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Supply Chain Stochastic Modeling: Analysis, Design and Optimization of Flows in Supply Networks

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Synopsis

Scope

In this thesis we propose analytic models for the exact numerical evaluation of production-inventory systems of practical significance. These models are used for an extensive numerical investigation of the respective networks with a view to gain an insight of the systems' behavior, and whenever possible, to draw conclusions of managerial importance. The two main directions of our research are the balance between inventory levels and customer satisfaction on the one hand, and the detrimental effect of uncertainty on system performance on the other.

Contribution

To a great extent the existing literature either uses simplified models that can be solved analytically, or it employs approximate methods with more realistic assumptions. The models proposed here offer exact numerical solutions, while at the same time retaining a realistic level of system complexity. Our approach based on matrix analytic methods combines mathematic clarity, high precision and easily programmable algorithms. The resulting computer programs are powerful evaluative tools. They can be used for a preliminary analysis of real systems, for optimization purposes in the frame of an optimization algorithm, or as evaluative tools for the investigation of general system characteristics. Moreover, our analysis focuses on the effects of variability on system performance, addressing the ever present but hard to model problem of supply and demand uncertainties.

Methodology

Our analysis is based on Markov theory and the systems under consideration are modeled as continuous time – discrete space Markov chains. We apply matrix analytic methods, exploiting the characteristic structure of the infinitesimal generator matrix. We propose an algorithm for the construction of the infinitesimal generator matrix for any given set of parameters and the formulation and solution of the corresponding system of linear equations. The performance measures are calculated algorithmically from the stationary probabilities vector. The validity of each model is confirmed using a simulation model of the respective system. The investigation is based on the numerical evaluation of a wide range of scenarios, while optimal policies are determined through an exhaustive enumeration of all policies within prescribed bounds. The analytic solution algorithm is programmed in Matlab, while simulations are executed with Arena simulation package.

Assumptions

Our models take into consideration both supply and demand uncertainties in the form of stochastic lead times and uncertain external demand. In all models lost sales are assumed for external demand that cannot be met from inventory on hand. To keep a tractable level of complexity we make the common assumption that at most one outstanding order can be in transit between any two different nodes of the system. Other general assumptions include reliable stations, never starving uppermost stations, and zero information lead times.

The stochastic nature of the events is modelled using mainly the exponential distribution. The exponential distribution is known to provide good fit for real systems, it has desirable properties, and it is extensively used in the models found in the literature. To make our models more realistic, whenever appropriate, more general distributions are used. Compound Poisson has been used for the external demand, while phase type distributions have been used for lead times. Compared to real life systems, the exponential distribution may be considered a simplification, but its use does not impair the value of our models since it is a good approximation for many instances and its properties make it appropriate for the general investigation of many system classes.

This thesis addresses three different system configurations. A serial, two stages, push-pull system; a serial, three stages Vendor-Managed-Inventory system; and a three tier, arborescent, pull system. All three systems present modeling challenges, practical interest and are the focus of extensive research literature.

Part 1: Push-Pull system

The scope of our research was the investigation of the interactions between the push and the pull segments of the system. The push-pull boundary is a strategic decision for a supply network and understanding the associated dynamics would be of value during the design phase of a supply chain. The proposed model captures relationships between variables, offers insight on key features of the system at hand, and can be used as a design tool for the evaluation of appropriate systems and the determination of optimal parameter values.

We investigate a linear, horizontally integrated, push-pull system. A production station feeds a finite capacity buffer, which in its turn supplies a retailer working under a continuous review inventory control policy (s, Q). Transportation is modelled independently as a virtual station, while external demand is modelled as a compound poison process.

The model was used to investigate the effect of different policies in balanced systems. Our analysis indicates that higher s values are preferable from a global perspective. When the analysis was based on a cost function, the optimal policies were found to be robust for a wide range of cost parameters values. In general the system has a dynamic behavior, especially as the external demand variability increases; however, under certain conditions some performance measures can be described with good accuracy with simple relations. The effect of demand variability was also studied and its detrimental effect on system operation and performance was documented.

Part 2: VMI system
The aim of our research was to study the interrelations between the vendor and the retailer under different operating conditions, and how these affect overall system performance. We provide a quantitative model and general conclusions that can promote understanding between different members of the supply chain, while the model can serve as a test-bed for coordination mechanisms and new contracts design.

We investigate a three stages, single product, serial inventory system working under the Vendor-Managed-Inventory (VMI) logic. The vendor follows a continuous review inventory control policy (s, Q_2) and decides based on echelon information. Moreover, it keeps track of the retailer's inventory and whenever a reorder point (r) is crossed, a constant quantity replenishment order (Q_1) is dispatched. External customers arrive according to a pure Poisson process and the demand of each external customer follows a discrete empirical distribution. Lead times are modeled using a phase type (Coxian) distribution with two phases.

Systems with different relations between lead times and customer inter-arrival times were tested (balanced, supply constrained and demand constrained). Our investigation highlighted the interrelations between the decision variables and the importance of keeping inventory closer to the end customer. From a managerial point of view, policies with high Q_1 and low Q_2 values, as well as policies with a high value of r in relation to s were found to be preferable. The deleterious effect of increasing demand variance was also documented in a quantitative way.

Part 3: Arborescent system
The aim of our research was to study the dynamics of arborescent networks and offer an evaluative tool that could support the choice of optimal policies. Arborescent networks can be found in practice, but the presence of more than one member in a given echelon increases the complexity of the analysis. Our model offers exact solutions under relatively realistic assumptions and it is used in order to explore the potential for coordination between different network participants, and to investigate the way the network configuration may affect the performance of each separate member.

We study a three-tier pull system of arborescent structure. A Distribution Centre orders from a saturated plant and supplies a Wholesaler. In its turn, the Wholesaler supplies *n* independent retailers. All members follow a continuous review inventory control policy based on local information. Transportation processes are modeled as virtual stations and partial orders are allowed. External demand is modeled as a pure Poisson process, while lead times are assumed to be exponentially distributed.

The model was used to investigate the effect of the decision variables on the performance measures. Our analysis indicates a dynamic behaviour. There is inter-dependence between the different members of the network and interplay between system parameters. Our analysis suggests that it is possible to coordinate the system in the sense that for a given desirable service level there can be found a combination of policies that minimizes total inventory in the system. As in many instances local optima were observed, the fine tuning of the system was found to be beneficial. The problem of optimal number of retailers was also briefly addressed.

Publications related to this thesis (as of August 2020)

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But when he has done this, let him not say that he knows better than his master, for he only holds a candle in sunshine.
W.Blake

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Contents

1. Research motivation

Logistics is a vital activity of the economic life. For smooth production processes and so that products of acceptable quality are available to the customers, raw materials must be procured and transferred to the manufacturing centers; intermediate products should be produced, stored and transported; and final products must be manufactured in a planned fashion and then moved through several stages (distribution centers, wholesalers, retailers) to the points where they will be available to the end customers. To coordinate all these processes so that they take place in a timely manner and in a cost effective way demands a considerable effort from the associated businesses and puts a severe strain on their resources.

From a macroeconomic point of view, logistics can be of strategic importance for a national economy (Karavias and Anastasatos, 2018). It promotes economic growth (Hayaloglu, 2015), it contributes significantly to the National Gross Product of both developed and developing countries, and it employs a considerable part of the workforce (Rushton et al., 2014).

With regard to individual consumers, logistics contributes significantly to the place and time aspects of product utility, as it is perceived by end customers (Murphy & Knemeyer, 2018). Moreover, it enables and supports new retail channels and it is a factor contributing to the expansion of consumer options.

At business level, effective logistics is a prerequisite for most operations. Many companies have gone a step further and seek to gain a competitive advantage through their logistics functions. Supply Chain Management encompasses and expands logistics concepts. It adopts a more holistic approach that cuts through the boundaries of different organizations, and the resulting coordination between different stages has allowed the successful implementation of elaborate schemes such as Just in time, Lean management and Vendor managed Inventory. Companies are able to cut costs, increase revenue, improve quality, and release resources for investment, by making their logistics functions more efficient and more effective. As a result, supply chain management is the object of extensive research and a focus of interest for both academics and practitioners.

Although the key components of logistics management (transport, inventory, warehousing) are at least as old as the modern mode of industrial production, the field of supply chain management is an evolving one. Advances in technology, organizational changes, and changes in the competition, all have an impact on supply chain operations. Globalization has given rise to longer, more complex and more susceptible to risk supply chains. Meeting rising customer expectations, both in terms of quality and availability, necessitates integration and coordination between different stages and challenges the established operating methods. Advances in information technology and the use of the internet facilitate information sharing and allow for novel modes of cooperation. Intense competition, as well as sustainability issues, compels companies to eliminate waste and rethink packaging and transportation issues. Most important of all, the assignment of strategic importance to Supply Chain Management calls for improved effectiveness and higher efficiency, and creates an intense drive for innovation and change.

