is given by

$$Price = \mathbb{E}[\max(K - S(T), 0)]$$

Correlational Companies Algorithm

Initially, the Standard jump diffusion model causes jumps in one stock market and it does not affect other companies. In correlational jumps model, one jump among companies affects other stock prices of companies obeying correlation coefficients[11]. Therefore, the equations are given by

$$S_t = S_o \exp\left\{\sigma B_t + \left(\alpha - \beta\lambda - \frac{1}{2}\sigma^2\right) t\right\} \prod_{i=1}^{N_t}(Y_i \times \rho_{xy} + 1)$$

where ρ_{xy} denotes a correlation coefficient between the x-th company and y-th company.

As you can see in the next graph there are pair companies and the correlation coefficients between them. In the matrix bellow we see analytically the results of the correlation coefficients of all pair companies which we will use later to calculate option prices by correlational method of Monte Carlo.

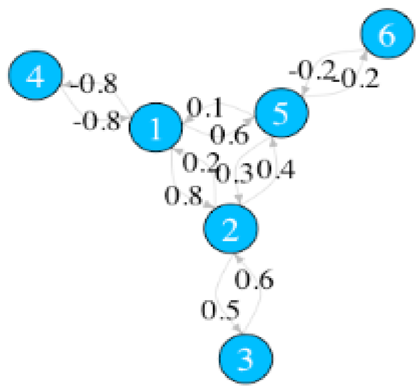

Figure 15. Circulation graph of the correlation coefficients between the companies.

The result of the correlation coefficients of the companies

	1	2	3	4	5	6
1	1	0.98	0.49	-0.8	0.92	-0.184
2	0.24	1	0.5	-0.192	0.52	-0.104
3	0.144	0.6	1	-0.1152	0.312	-0.0624
4	-0.8	-0.784	-0.392	1	-0.736	0.1472
5	0.16	0.38	0.19	-0.128	1	-0.2
6	-0.032	-0.076	-0.038	0.0256	-0.2	1

Table1. The correlation coefficients of the companies.

We will use the Jdmbs: an R package for Monte Carlo Option Pricing Algorithms for Jump Diffusion Models with Correlational Companies.

Methods

This package has three methods.

This is a normal model by Monte Carlo:

```
> price <- normal_bs (day =180 , monte_carlo =1000 ,
                          start_price , mu , sigma , K , plot = TRUE
                         )
```

Source: Snapshot1 from R-package. R-code for Option pricing with normal Monte Carlo model.

Jump diffusion mode by Monte Carlo:

```
> price <- jdm_bs (day =180 , monte_carlo =1000 ,
                      start_price , mu , sigma , lambda , K , plot = TRUE
                     )
```

Source:Snapshot2 from R-package. R-code for Option pricing with jump-diffusion model of Monte Carlo.

This is a correlational method by Monte Carlo. Companies_matrix must be required:

```
> price <- jdm_new_bs (companies_matrix , day =180 , monte_carlo =1000 ,
                       start_price , mu , sigma , lambda , K , plot = TRUE
                      )
```

Source:Snapshot3 from R-package. R-code for Option pricing with correlational method of Monte Carlo.

Let arguments be:

- companies_matrix: a matrix of a correlation coefficient of companies
- day: an integer of a time duration of simulation
- monte_carlo: an integer of an iteration number for Monte Carlo
- start_price: a vector of company's initial stock prices
- mu: a vector of drift parameters of Geometric Brownian Motion
- sigma: a vector of volatility parameters of Geometric Brownian Motion.
- lambda: an integer of how many times jump in unit time
- K:a vector of option strike prices
- plot: a logical type of whether plot a result or not

Let return be:

- price of a list of(call_price, put_price)

Simulation

It is a normal model by Monte Carlo:

```
> price <- normal_bs(day =100 , monte_carlo =10,
                     start_price =c (300 ,500 ,850),
                     mu =c (0.1 ,0.2 ,0.05) , sigma =c (0.05 ,0.1 ,0.09),
                     K=c (600 ,700 ,1200),
                     plot = TRUE
                    )
```

Source: Snapshot4 from R-package. R-code for Option pricing by normal model of Monte Carlo , simulation with numerical input.