In such an environment it is imperative to gain a deep understanding of the systems under consideration, to predict their behavior with changing conditions, and evaluate the effect of the uncertainty ingrained in all inventory networks. Towards these ends, the application of theoretic models can be useful. Although such models are abstractions of the real systems, by focusing on key features they bring forth the basic relations between quantities of interest and allow us an insight of the deepest workings of the systems despite their complexity. With theoretic models we can improve our control over systems by understanding their dynamic nature; we can promote supply chain integration by exposing the interconnection of supply chain members and the advantages of closer cooperation; we can also facilitate change and the adoption of novel approaches to supply chain management by exhibiting in a quantitative way the potential benefits for the companies.

In this thesis we present three different stochastic models of inventory networks working under a lost sales assumption. Our study is based on Markov theory and the matrix analytic approach. First a short introduction to the related theory is given, and then the presentation of each model follows. The solution for each model is given in detail, while the resulting algorithm is programmed in Matlab and then used to numerically investigate the system so that conclusions of managerial interest can be drawn.

In the next chapter we discuss briefly some key elements of modeling and we make a short introduction to management science and its methodology.

In chapter 3 some basic concepts of supply chain management are defined. Mention is made to the factors that affect supply chain performance and commonly used supply network metrics are given. The role of inventory in supply chains and the basic inventory control policies are also briefly discussed.

In chapter 4 the theory behind our methodology is presented. Basic definitions are given and elements of Markov Chains theory are briefly discussed. The outline of our methodology is also given.

In chapter 5 we present a model of a linear, horizontally integrated, push-pull system. A production station feeds a finite capacity buffer, which in its turn supplies a retailer working under a continuous review inventory control policy. Transportation is modelled independently as a virtual station, while external demand is modelled as a compound poison process. The focus of our analysis is the interactions between the push and the pull segment of the system. The push-pull boundary is a strategic decision for a supply network, and understanding the associated dynamics would be of value during the design phase of a supply chain.

In chapter 6 we investigate a three stages, single product, serial inventory system working under the Vendor-Managed-Inventory (VMI) logic. The vendor follows a continuous review inventory control policy and decides based on echelon information. External customers arrive according to a Poisson process and the demand of each external customer follows an empirical distribution. To better capture transportation uncertainties, a phase type distribution is used for lead times. A VMI venture means developing a strategic partnership between different supply network members, and the proposed model can help clarify the new role of each part and quantify the expected overall benefits.

In chapter 7 a three-tier pull system of arborescent structure is studied. A Distribution Centre orders from a saturated plant and supplies a Wholesaler. In its turn, the Wholesaler supplies *n* independent retailers. All members follow a continuous review inventory control policy based on local information. Transportation processes are modeled as virtual stations and partial orders are allowed. External demand is modeled as a pure Poisson process, while lead times are assumed to be exponentially distributed. The model is used in order to explore the potential for coordination between different network participants, and to investigate the way the network configuration may affect the performance of each seperate member.

Finally, in chapter 8 we sum up our research. Some general conclusions about the investigated systems are drawn, and some directions of possible future research are given.

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2. Introduction to Models and Management

2.1 On Models

The modeling process is a fundamental function of the way we understand the world around us. Implicitly or explicitly, we continuously build models and employ them as tools to interpret the phenomena that confront us and define actions that we regard beneficial to our motives and purposes. When it comes to science, models form an integral part of the scientific method, creating and using them is a standard part of problem solving in science (Nersessian, 2006), while in certain fields models may be employed even in the absence of a related theory.

Despite the ubiquity of models, or perhaps because of it, several important aspects of them are still subject to debate. Amongst others, the use of models raises semantic, ontological and epistemological questions, and several schools of thought have been developed to address each one of them (Frigg and Hartmann, 2012).

Semantic questions are concerned with the representational function of models (what do they represent). Models may be representations of a selected part of the world, or target system. As such they may be models of phenomena, or models of data (Frigg and Hartmann, 2012). Models of phenomena are based on some kind of isomorphism with the target system, or on more relaxed terms, on some similarity or analogy with it. One issue that must be pointed out here is the fact that models cannot capture all the aspects of the target system, but focus only on those features that are of concern in the particular investigation. Even then, an idealization, or approximation process may be needed in order to build a working model. Such simplifications do not necessarily compromise the value of the resulting model as long as the assumptions that are made are consistent with the purposes of the analysis. In fact, under certain circumstances, these "simplified" models can be more useful than more detailed and less idealized ones (Batterman, 2009).

On the other hand, models of data describe data that have been gathered during experiments. Their development usually includes a first step of data reduction and then a second step of some kind of curve fitting (Harris 2003). Data models may incorporate elements of theory, or they may even exist without an underlying theory where the mere analysis of data allows us to make reliable and useful predictions about the world (Napoletani et al, 2011).

The nature of models is the second fundamental question that must be addressed. Physical objects or material models, such as scale models, are a straightforward and well defined category. However, for models involving more abstraction the process of defining what exactly is a model becomes more confused. Several points of view have been proposed. Models have been treated as a variety of entities including as fictional objects, set-theoretic structures, descriptions, equations, or even mixtures of different classes of objects (Frigg and Hartmann, 2012). A special mention must be made to mathematical modeling. Mathematical models are idealized, abstract models based on a number of different mathematical techniques which are deployed as suits the specific problem at hand (Giere, 1999).

A final question about models is how we can use them to gain knowledge about the world. It is an indisputable fact that much of scientific (and non-scientific) inquiry is done on models and not on real systems. The process of learning through models has three stages (Frigg and Hartmann, 2012). First the model is constructed and its relation with the real system is established. Then, the characteristics and the behavior of the model are explored through its manipulation. The term manipulation is not strictly defined and its exact meaning obviously depends on the specific model under consideration. With regard to mathematical models, manipulation could entail the analytic solution of equations, the gathering of numerical results, or where intractable systems are concerned, the execution of computer simulations. In a final step, the knowledge about the model has to be translated back into conclusions about the real system under investigation.

Closing the introduction on models, we must make mention to the specific functions of models and refer to why we model. Models are flexible in regard with the level of complexity in our analysis. They allow experimentation where high costs, accessibility issues, or ethical reasons forbid direct analysis of the real system. They are important tools for discovery allowing researchers to experiment with different variables, to test the behavior of a system under different conditions, and to explore the relationships between its various features (Truran, 2013). Prediction is the most obvious motive for such investigations, but it is not the only one. Epstein (2008) cites sixteen reasons other than prediction of why we build models, including amongst others to explain, to guide data collection, to discover new questions, to demonstrate tradeoffs and to educate.

2.2 Models as a Management Tool

Managing an organization is about planning, organizing, leading, and controlling of resources with a view to achieving the organizational goals efficiently (in terms of costs) and effectively (in terms of results) (Jones & George, 2015). Inevitably, the management of any complex system requires sound decision making at different levels, from the conception of an idea, to the planning of its execution, up to the details of its realization in a day to day horizon. Decisions fall into three broad categories depending on time scale, scope and tolerance of error. Strategic decisions have wide scope, a long-term impact, and they require substantial investment and usually a long term commitment on the part of the organization. The tactical level is in-between strategic and operational levels. Tactical decisions have a shorter time horizon, a more restricted scope, while they offer some flexibility in case corrections are required. Finally, operational decisions have local scope, short term impact and are relatively easy to adjust in case of error.

Management science is the discipline that attempts to aid managerial decision making by applying a scientific approach to managerial problems that involve quantitative factors (Hillier and Hillier, 2014). Although the first systematic approaches to management can be traced as far back as the work of Adam Smith in the eighteenth century, the consensus holds the birth of scientific management in the work of F.W. Taylor at the beginning of the twentieth century. Management Science in its present form originates from the second World War and the multi-disciplinary teams established by the British military to cope with the complex problems arising in the operational theatres of the war. The alternative term "Operations research" is a legacy of this period. After the war, management science techniques found their way in the field of business management, while their proven effectiveness in terms of increased efficiency and productivity, led to an ever-expanding body of research. A major boost was the development and propagation of computers which allowed for the introduction of more sophisticated techniques based of iterative algorithms that took advantage of the increasing computational power.

Management science is based upon organized principles of knowledge and a systematic analysis of empirical data so that repeatable results can be obtained (Kumar and Suresh, 2009). In the core of the scientific approach to management is the development of mathematical models that identify the variables of concern and their relations (Boddy, 2017), and allow the decision makers to establish the relationships between the actions they may take and the results they might expect (Evans, 2016). The employment of models for managerial purposes is cost effective, it facilitates a better understanding of the system under investigation, it promotes the quantification of the problem and the application of rigorous mathematical techniques, and it offers managers a way to evaluate what-if scenarios in a standardized manner (Stevenson, 2018).

On the other hand, it must always be kept in mind that models do not decide, but are tools that support decision making, and as such, their employment must always be subject to the critical evaluation of the decision maker. By their very nature, models do not capture all the aspects of a problem, especially when qualitative factors are concerned, nor the results are always unambiguous. The human factor is critical, both during the development of the model and during the interpretation of the end-results.

Of major importance are the model inputs, which parameters of the system will be included, as well as the quality of the information about them. For most models, inputs fall into three categories: Data, Uncontrollable variables, and Decision variables (Evans, 2016). Data include elements such as costs and capacities that can, with relative safety, be assumed to be constant in the context of the model. Uncontrollable variables are quantities that do change, but their change is beyond the control of the decision maker. Product demand is a typical example of such a quantity. Finally, decision variables are those parameters that can be directly influenced by the organization and whose optimal value is a usual objective of the investigation. Inventory policy parameters and staffing levels are examples of this kind of input.

When all model parameters are assumed to be known with certainty, the model is characterized as deterministic. Deterministic models can be useful, but in many cases an oversimplification of the problem is required. In contrast, stochastic models use elements of probability to take into consideration the uncertainty, or randomness, that is ingrained into many system parameters. Stochastic models are harder to build and solve, but they are significantly closer to real systems.

In any case the appropriate level of model complexity depends on the specific objectives of the research. A good model is a finely tuned compromise between the conflicting objectives of the modeling process. It must be simple and easily understood, but at the same time it must include all the significant aspects of the problem at hand. It must be flexible and adaptive to reasonable changes of the inputs, while it should be easy to use and offer a friendly interface to the user. Moreover, as the model will be used to support decision making, its results must offer insights of the problem and have a direct bearing on the decision process, while they must be available at the lowest possible cost and in a time-frame appropriate for decision making (Daellenbach, 2005).

2.3 Methodology of Operations Research

A successful Operational Research study requires flexibility and critical judgment on the part of the modeler. In practical research each problem is unique and its specific traits will dictate the appropriate solution approach, so a detailed prescription valid for every case cannot be given. The basic steps are generally agreed upon, but different researchers put emphasis on different aspects of the proposed method. A succession of five steps, reflecting the systematic approach to problem solving, can be mentioned as a general guideline of Operational Research methodology (Taha, 2017). The sequence is not rigid, but overlapping and loops between the steps usually occur.

2.3.1 Definition of the problem

Every study starts with the recognition and description of a problem, risk, or opportunity. The detection is based on a systematic observation of the system and draws on the experience and knowledge of both managers and researchers (Taylor, 2016). At this stage the problem (in a broad sense of the term) must be delimited and concisely defined, the relevant variables must be identified, and the proper level of detail should be decided. The appropriate objectives of the research are set, the feasible alternative courses of action are determined, and the constraints on the decision choices are specified. The alternatives could be in the form of a list of options, or a set of decision variables, while the necessary trade-offs between conflicting objectives must be taken into account (Daellenbach, 2005). The gathering of relevant data also occurs at this stage, a process that can be surprisingly challenging and time consuming, especially when too little or too many data are available (Hillier, 2014).

2.3.2 Construction of the model

The model translates the problem definition into mathematical relationships (Taha, 2017). It should capture the essence of the problem, abstracting relevant variables from the real system in a way appropriate to the objectives of the research (Kumar, 2009). The exact formulation of the model may fit some of the standard Operational Research techniques, or else a special-purpose model may be needed. It should be noted that the complexity of most real life situations means that there is no single correct model, but there exist several alternative approaches for the solution of the same problem. Moreover, the modeling process is an evolutionary process where the model becomes more complete as the knowledge and the understanding of the modeler increases (Hillier, 2014).

2.3.3 Solution of the model

The solution of the model is considered the simplest phase of an operational research study since a wide range of well established solution techniques is available. The technique of choice depends on the nature of the developed model, as well as possible time and cost limitations. In simple cases a closed-form mathematical expression may be obtained, but in most cases the solution involves a systematic procedure of successive steps (algorithm) (Evans, 2016). Since for an iterative procedure extensive calculations may be needed, the application of the algorithm is usually done with a computer. An important issue about the solution of the model is sensitivity analysis. Given the unavoidable simplifications of the modeling process and the uncertainty that characterize many of the system parameters, it is important to evaluate the robustness of the proposed solution before any decision is implemented (Taha, 2017).

2.3.4 Validation of the model

Before reaching any conclusion about the system, the validity of the developed model must be established. First the internal validity of the model must be tested (verification). Verification addresses the internal consistency and the logical coherence of the model (Truran, 2013). Its aim is to verify that the assumed relationships are correctly represented by the respective mathematical expressions and that these mathematical expressions have been correctly implemented in the computer algorithm (Daellenbach, 2005). The manual checking of numerical results and the confirmation of the dimensional consistency of the model values are two methods that can be applied at this stage.

On the other hand, validation or external validity seeks to establish that the model provides reliable information about the real system as it was intended to do. It should be ascertained that the model is a sufficiently accurate representation of the problem, given the objectives of our study. At a first stage, the behavior of the system with changing inputs can be investigated and any unreasonable solutions, or counter-intuitive results be evaluated. For a more thorough validation procedure, if possible, the results of the model are compared to historical data concerning the system. In the absence of such data, simulation may be used as an independent tool for validating the output of the mathematical model (Taha, 2017).

2.3.5 Implementation of the solution

By manipulating the model we can generate and evaluate alternative solutions to the problem. Then, using an optimization algorithm, or by exploring the set of feasible solutions (solution space), the most appropriate solution can be determined. In a last (but not final) step the results must be applied in the real world. Implementation of the solution involves the translation of the results into understandable operating instructions to guide those who administer the actual system (Taha, 2017). The realization of the proposed changes should be audited and the actual results monitored. The proper use of the solution must be ensured, any deviations should be timely detected and corrected, and any opportunities for further performance enhancement should be exploited. Attention should be paid so that changes in the organization and its environment will not invalidate the appropriateness of the proposed solution, while it must always be kept in mind that improvement is a continual process and that no solution, however successful, should be considered final.

2.4 Operations Research in Practice

The effectiveness of operations research techniques in improving efficiency and productivity has been proven by a long string of documented success stories. As a result, over the years substantial effort has been expended to further the knowledge and develop the practice of management science. Today Operational Research is a body of well established models and techniques, comprising a discipline in itself (Ravi Ravindran, 2008). Its most frequently used tools include linear programming, integer programming, network models (including supply chain models), and simulation (Anderson et al., 2019). Inventory models, waiting-line and queuing models, and Markov Process models are also part of the standard operational research arsenal.

Management science is applied in a variety of fields, including the military (logistical planning, war gaming), the government (emergency services, policy testing), and the area of healthcare (healthcare delivery, disease modeling) (Hillier, 2014). With regard to business and industry, the techniques of operational research can be employed in practically any decision involving quantitative factors. As examples we can mention financial services, human resources management, production planning and control, marketing, inventory control, and distribution network design.

In this thesis we have used Markov chains to analyze inventory networks and evaluate their performance under different working conditions. Some basic elements of inventory networks will be covered in chapter 3, while a brief presentation of Markovian theory will be made in chapter 4.

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3. Supply Networks

3.1 Definitions

Logistics management can be defined as the set of activities that "plans, implements, and controls the efficient, effective, forward and reverse flow and storage of goods, services and related information between the point of origin and the point of consumption in order to meet customers' requirements" (CSCMP, 2013). Typically, logistics refers to activities that occur within the boundaries of a single organization and focuses on procedures such as procurement, distribution, maintenance, and inventory management (Hugos, 2011).

On the other hand, the terms "supply chain" and "supply chain management" have a wider scope. A supply chains extends from raw materials to the end products being available to consumers and includes all the parties that directly, or indirectly are involved in fulfilling a customer request (Chopra, 2016). A supply chain can be viewed as a "combination of processes, functions, activities, relationships, and pathways along which products, services, information, and financial transactions move in and between enterprises from original producer to ultimate end-user or consumer" (Murphy & Knemeyer, 2018). In the context of a supply chain each company focuses on its core competences and trusts in close cooperation and operative integration with the other members of the chain, so that the whole network can work as a single efficient and effective organization.