The algorithm above for given stock prices, diffusion component and strike prices gives the Option price for time period 100 and simulation size 10.

Jump Diffusion by Monte Carlo:

```
> price <- jdm_bs(day =100 , monte_carlo =10,
                  start_price =c (5500 ,6500 ,8000),
                  mu =c (0.1 ,0.2 ,0.05) , sigma =c (0.11 ,0.115 ,0.1),
                  lambda =2,
                  K=c (6000 ,7000 ,12000),
                  plot = TRUE
                 )
```

Source: Snapshot5 from R-package. R-code for Option pricing by jump-diffusion model of Monte Carlo, simulation with numerical input

In the Jump diffusion model by Monte Carlo, we can get the option prices by the same way by adding an extra parameter λ which symbolizes the intensity of the jump process.

It is a correlational method by Monte Carlo. Companies_matrix must be required:

```
> corr_matrix <- matrix (c (0.1 ,0.2 ,0.3 ,0.4 ,0.5 ,0.6 ,0.7 ,0.8 ,0.9) , nrow =3 , ncol =3)

> price <- jdm_new_bs(corr_matrix,
                      day =100 , monte_carlo =10,
                      start_price =c (5500 ,6500 ,8000),
                      mu =c (0.1 ,0.2 ,0.05) , sigma =c (0.11 ,0.115 ,0.1),
                      lambda =2,
                      K=c (6000 ,7000 ,12000),
                      plot = TRUE
                      )
```

Source: Snapshot6 from R-package. R-code for Option pricing by correlational method of Monte Carlo, simulation with numerical input

In the figures we can see how option prices are produced for related companies.

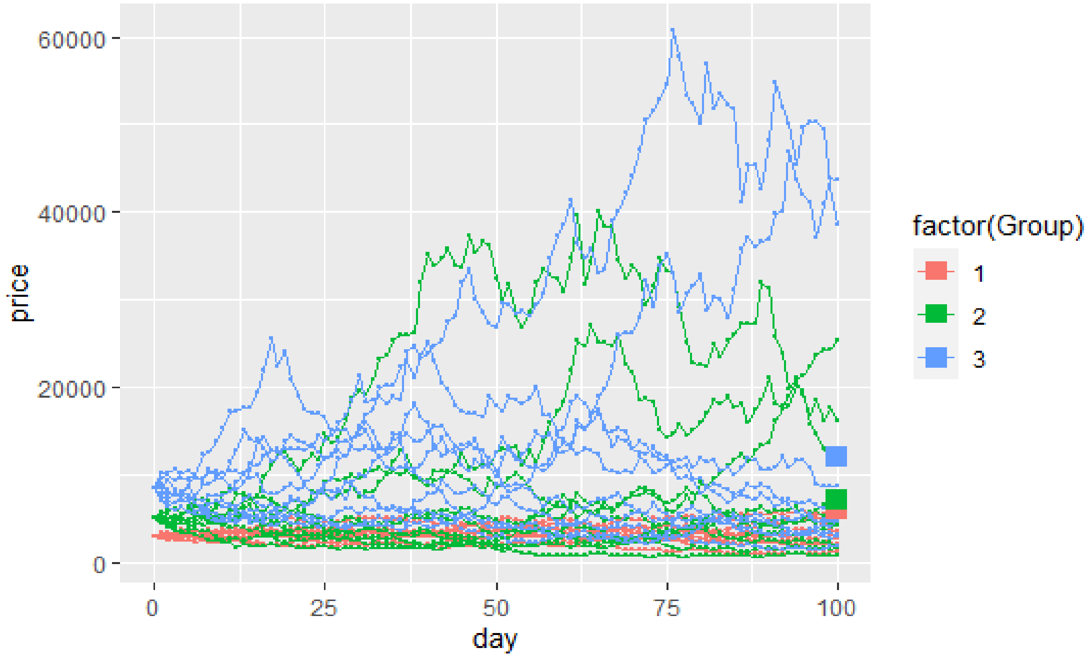

Figure16. Three stock prices. Square points are Strike prices. A normal model by Monte Carlo.