Supply chain management integrates the logistics with other functional areas of an organization such as marketing, new product development, finance, and customer service. Compared to traditional logistics management Rushton et al. (2014) cites four basic differences: Supply chain management integrates suppliers and end users and views the whole supply chain and the constituent organizations as a single entity; its scope is mostly strategic than operational; it adopts a different view about inventory, treating it as a last resort to balance an integrated flow rather than as a means of decoupling successive member of the supply chain; and finally, decisions are based on information about the whole supply chain accessed through integrated information systems rather than having each member acting independently and with limited knowledge.

The objective of supply chain management is to maximize the total value created by the supply chain. This is related to the profitability of the supply chain and can be defined as the difference between the revenue from end customers and the total cost incurred in the process of meeting customers' requests. The appropriate configuration of the supply chain must take into account the characteristics of the product, as well as customers' expectations (Huang, 2013). The right balance between the five operations performance objectives of quality, speed, dependability, flexibility and cost must be found (Slack et al., 2007), and the supply chain must be managed accordingly. Decisions must be made with a view to global optimization and they must address the issues of efficient asset management and successful coordination of material, information and finance flows through the supply chain. Goods in various forms flow mainly downstream from the supplier towards the end consumer, but upstream flow of product is also possible (reverse logistics). Funds flow upstream, starting from the end customer who is the main source of revenue, and financing the successive supply chain members as they move. The flow of information is more complex as there must be an exchange of crucial information amongst all the members of the supply chain as appropriate (Taha, 2017).

3.2 Participants in the Supply chain

The exact configuration of a supply chain depends on both the product and the market that is targeted. A successfully operating supply chain consists of a combination of companies performing different functions in close cooperation with each other. The central role is played by the primary participants in a logistics channel: producers, distributors, wholesalers, retailers and of course end customers. In addition to them there are facilitators, or channel intermediaries, that play minor but essential roles (Murphy, 2018). These are usually service providers that support and smooth out supply chain operations, and occasionally make up for the inefficiencies in supply chain integration. An example of an extended supply chain is given in figure 3.1

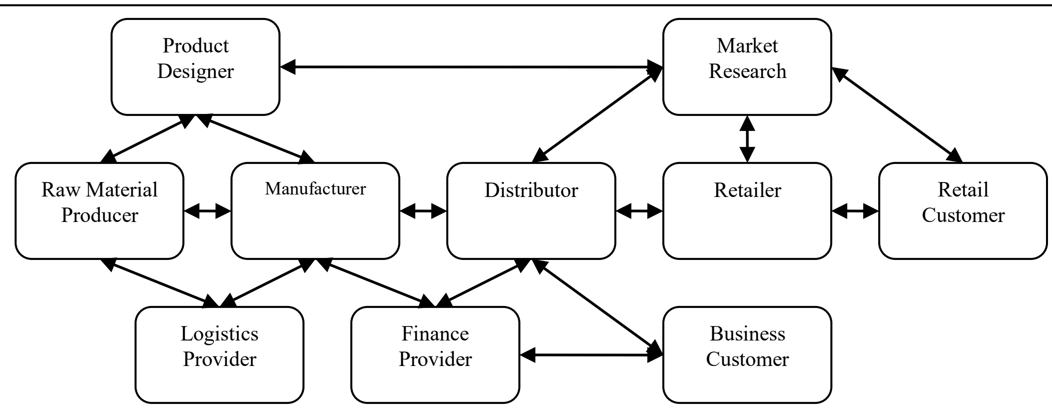

Figure 3.1: Example of an extended Supply chain (Hugos, 2011)

With regard to material flow, there are three basic network structures: Serial systems, assembly networks, and distribution systems (Figure 3.2). In serial systems every facility has at most one upstream supplier and one downstream customer and only one participant exists at each stage (echelon). It is the simplest configuration and it can be considered as a special case of both assembly and distribution systems. In assembly networks, each installation has at most one immediate successor. Each downstream facility is supplied by multiple suppliers which in turn have multiple suppliers themselves. Such systems are common in process industries, and at the end of the production chain in mechanical industries. Finally, distribution systems are systems where upstream facilities supply multiple downstream customers. If each facility has at most one supplier and many possible customers, the term arborescent distribution network is used. (Axsäter, 2015; Azadivar & Rangarajan, 2008). A special case of material flow is lateral supply, where a facility may receive products from another facility at the same stage. Such arrangements can usually be found in multi-location retailers.

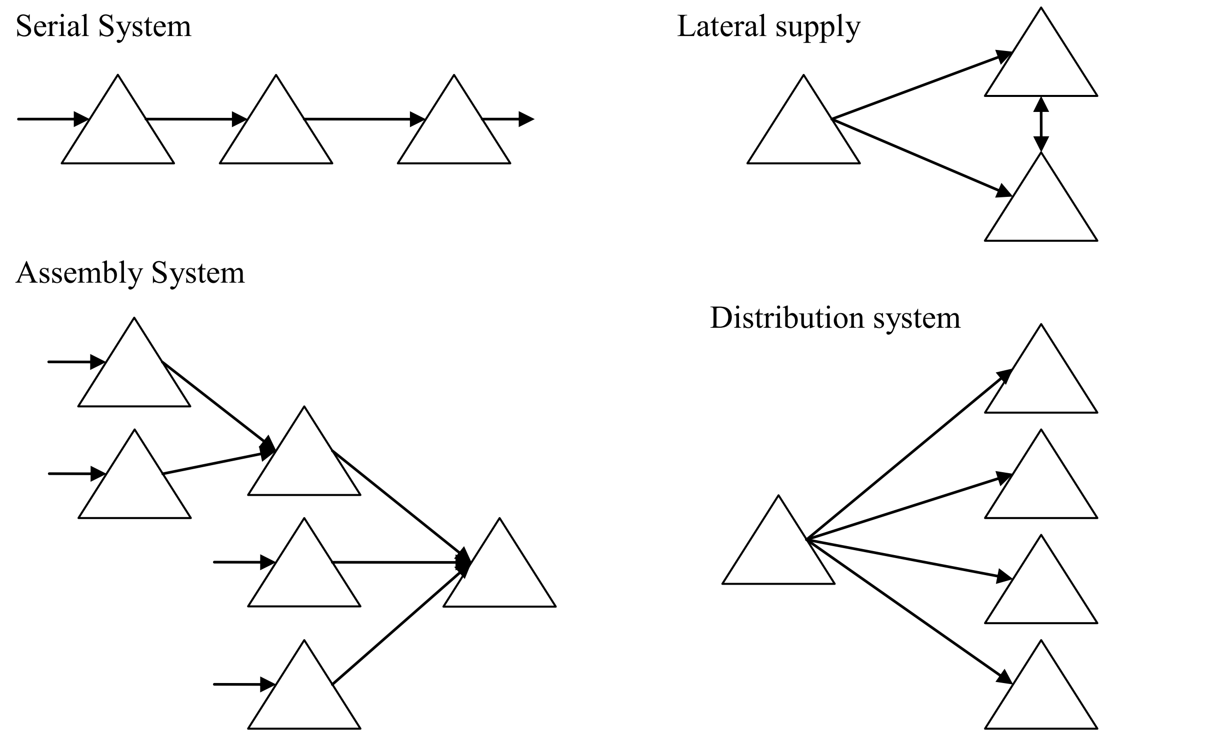

Figure 3.2: Common supply network structures (Adapted from Azadivar & Rangarajan, 2008)

Below, the various participants of a supply chain are briefly discussed.

Producers: Producers or manufacturers are those organizations that make a product in the form of raw material, intermediate product, or finished goods. The product may be intangible, or it may have the form of a service.

Distributors: Distributors or wholesalers receive products in bulk from the producers and distribute them to other businesses typically in larger quantities than an individual consumer would demand. Their operations may include inventory management, warehousing, product transportation, as well as customer support (Hugos, 2011). A distribution centre may consolidate freight that comes in small shipments, it may break large shipments into smaller ones, or it may just re-sort goods and forward them immediately for transportation to customers (cross-docking) (Schroeder and Goldstein, 2016).

Retailers: Retailers hold inventory and sell to the general public. As they interface with the end customers, they play a central role in defining demand characteristics and customer requirements.

Customers: Customers are those organizations or individual consumers that purchase and use the product. The customers are the main source of revenue for the supply chain and to meet their requirements is one of its main objectives. Customer demand is affected by the type of the product, the current phase of its life-cycle and of course the competition. It may be static or it may vary with time and in many cases it is a significant source of uncertainty for the supply chain.

Service providers: These organizations can be found across the supply chain. They have specialized in and are able to provide effectively and at low cost functions supportive to the core supply chain operations. Typical services offered include amongst others transportation, warehousing, financial services, market research and advertising, product design, legal services, and information technology and data collection services (Hugos, 2011). The degree that they participate in a supply network depends on the individual characteristics of each chain.

3.3 Drivers of performance

A successful supply chain must be effective in meeting customer requirements and simultaneously its members must operate efficiently, at the lowest cost for a given level of performance. In many cases conflicting needs arise and supply chain management involves much trading-off so that the resulting policies are consistent with the competitive strategy of the chain. The overall objective is to find the appropriate balance between efficiency and responsiveness. Chopra & Meindl (2016) cite six drivers on whose interactions the overall performance of the supply chain is dependent: Facilities, inventory, transportation, information, sourcing, and pricing.

Facilities
Facilities concern the physical locations in the supply network where goods are produced, processed, or stored. The level of centralization must be decided, the location of each facility must be determined and its role defined. For production sites, the degree of specialization must be decided, as well as if they will be focused on products or on specific functions. In a similar fashion, an appropriate warehousing approach should also be defined (Stock Keeping Unit - SKU storage, job lot storage, cross-docking) (Hugos, 2011). In any case, capacity issues must be addressed. Excess capacity offers flexibility to demand fluctuations, but it is detrimental to efficiency, so a suitable balance must be found.

Inventory
Inventory in the supply chain can take many forms (raw materials, work in process, finished goods) and some kind of inventory is kept practically by all supply chain members. Decisions

that must be made concern what kind of inventory must be kept, at which point of the supply chain, and at what levels. Inventory policies define when to reorder and how much to reorder, and they are commonly used to affect the efficiency-responsiveness balance. As inventory management has a central role in this thesis, inventory issues will be covered in greater detail in the next section.

Transportation

Transportation relates to how inventory should be moved from one location of the supply chain to another. Transportation costs can be as much as a third of the operating cost of a supply chain (Hugos, 2011), so careful and informed decisions are necessary. The mode of transportation must be decided and appropriate routes must be designed. The transportation to the point of demand can be direct, or intermediate consolidation points may be used (Chopra & Meindl, 2016).

Information

Information entails the collection, analysis and dissemination of data concerning facilities, inventory, transportation, costs, prices, and customers throughout the supply chain. Information provides the link between different supply chain members and it is the basis on which decisions about the other performance drivers are made. It enables the coordination of the supply chain and the optimization of its performance, while it is the only driver that can improve both efficiency and responsiveness simultaneously. Appropriate information is based on data that is accurate, timely and complete and offers a wide range of advantages: It helps to reduce variability in the supply chain; in enables better forecasts; it allows for the coordination of manufacturing and distribution systems; it promotes improved customer satisfaction; it enables a faster reaction and adaptation to supply problems; and it enables lead time reductions (Levi, Kaminsky & Levi, 2004).

Sourcing

Sourcing refers to the processes required for the acquisition of the inputs that are necessary for the realization of the product. Outsourcing decisions also fall into this category. For each purchase the company must decide on the number of suppliers and the contribution of each one. Selection criteria must be set, as well as processes for performance monitoring and evaluation. The procurement process should improve efficiency and coordination in the supply chain and the necessary trading-offs must be made with concern to both short-term (quality, responsiveness, cost) and long-term (innovation, commitment, risk sharing) factors. (Chopra & Meindl, 2016; Hugos, 2011; Huang, 2013)

Pricing

Pricing is the process by which a firm decides how much to charge customers for its goods and services (Chopra & Meindl, 2016). Price is an important aspect of the competitive strategy of a company, it usually correlates strongly with demand, and it affects the perceived customer satisfaction. Price is also one of the main ways to match supply and demand on a short term basis. If economies of scale occur, these must be recognized and price incentives must be given to customers so that an optimal output can be achieved.

3.4 The role of inventory

Inventory bridges the distance, in terms of both space and time, between different supply chain functions. All organizations, no matter the nature of their operations and their position in the supply chain, are forced to keep some kind of inventory. Even organizations specifically designed to work without inventories (Just-in-time philosophy) will in practice hold some, though minimal, stock.

The principal reasons behind keeping inventory are the gap between supply and demand, the uncertainties inherent in most supply chain functions, and the drive for cost reduction.

The perfect coordination of supply and demand is usually unfeasible and in many cases economically unwanted, so keeping inventory is necessary as a buffer for a fast response to customer demand, or as a way to accommodate demand variations. Inventories are indispensable in markets where the products must be available in anticipation of demand, as well as where the supply or the demand of a product has seasonal characteristics.

Uncertainty concerns both ends of an operation. With regard to demand uncertainty, higher inventory levels can be used to avoid costly stock-outs and loss of sales and revenue. As far as supply uncertainty is concerned, inventory decouples successive operations and permits smooth production and distribution runs without delays due to raw material shortages. For critical raw materials, strategic inventory may be needed as a precaution against unforeseen events.

Finally, economic motives for keeping inventory may relate to economies of scale or operational reasons. Larger replenishment orders and longer production runs are often associated with lower cost per unit due to inelastic re-order and set-up costs, while suppliers' discount policies and price fluctuations may also favor stock accumulation. Beyond these, inventory can facilitate higher utilizations of a company's resources. It upkeeps the independence of operations and it can be used to avert blocking or stock-out at potential bottlenecks in a process. It also allows for more flexible scheduling and the keeping of a given output with minimal capacity. (Rushton et al., 2014; Azadivar & Rangarajan, 2008; Schroeder & Goldstein, 2016; Huang, 2013)

On the downside, excessive inventory can have a negative impact on an organization and its supply chain. Keeping inventory has associated costs and entails certain risks (section 3.4.2). Inventory is not productive in an economic sense and it ties up capital that could be used in a more efficient way. Although it hedges against uncertainties, at the same time it obscures system inefficiencies and obstructs the detection of problems and the initiation of corrective actions.

Finally, positioned between different members of a supply chain, it can insulate organizations, hampering effective communication and supply chain integration.

3.4.1 Types of inventory

Inventory in an organization may take various forms. Raw material, components, and packaging material are necessary inputs for manufacturing operations. Finished products are held at the end of the production lines, or at warehouses waiting transportation to customers. Semi-processed stock may build up between different manufacturing processes. Moreover, general stores and consumables used to support operations, as well as spare parts for critical infrastructure, may also have to be kept. The specific purpose of each kind of inventory designates the appropriate criteria for the associated decision making. So it useful to categorize inventory according to the characteristic role it plays for the functions of a company or a supply chain (Rushton et al.,2014; Sürie & Wagner, 2008) :

Cycle Stock or Production lot-sizing stock: It is the inventory necessary for the satisfaction of normal demand between successive replenishment orders or production runs. The exact level of cycle stock may be decided for economic reasons, or it may be imposed by technical issues such as the availability of transportation, or the use of the same infrastructure for different products.

Inventory in transit or Pipeline inventory: It refers to products that are en route between different facilities of the supply network. These products are not available to sell or use and their ownership is usually defined in the selling contract between supplier and buyer. Pipeline inventory level is a function of transportation time and demand, and it is generally independent of the replenishment orders quantity, or the frequency of reordering (Sürie & Wagner, 2008).

Safety Stock or Buffer stock: It is kept to protect against the uncertainty stemming from unknown demand, unforeseen production disruptions and uncertain supplier lead times. Decisions about safety stock must take into account the stochastic nature of supply networks. Its appropriate levels depend on the desirable customer service level and the cost of an associated stock-out.

Seasonal stock: It concerns goods whose demand has seasonal characteristics. It is used to satisfy expected large increases in demand and to compensate for periods when demand exceeds production capacity. Its accumulation helps to avoid working overtime costs and costs associated with unused equipment.

Work in Process: It refers to products undergoing treatment, or waiting for treatment between the different stages of a production process. It prevents the starvation of bottleneck machines and allows for a smoother production flow and a higher utilization of resources. Work in Process relates to process Throughput (products finished per unit of time) and total time in the system (Cycle time) according to Little's Law: Throughput = Work in Process / Cycle time.

Speculative inventory: It is inventory kept so that the company may guard against and profit from uncertain contingencies such as projected price increases and potential shortages of raw materials. The strategic inventory of critical products may also fall into this category.

Phychic Stock: It refers to inventory kept in order to stimulate rather than satisfy demand. It is associated with retail stores and its levels have more to do with marketing than inventory management (Murphy & Knemeyer, 2018).

Dead Stock: It is inventory that is no longer in demand. Such inventory is a liability for a company and may entail significant economic loss (Abbasi, 2011).