The price for call option for the first stock is zero, the call option price for the second stock is 3172 and for the third stock 5811. Furthermore, the put option price for the first stock is 2551, for the second 2353 and for the third stock 5911.

```
[1] "Call Option Price:"
option_ 1 option_ 2 option_ 3
    0.000  3172.869  5811.420
[1] "Put Option Price:"
option_ 1 option_ 2 option_ 3
 2551.030  2353.425  5911.238
```

Table2. Option prices by a normal model of Monte Carlo.

Bellow we can see the results for the other two methods presented.

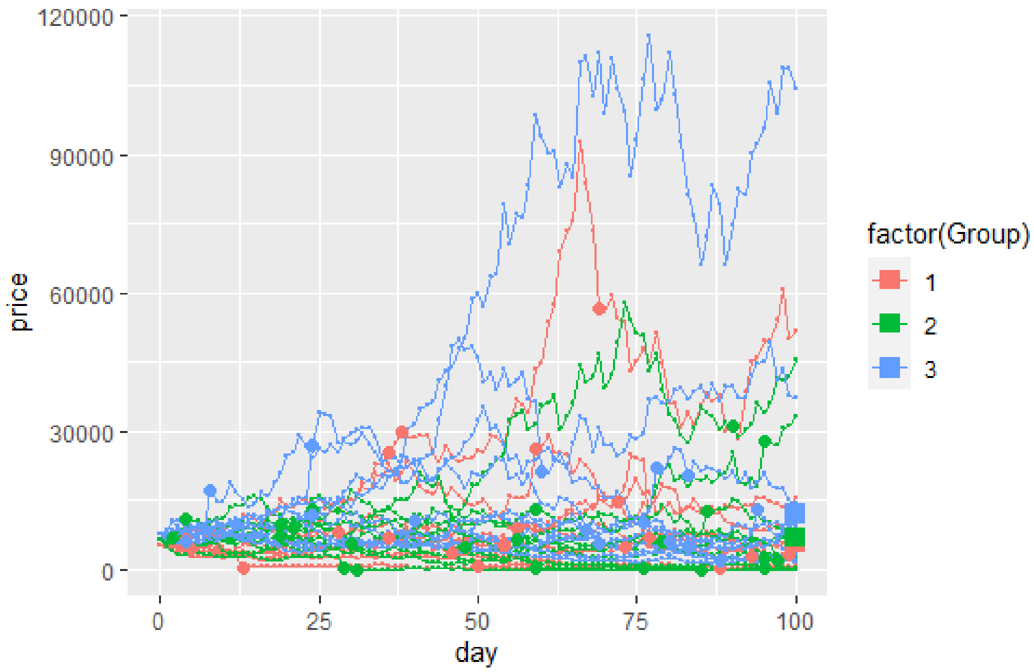

Figure17. Three stock prices. Square points are strike points. Big round points are jump points. Jump diffusion by Monte Carlo.

```
[1] "Call Option Price:"
option_ 1 option_ 2 option_ 3
 5803.615  7031.288 12000.212
[1] "Put Option Price:"
option_ 1 option_ 2 option_ 3
 2163.071  2757.448  4778.676
```

Table3. Option prices by Jump diffusion model of Monte Carlo.