3.4.2 Inventory costs

Beyond its operational significance, inventory is also important from an economic point of view. A typical firm can have up to 30 percent of its current assets and as much as 90 percent of its working capital invested in inventory (Stevenson, 2018). Moreover, holding inventory entails significant costs that can be as high as 35 percent of its value (Jacobs & Chase, 2018). Current trends such as reduced product life cycles, product proliferation, rising customer expectations, demand volatility, extended supply chains, and just-in-time responsibilities tend to increase these costs (Rushton et al., 2014) and bring into focus the need for better inventory management. Inventory associated costs that must be taken into consideration during decision making include the item cost, the ordering cost, the setup cost, the holding cost, and the stock-out cost (Chopra & Meindl, 2016; Schroeder & Goldstein, 2016; Jacobs & Chase, 2018; Rushton et al., 2014).

Item cost is the cost paid to a supplier to buy an item of a specific product (purchase cost), or in the case of production, the cost of producing an item of the product. It is usually the largest of the inventory costs. Purchase cost may vary with the order quantity as many suppliers offer quantity-related discounts.

Ordering cost is the cost incurred when placing a purchasing order. It is generally independent of the order size and includes all the managerial and clerical costs. Amongst other it may account for buyer time, transportation costs and receiving costs.

Setup cost is the cost of preparing the equipment for a production run in the case where the company produces its own inventory. It is analogous to the ordering cost and it refers to the production batch as a whole, independently of its specific size. Setup cost may include managerial costs as well, for example costs associated with creating a production order and record keeping.

Holding or carrying cost is the cost incurred by physically keeping items in storage. It is the sum of four principal components: the cost of capital being tied up in inventories; service costs associated with stock management and insurance; storage costs accounting for space occupancy, handling costs, and warehousing costs; and risk costs associated with obsolescence, deterioration, damage, or theft. The importance of each component depends on the specific characteristics of the stocked products, amongst others, the cost of their acquisition, their self life, and the need for specialized storage conditions. Holding cost is usually expressed as a percentage of the item cost per year. Typically values between 15 and 30 percent per year are quoted (Schroeder & Goldstein, 2016), but higher figures can also be found (Stevenson, 2018).

Stock-out cost is the result of not having enough inventory to meet customer demand. Amongst others it includes lost profits due to lost sales, penalties for late deliveries, backorder costs, and costs due to urgent deliveries. A significant, but difficult to quantify, part of stock-out cost is the loss of reputation and customer goodwill which in its turn may lead to loss of future sales. When a product is intended for internal use, lost production and downtime due to product shortages can also be considered as a stock-out cost. Stock-out costs are usually difficult to measure and they are a major source of uncertainty for inventory decisions.

The cost of inventory increases as its value increases through manufacturing and processing (Murphy & Knemeyer, 2018). Finished goods inventories are more expensive than raw materials, or work in process inventories, so there is a strong incentive to keep inventory upstream in a supply chain. However, such a practice can lead to longer lead times and may increase the probability of a stock-out. In multi-echelon inventory systems a decision must be made regarding the levels of safety inventories carried at different stages (Chopra & Meindl, 2016).

3.4.3 Inventory control policies

The ultimate objective of an inventory replenishment system is to achieve an appropriate customer service level at the lowest possible cost. To do this, a trade-off between the costs mentioned above must take place and an inventory control policy consistent with the strategic objectives of the company must be defined. The two main questions about an inventory policy is when to order and how much to order. With regard to these questions, two main approaches exist: The periodic review inventory control policy and the continuous review inventory control policy. In both cases decisions are based on the inventory position, defined as the inventory on hand plus the inventory on order, minus any backorders.

3.4.3.1 Periodic review inventory control system

In a periodic review, or fixed order period system the inventory status is checked at fixed time intervals. After each review an order is placed so that the inventory position is raised to a specified threshold. This target level inventory must be high enough to cover demand until the next periodic review plus the delivery lead time, and it must take into account the appropriate safety stock. The time of ordering is known, but the level of the replenishment order varies. A company that follows a periodic review policy must decide on two parameters: the review interval and the order-up-to level.

Periodic review systems are simpler and easier to apply. They demand less managerial effort and they allow a better coordination of transportation operations. On the downside, they generally tend to accumulate more inventory in the system. (Chopra & Meindl, 2016; Schroeder & Goldstein, 2016; Jacobs & Chase, 2018)

3.4.3.2 Continuous review inventory control system

In a continuous review, or fixed order quantity system inventory position is monitored continuously, or after each transaction. Whenever the inventory position crosses a specified reorder point, a replenishment order of fixed quantity is placed. Continuous review systems are event triggered in the sense that an order is initiated when the event of reaching a specified inventory level occurs. The order quantity is known and can be optimized, but the time of ordering varies depending on how soon the reorder point is reached. The reorder point must take into account the projected demand during the replenishment lead time as well as the appropriate safety stock.

Continuous review systems require up to date records, and without an information system the whole process of recording inventory every time that a change occurs can be time consuming. On the other hand, continuous systems require lower inventories to achieve a given customer service level. They are appropriate for high cost products, as well as for critical products whose management will be benefited from closer monitoring and quicker responses (Jacobs & Chase, 2018, Schroeder & Goldstein, 2016; Chopra & Meindl, 2016; Slack et al., 2007).

3.5 Supply networks metrics

Performance measures are vital for effective supply chain management. They provide a clear picture of overall performance and they help to identify problems and their causes, as well as opportunities for improvement. The appropriate metrics promote understanding and provide an insight into the nature and workings of processes; they are an effective way of communication and help shape the behavior of both managers and workers; and they are an indispensable management tool for setting, monitoring and attaining well defined targets, consistent with the strategic vision of a company. The correct measurement of the right things is the basis of effective decision making, but to be of value information must be accurate, relevant and delivered in a timely manner.

Traditionally five performance areas are considered critical for logistics operations: Asset management, cost, customer service, productivity, and quality (Fawcett et al., 2014).

Assets include facilities, equipment and inventories. For facilities and equipment capacity utilizations are used, with higher utilizations being preferable. With regard to inventory, the ratio of demand to available inventory (days of supply), and the inventory turnover (Cost of goods sold / Average inventory value) can be used. More general metrics such as Cash-to-Cash cycle time and return on assets are also useful from a managerial point of view.

Cost relates directly to profit, so its tracking is an obvious necessity. Costs may refer at functional areas or at specific activities, and they may be expressed as a percentage of sales or as cost of goods sold.

Customer service depends on product availability, responsiveness to customer demands, and customer satisfaction. Product availability is commonly measured with Product fill rate expressed as the ratio of products delivered to products ordered, and Order fill rate expressed as the ratio of complete orders delivered to the total number of customer orders. In make-to-order situations, on-time delivery is the percentage of orders delivered complete and on the date requested by the customer (Schroeder & Goldstein, 2016). Other measures include the frequency of stock-out (when the company has no inventory to meet expressed demand), the frequency and duration of back orders, and the average number of back-ordered items. With regard to responsiveness, cycle time measuring the time between order receipt and order delivery is widely used. The time needed to change the output volume by a fixed amount (volume flexibility) and the time to change the mix of delivered products (mix flexibility) are also important measures, especially where high levels of agility are expected. The overall satisfaction of the customer is usually evaluated through customer surveys, while the number of customer complaints is also a good, though somewhat paradoxical, indicator. (Fawcett et al., 2014; Schroeder & Goldstein, 2016)

Productivity measures the output of a process against the resources that were consumed. Labor productivity expressed in units produced per labor hour is a common metric, especially for labor intensive industries. Although it is important, a focus on productivity measurements can be misleading when it is not done properly. Productivity should not be measured in isolation, but the effect of productivity changes on other performance metrics must be taken into consideration.

The definition of quality can be elusive. Usually it is defined as the conformance with customer requirements. With regard to products, the percentage of defective items is a common indicator. In the context of logistics, quality is related to service reliability. Usual metrics track the percentage of an activity performed correctly to the total number of times the activity was carried out. Examples include the frequency of wrong deliveries, the frequency of errors in invoicing and the percentage of damages during transportation.

The performance measures that are most relevant to the models developed in this thesis are presented below. The performance metrics are categorized according to the associated performance driver along the lines of Chopra and Meindl (2016):

Facilities

- Capacity: it is the maximum amount a facility can process.
- Utilization: It measures the fraction of the capacity that is being used in the facility. In general, high levels of utilization are preferable from an economic standpoint, but longer delays and higher inventories may have to be traded off.
- Actual average flow/cycle time: It is the average time that a product unit spends in the system. It includes the theoretical time due to processing and any delays that may occur.
- Average production batch size: It measures the average amount produced in each production batch.

Inventory

- Average inventory: It measures the average amount of inventory carried. It can be measured in units, days of demand, or financial value.
- Average replenishment batch size: It measures the average amount in each replenishment order.
- Average safety inventory: It refers to the average amount of inventory on hand when a replenishment order arrives.
- Order Fill rate: It measures the fraction of orders that were fully met on time from inventory. Fill rate should be averaged not over time but over a specified number of units of demand.
- Product Fill Rate or Type II service Level: It refers to the fraction of demand in terms of product units that is met from inventory on hand.
- Fraction of time out of stock: It is the fraction of time that a particular product had zero inventory. It can be used to estimate the lost sales during the stock-out period.

Transportation

- Average inbound transportation cost: It refers to the cost of bringing a product unit into a facility.
- Average incoming shipment size: It measures the average number of units in each incoming shipment at a facility.
- Average inbound transportation cost per shipment: It measures the average transportation cost of each incoming delivery. Along with the incoming shipment size, this metric allows the identification of potential economies of scale in inbound transportation.
- Average outbound transportation cost: It is the cost of sending a product unit out of a facility to the customer. Usually it is measured separately for each customer.

- Average outbound shipment size: It measures the average number of units on each outbound shipment at a facility.
- Average outbound transportation cost per shipment: It measures the average transportation cost of each outgoing delivery. Along with the outgoing shipment size, it allows the identification of potential economies of scale in outbound transportation.

Information

- Ratio of demand variability to order variability: It measures the standard deviation of incoming demand and supply orders placed. A ratio less than 1 may indicate the existence of the bullwhip effect.

Sourcing

- Average purchase price: It is the average price at which a product was purchased. It is obtained by weighting each price by the quantity purchased at that price.
- Supply lead time: It measures the average time between when an order is placed and when the product arrives.
- Supplier reliability: It refers to the variability of the supplier's lead time, as well as the delivered quantity relative to plan.

Pricing

- Incremental fixed cost per order: It measures the incremental costs that are independent of the size of the order. These include changeover costs at a manufacturing plant, or order processing or transportation costs that are incurred independently of shipment size.
- Incremental variable cost per unit: It measures the incremental costs that vary with the size of the order such as variable production costs at a manufacturing plant.
- Average sale price: It is the average price at which a product is sold to a customer. It is obtained by weighting the price with the quantity sold at that price.
- Average order size: It measures the average quantity per order placed from the customer.

3.6 Modelling of Supply networks

Supply networks are complex systems with multiple agents often acting according to their own interests. Despite its acknowledged importance, the coordination of information, material and finance flows is often an elusive objective. To achieve supply chain optimization, the behavior of the systems must be understood, and the ultimate benefits must be made clear to those who are called to make decisions that often seem contrary to their short-term benefit. Complexity and uncertainty render both prerequisites hard to get. Mathematical models have been proven useful tools, they provide scientific methods to address supply network management problems, and they can offer the necessary insight of the systems under consideration.

Although the advances in computing and information technology have allowed for more powerful models, the sheer complexity of the problems means that usually only a limited part of this complexity can be addressed. This does not detract from the value of the models. It is acknowledged that all the models are not applicable in all situations and even though real-world applications often use highly sophisticated methods (heuristics, simulations etc) to manage their operations, most of these approaches are based on theoretical, idealized models (Azadivar & Rangarajan, 2008).

In this thesis models of inventory systems are proposed. Inventory decisions are concerned with inventory needs and aim to coordinate production and stocking decisions throughout the supply chain. Inventory models continue to receive significant attention from researchers as they are of practical value and represent significant theoretical challenges. Even under deterministic assumptions, optimal policies for multi-echelon models are lacking for many network configurations and their identification would be of interest. The field is even less explored in the case of stochastic multi-echelon inventory management and it is in such systems that this thesis is focused on.

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4. Tools and Methodology

4.1 Basic Concepts

4.1.1 Stochastic Processes

The concept of stochastic processes expands the concept of random variables so that time can be included. Stochastic processes deal with the dynamics of probability theory (Ibe, 2009) and they are a powerful tool to model systems that are characterized by both uncertainty and time evolution. They find application in the analysis of a wide range of systems, helping us to understand the variability inherent in the underlying processes, permitting predictions, offering valuable insights for effective design and control, and facilitating decision making based on quantitative parameters (Xu, 2008).

If X a random variable, then we can define the collection, or family, of random variables $\{X(t), t \in T\}$ as a stochastic process with index set T. The index t is usually interpreted as time, so the stochastic process could be described as a time-dependant family of X. The values of $X(t)$ are called the states of the stochastic process, while the set of all possible values of $X(t)$ forms the state space of the process S.

Stochastic processes can be classified into four board categories depending on the nature of the index set T and the state space S. When T is a countable set, then the process is characterized as a discrete time process. On the other hand, if T is an interval of the real line the process is characterized as a continuous time process. In a similar fashion, when S is discrete the process is called a discrete state process, while in the case where S is continuous the process is called a continuous state process.

Other characterizations of a stochastic process can be made according to its dependence on time, the statistical dependence of its developments over disjoint time intervals, and the influence of its history on its future evolvement (Beichelt, 2006).

A stochastic process $X(t)$, $t \geq 0$, is called a counting process if it represents the total number of "events" that have occurred in the interval $[0,t)$. $X(t)$ is a non-decreasing function that takes non-negative integer values, while $X(0)=0$. $X(t_2)-X(t_1)$ represents the number of events that occur in the interval $[t_1, t_2]$.

A counting process is characterized as an independent increment process if the number of events that occur in disjoint time intervals is an independent random variable. Moreover, a counting process $X(t)$ is defined to possess stationary increments if for every set of time instants $t_0=0<t_1<t_2<..<t_n$ the increments $X(t_1)-X(t_0)$, $X(t_2)-X(t_1)$,.., $X(t_n)-X(t_{n-1})$ are identically distributed (Ibe, 2009).

4.1.2 Distributions of interest

We denote *f(s)*:the probability density function, or probability mass function, *F(s)*:the cumulative density function, *E(X)*: the mean, *Var(X)*: the variance, *StDev*: the standard deviation and *CV*: the coefficient of variation.

4.1.2.1 The exponential distribution

The random variable X has an exponential distribution with parameter (rate) λ if its probability density function f(s) is given by

$$f(s) = \begin{cases} \lambda \cdot e^{-\lambda \cdot s}, & s \geq 0 \\ 0, & s < 0 \end{cases}$$

It follows that

$$F(s) = \begin{cases} 1 - e^{-\lambda \cdot s}, & s \geq 0 \\ 0, & s < 0 \end{cases}$$

$$E(X) = \frac{1}{\lambda}$$

$$Var(X) = \frac{1}{\lambda^2}$$

$$StDev(X) = \frac{1}{\lambda}$$

$$CV = 1$$

One important feature of the exponential distribution is that it is the only continuous distribution that contains no memory. The so called memoryless property means that the time until the occurrence of the next event is probabilistically always the same:

$$Pr(X > t + s | X > t) = \Pr(X > s)$$

Other useful properties of the exponential distribution are (Xu, 2008):

- The sum of a fixed number of i.i.d exponential variables follows a Gamma (Erlang) distribution. If $S_n = T_1 + T_2 + \cdots + T_n$, where T_i are iid exponential random variables with rate λ, then the random variable S_n has probability density function $f(t) = \lambda \cdot e^{-\lambda t} \cdot \frac{(\lambda t)^{n-1}}{(n-1)!}$, for t≥0.
- The minimum of independent exponential variables is still an exponential random variable. If T_i, i=1,2,…,n are independent exponential variables with respective rates λ_i, then the time when the first of the n event occurs, T=min(T_1,T_2,…,T_n), is an exponential random variable with rate $\sum_{i=1}^{n} \lambda_i$
- The probability that the i[th] event is the first to occur amongst the n events is proportional to λ_i. $P(T_i = T) = \frac{\lambda_i}{\sum_{i=1}^{n} \lambda_i}$

4.1.2.2 The Poisson distribution

A random variable X follows a Poisson distribution with parameter (rate) λ, $0<\lambda<\infty$, when it has probability mass function

$$f(x) = \frac{\lambda^x e^{-\lambda}}{x!} \text{ , x=0,1,…}$$

It follows

$$E(X) = \lambda$$
$$Var(X) = \lambda$$

The Poisson distribution is the most important distribution in stochastic modeling. Amongst others, it can be used as an approximation of the binomial distribution for large n (population) and small p (probability of occurrence in each individual) (Feldman & Flores, 2010; Blumenfeld 2009).