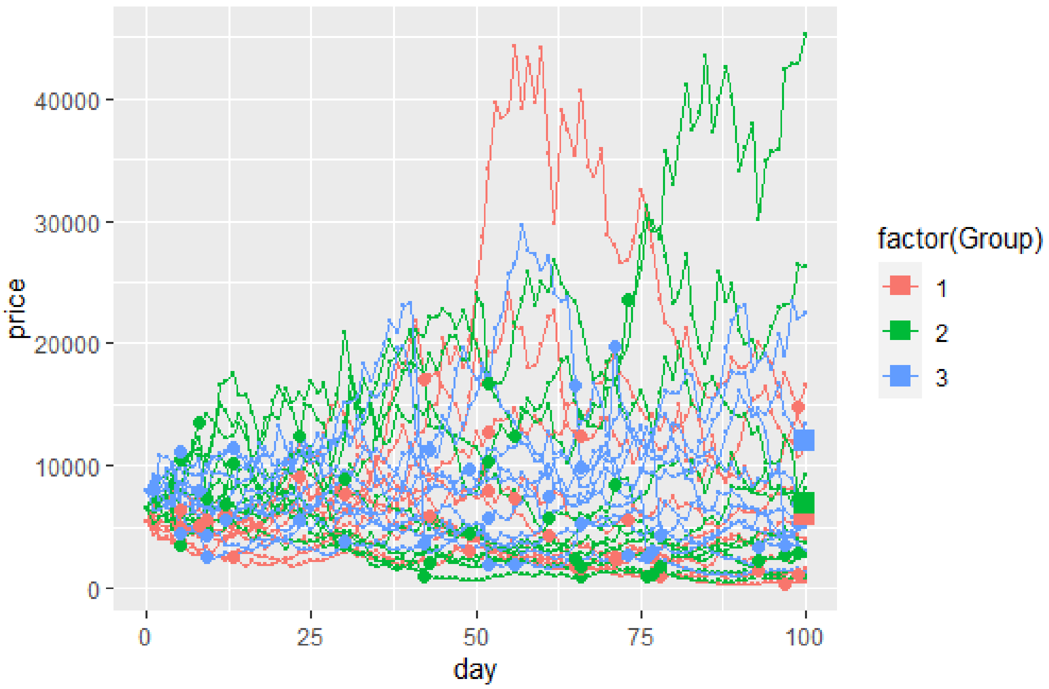

Figure18. Three Stock prices. Square points are strike prices. Big round points are jump points. Correlational method by Monte Carlo.

```
option_ 1 option_ 2 option_ 3
 2508.817  6089.641  1193.234
[1] "Put Option Price:"
option_ 1 option_ 2 option_ 3
 2011.371  2460.516  5909.469
```

Table4. Option prices by correlational method of Monte Carlo.

19. Hedging the jump risk

In the Black-Scholes model, the delta-hedging strategy completely eliminates the risk of an option position. This strategy consists in holding the amount of stock equal to $\frac{\partial C}{\partial S}$, the sensitivity of the option price with respect to the underlying. However, in presence of jumps, delta-hedging is no longer optimal.

Since typically the jump size is not known in advance, the risk associated to jumps cannot be hedged away completely, because we are in an incomplete market. In this setting, the hedging becomes an approximation problem: instead of *replicating* an option, one tries to *minimize* the residual hedging error. Strategies using only stock lead to high levels of residual risk, and to obtain realistic hedges, liquid options should be added to the hedging portfolio.

In this section we show how to *compute optimal hedging strategies in presence of jumps*. First, we treat the case when the hedging portfolio contains only stock and the risk-free asset. Let S_t denote the stock price and ϕ the quantity of stock in the hedging portfolio, and suppose that S satisfies (14.3) with Lévy measure of the jump part denoted by v. Then the self-financing portfolio evolves as

$$dV_t = (V_t - \varphi_t S_t) r dt + \varphi_t dS_t$$

The 'forward' values of the stock and the portfolio

$$S_t^* = e^{r(T-t)} S_t \quad \text{and} \quad V_t^* = e^{r(T-t)} V_t$$

satisfy

$$V_T^* = e^{rT} V_o + \int_0^T \varphi_t dS_t^*$$

We would like to compute the strategy which minimizes the expected squared residual hedging error under the martingale probability

$$\varphi^* = arginf \mathbb{E}[(V_T - H_T)^2] = arginf \mathbb{E}\left[\left(e^{rT} V_o + \int_0^T \varphi_t \, dS_t^* - H_T\right)^2\right]$$

The initial capital minimizing the hedging error is $V_o = e^{-rT} \mathbb{E}[H_T]$

In the case that the residual hedging error is non-zero (and the market is incomplete), it's minimized by

$$\varphi^*(t, S_t) = \frac{\sigma^2 \frac{\partial C}{\partial S} + \frac{1}{S_t} \int v(dz) z (C(t, S_t(1+z)) - C(t, S_t))}{\sigma^2 + \int z^2 v(dz)}$$

The optimal quadratic hedging strategy is a weighted sum of two terms: the sensitivity of option price to infinitesimal stock movements, and the average sensitivity to finitely-sized jumps. In a pure-jump Lévy model the first term disappears and the hedge ratio does not involve the derivative of the stock price.

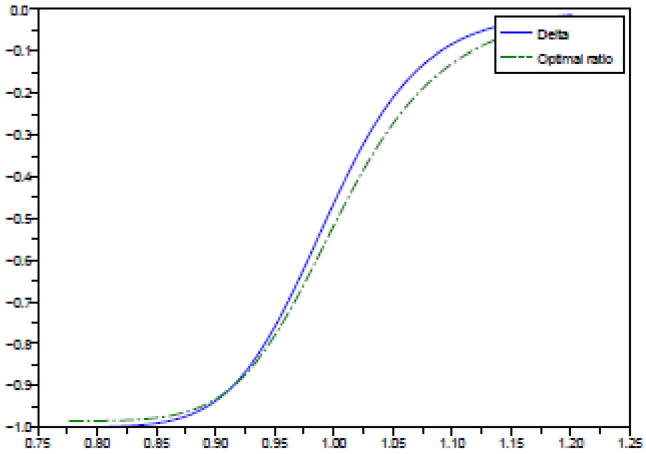

Figure19. Delta-hedging strategy and optimal quadratic hedging strategy ratios as a function of stock price

The figure above shows the difference between the optimal strategy and the delta-hedging strategy. These data were obtained in Merton's jump diffusion model as it is described in Tankov.P [9].

As we see, the two strategies are not so different. The residual hedging errors are also similar. In conclusion,

- Hedging with stock only in presence of jumps eliminates a large part risk but still leads to an important residual hedging error.
- Performances of delta hedging and of the optimal quadratic hedging with stock only are very similar.

To eliminate the remaining hedging error, a possible solution is to introduce liquid options into the hedging portfolio. If, in addition to the stock, the hedging portfolio contains a European option, then the risk due to jumps becomes negligible.

20. Risk management in jump models

In this section, we review an application of Lévy processes to computing risk measures of dynamically managed portfolios (developed in [12]). We are interested in one of the most widely used portfolio insurance strategies: the constant proportion portfolio insurance (CPPI) introduced by Black and Jones [13]) for equity instruments [14] and for fixed income instruments. Under this strategy, the exposure to the risky asset is equal to the constant multiple m>1 of the *cushion,* i.e., the difference between the current portfolio value and the guaranteed amount. In theory, that is, in the Black-Scholes model with continuous trading, this strategy has no downside risk, whereas in the real markets this risk is non-negligible and grows with the multiplier value.

The CPPI strategy *is a self-financing strategy* such that at every moment t, a fraction of the portfolio is invested into the risky asset S_t and the remainder is invested into zero-coupon bond with maturity T and nominal N, whose price is denoted by B_t

- If $\boldsymbol{V_t > B_t}$, the risky asset exposure (amount of money invested into the risky asset) is given by $\boldsymbol{mC_t \equiv m(V_t - B_t)}$, where $\boldsymbol{C_t}$ is the 'cushion' and

  m>1 is a constant multiplier.

- If $\boldsymbol{V_t \le B_t}$, the entire portfolio is invested into the zero-coupon.

We suppose that the price processes for the risky asset S and for the zero-coupon B may be written as

$$\frac{dS_t}{S_{t-}} = dX_t \quad \text{and} \quad \frac{dB_t}{B_t} = rdt$$

where X is a Lévy process with $\Delta X_t > -1$ almost surely.

Let $\tau = \inf\{t: V_t \le B_t\}$. Then, since the CPPI strategy is self-financing, up to time r the portfolio value satisfies

$$dV_t = m(V_{t-} - B_t)\frac{dS_t}{S_{t-}} + \{V_{t-} - m(V_{t-} - B_t)\}\frac{dB_t}{B_t}$$

which can be rewritten as $\frac{dC_t}{C_{t-}} = mdX_t + (1-m)dR_t$

Where we recall that $C_t = V_t - B_t$ denotes the cushion.

Probability of loss

A PCCI-insured portfolio incurs a loss (breaks through the floor) if, for some $t \in [0,T]$, $V_t \le B_t$. The event $V_t \le B_t$ is equivalent to $C_t^* \le 0$.

Proposition 1.

Let X be a Lévy process with Lévy measure v. Then the probability *of going below the floor* is given by

$$\mathbb{P}[\exists t \in [0,T]: V_t \le B_t] = 1 - exp\left(-T\int_{-\infty}^{-1/m} v(dx)\right) \text{(20.1)}$$

21. Application to an exponential Lévy model

In this example we compute the loss probability of a CPPI-insured portfolio supposing that the risky asset follows the Kou's model[15], that is, an exponential Lévy model where the driving Lévy process has a non-zero Gaussian component and a Lévy density of the form

$$v(x) = \frac{\lambda(1-p)}{\eta_+} e^{-x/\eta} 1_{x>0} + \frac{\lambda p}{\eta_-} e^{-|x|/\eta} 1_{x<0}$$

Here, λ is the total intensity of positive and negative jumps, p is the probability that a given jump is negative and η_- and η_+ are the characteristic lengths of respectively negative and positive jumps.

The parameters of Kou's jumps diffusion model were estimated by maximum likelihood for daily time series of French CAC40 index and of the Microsoft Corporation (MSFT) share price. For both series, 10 years of data, from December 1st 1996 to December 1st 2006 were used, making a total of 2500 data points for each series. The jump intensity parameter λ was bounded from above by 250.The estimated parameter values are shown in table1of [9].

For Kou's exponential Lévy model the equation 21.1 for loss probability yields

$$\mathbb{P}[\exists t \in [0,T]: V_t \leq B_t] = 1 - \exp\left(\left(-Tp\lambda\eta_-(1-\frac{1}{m}\right)^{1/\eta_{-\_}}\right)$$

Figure20 shows the dependence of the loss probability on the multiplier for a CPPI portfolios containing Microsoft stocks and CAC40 as risky asset.

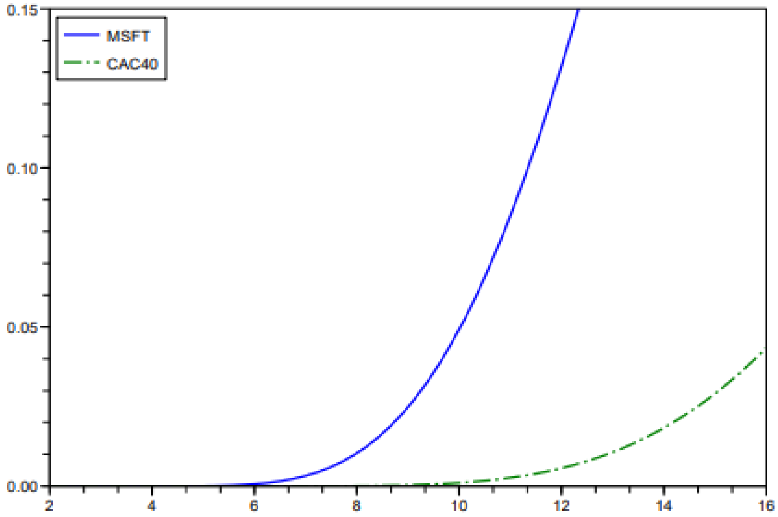

Figure20.Probabilty of loss as a function of the multiplier.

R code for Monte Carlo simulation for various Stochastic processes.

```
#generation of random numbers from uniform distribution
ploter<-function(u,iter)
{v<-vector(length=iter-1)
w<-vector(length=iter-1)
for(i in 1:iter-1)
{v[i]=u[i+1]
  w[i]=u[i]}
plot(v,w)}
lcg<-function(iter,a,x0,m)
  {u<-vector(length=iter)
  u[1]<-x0/m
  for(i in 1:iter)
   {x1<-(a*x0)%%m
  x0<-x1
  u[i+1]<-x1/m}
  hist(u,col="light cyan",breaks=50)
  ploter(u,iter)}
iter=100000
  lcg(iter,16807,1,2^31-1)
  cat("\n")
  lcg(iter,40692,1,21474833399)
  cat("\n")
  lcg(iter,40014,1,2147483563)
```

```
#generation of random numbers from normal distribution
box<-function(mean,variance)
{u1<-runif(5000)
r<--2*log(u1)
u2<-runif(5000)
th<-2*pi*u2
z1<-sqrt(r)*cos(th)
z2<-sqrt(r)*sin(th)
z<-vector(length=10000)
i=1
j=1
while (i<=10000)
{z[i]=z1[j]
i=i+1
z[i]=z2[j]
j=j+1
i=i+1}
z<-z*variance+mean
hist(z,main="box-mueller",breaks=50)
cat(mean(z),"  ",var(z),"\n")}
system.time(box(5,5))
```

```
# Brownian motion simulation
Brownian<-function()
{paths<-10
count<-5000
interval<-5000/count
sample<-matrix(0,nrow=(count+1),ncol=paths)
for(i in 1:paths)
{sample[1,i]<-5
for(j in 2:(count+1))
{sample[j,i]<-sample[j-1,i]+interval*0.06+((interval)^.5)*rnorm(1)*0.3} }
cat("E(W(2)]=",mean(sample[2001, ]),"\n")
cat("E[W(5)]=",mean(sample[5001, ]),"\n")
matplot(sample,main="Brownian",xlab="Time",ylab="Path",type="l")}
Brownian()

#standard brownian motion simulation
StandardBrownian<-function()
{paths<-10
count<-5000
interval<-5/count
sample<-matrix(0,nrow=(count+1),ncol=paths)
for(i in 1:paths)
 { sample[1,i]<-0
for(j in 2:count+1)
{sample[j,i]<-sample[j-1,i]+((interval)^.5)*rnorm(1)}}
cat("E(W(2)]=",mean(sample[2001,]),"\n")
    cat("E(W(5)]=",mean(sample[5001,]),"\n")
        matplot(sample,main="Standard Brownian",xlab="Time",ylab="Path",type="l")}
StandardBrownian()
```

```r
#geometric brownain motion simulation
GeometricBrownian<-function()
{paths<-10
count<-5000
interval<-5/count
mean<-0.06
sigma<-0.3
sample<-matrix(0,nrow=(count+1),ncol=paths)
for (i in 1:paths)
{sample[1,i]<-100
for(j in 2:(count+1))
{sample[j,i]<-sample[j-1,i]*exp(interval*(mean-((sigma)^2)/2)+((interval)^.5)*rnorm(1)*sigma)}}
cat("E[W(2)]=",mean(sample[2001, ]),"\n")
cat("E[W(5)]=",mean(sample[5001, ]),"\n")
 matplot(sample,main="Geometric Brownian",xlab="Time",ylab="Path",type="l")}
GeometricBrownian()
```

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