4.1.3 Poisson processes

The counting process $X(t)$, $t\geq 0$ is a Poisson process with rate (intensity) λ , $\lambda>0$, if

i) X(0)=0
ii) The process has independent increments
iii) The number of events in any interval of length t is Poisson distributed with mean λt
$P\{X(t+s) - X(s) = n\} = e^{-\lambda t} \cdot \frac{(\lambda t)^n}{n!}$, n=0,1,..

If $o(\Delta t)$ is a function of Δt that goes to zero faster than Δt, $\lim_{\Delta t \to 0} \frac{o(\Delta t)}{\Delta t} = 0$, then from the above definition follows:

iv) $P\{X(t+\Delta t) - X(t) = 1\} = \lambda \Delta t + o(\Delta t)$: The probability of exactly one event within a small interval Δt is approximately $\lambda \Delta t$
v) $P\{X(t+\Delta t) - X(t) \geq 2 = o(\Delta t)$: The probability of more than two events within a small time interval Δt is negligible ($o(\Delta t)$).
vi) $P\{X(t+\Delta t) - X(t) = 0\} = 1 - \lambda \Delta t + o(\Delta t)$

Statements iv-vi for a counting process with stationary and independent increments can be used as an alternative definition of a Poisson process (Ibe, 2009).
With regard to the mean and the variance of a Poisson process:

$$E[X(t)] = \lambda t$$
$$Var[X] = \lambda t$$

An important property of a Poisson process with rate λ is that the inter-arrival times are exponentially distributed with mean time between arrivals $1/\lambda$. Conversely, an arrival process with exponentially distributed inter-arrival times is a Poisson process (Feldman and Flores, 2010). The Poisson process at any point in time probabilistically restarts itself and from any point on it is independent of all that has previously occurred (Ross, 2010).

Two other interesting properties concern the superposition and the decomposition of Poisson processes. The superposition property states that if $X_i(t)$, $t \geq 0$, i=1,2,..n are independent Poisson processes with respective rates λ_i, i=1,2,..n, then the composite process $\{\sum_{i=1}^{n} X_i(t)\ , t \geq 0\}$ is a Poisson process with rate $\sum_{i=1}^{n} \lambda_i$.

With regard to the decomposition property, we define $\{X(t)\ t \geq 0\}$ a Poisson process with rate λ and each time an event occurs, independent of all else, we classify it as a type *i* event with probability p_i, $\sum_{i=1}^{n} p_i = 1$. If $\{X_i(t),\ t \geq 0\}$ is the arrival process of type *i*, then $\{X_i(t),\ t \geq 0\}$, i=1,2,..,*n* are *n* independent Poisson processes with respective rates $\lambda \cdot p_i$ (Xu, 2008)

4.1.2.3 The compound Poisson Process

We can generalize the Poisson process by relaxing the assumption that arrivals occur one at a time. If $\{N(t), t \geq 0\}$ a Poisson process with arrival rate λ; $\{Y_i$, i=1,2,..$\}$ a family of independent and identically distributed random variables; and the Poisson process $\{N(t), t \geq 0\}$ and the sequence $\{Y_i$, i=1,2,..$\}$ are independent, then the process X(t), $t \geq 0$ defined by $X(t) = \sum_{i=1}^{N(t)} Y_i$ is called a compound Poisson process.

The compound Poisson process involves two kinds of randomness, the randomness of the main process, also called the Poisson point process, and an independent randomness associated with its rate (Ibe, 2009). Intuitively, the compound Poisson process can be regarded as describing arrivals in "batches", but the values Y_i need not necessarily be integers. (Feldman and Flores, 2010). With regard to mean and variance:

$$E[X(t)] = \lambda \cdot t \cdot E[Y]$$
$$Var[X(t)] = \lambda \cdot t \cdot E[Y^2]$$
$$Var[X(t)] = \lambda \cdot t \cdot \{Var[Y] + (E[Y])^2\}$$

4.2 Markov Processes

A Markov Process is a stochastic process whose future is independent of its past given the present. More formally, a stochastic process $\{X(t), t \in T\}$ is called a first order Markov process if for any $t_0 < t_1 < \cdots < t_n$ the conditional Cumulative Density Function of $X(t_n)$ for given values of $X(t_0)$, $X(t_1)$,.., $X(t_{n-1})$ depends only on $X(t_{n-1})$:

$$P[X(t_n) \leq x_n | X(t_{n-1}) = x_{n-1}, X(t_{n-2}) = x_{n-2}, \ldots, X(t_0) = x_0]$$
$$= P[X(t_n) \leq x_n | X(t_{n-1}) = x_{n-1}]$$

This property of the Markov process is referred to as the Markov property (Ibe, 2009).

Similarly to other stochastic processes, Markov processes can be categorized according to the nature of the time parameter (discrete time - continuous time) and the nature of the state space (discrete space - continuous space). In this thesis we are concerned with discrete state processes. Next such processes will be briefly discussed for discrete and continuous time.

4.2.1 Discrete Time Markov Chains

The discrete process $\{X_k$, k=0,1,2..$\}$ is called a Markov Chain if for all i,j,k,…,m:

$$P[X_k = j|X_{k-1} = i, X_{k-2} = n, \dots, X_0 = m] = P[X_k = j|X_{k-1} = j] = p_{ijk}$$

p_{ijk} is called the one step transition probability and stands for the conditional probability that the process will be in state j at time k given that it is in state i at time k-1. When these probabilities depend on time then the process is called a non-homogeneous Markov Chain. On the other hand, when the transition probabilities are independent of time then the process is called homogeneous and p_{ijk}=p_{ij}.(Ibe, 2009)

Since probabilities are non-negative and since the process must make a transition into some state it follows:

$$0 \le p_{ij} \le 1$$

$$\sum_j p_{ij} = 1$$

The transition probabilities are usually given in the form a matrix P, where p_{ij} is at the i^{th} row and the j^{th} column. A discrete time Markov chain is completely specified by the probability transition matrix P and the initial distribution a=$\{a_i\}$, where a_i=P[X_0=i] is the probability that the chain starts in state *i*.(Xu, 2008)

4.2.1.1 Transient analysis

Transient analysis is concerned with probability statements about the possible realization of the discrete time Markov Chain at time *n*. The transition matrix gives direct information about the one-step transition probabilities. It can also be used as a basis to compute the n-step transition probabilities $p_{ij}^{(n)} = P(X_n = j|X_0 = i), n = 1,2,..$

The analysis is based on the Chapman-Kolmogorov equation that states that for the chain to be in state j at time (m+n) first it must go through some intermediate states

$$p_{ij}^{(m+n)} = \sum_{k=0}^{\infty} p_{ik}^{(m)} \cdot p_{kj}^{(n)}$$

If we denote $P^{(n)}$ the n-step probability transition matrix,

$$P^{(n+m)} = P^{(n)} \cdot P^{(m)}$$

It follows that

$$P^{(n)} = P^{(n-1)} \cdot P = P^{(n-2)} \cdot P \cdot P = P^n$$

$$a^{(n)} = a^{(0)} \cdot P^n$$

4.2.1.2 Classification of States

The states of a Markov process can be classified based on the transition probability p_{ij}. A state *j* is accessible from state *i* when it is possible to travel from state *i* to state *j* with non zero probability in a random but finite number of steps: $P(X_n = j|X_0 = i) > 0$. If two states *i* and *j* are mutually accessible, then the two states communicate. Two states that communicate are said to belong in the same class, while by definition two classes of states are either identical or disjoint (Ross, 2010). A Markov chain whose state space is made of a unique class of states is characterized as irreducible. The concept of equivalent classes is helpful as many state properties are class properties as well.

A set of states C is said to be closed when $p_{ij} = 0$ for all $i \in C, j \notin C$. The interpretation of a closed set is that once the chain takes a value in the set C, then it can never leave C (Grimmett & Stirzaker, 2001). For a closed set also holds $\sum_{j\in C} p_{ij} = 1$ for all $i \in C$. An irreducible set can alternatively be defined as a closed set that contains no proper subset that is also closed (Feldman & Flores, 2010). A state that forms an irreducible set is called absorbing. An absorbing state is certain to return to itself in one transition, $p_{jj}=1$.

Another important distinction is between transient and recurrent states (Xu, 2008; Ibe, 2009). A state is transient if starting from the state the process will revisit it only a finite number of times before eventually leaving it and never return. Equivalently, there is a positive probability that the process starting from the state will never return to it. On the other hand, the recurrent state will be surely revisited again and again, an infinite number of times. More formally, for a transient state:

$$\lim_{n\to\infty} p_{ii}^{(n)} = 0$$

$$P(X_n = i, n \geq 1 | X_0 = i) < 1$$

Correspondingly, for a recurrent state

$$\lim_{n\to\infty} p_{ii}^{(n)} > 0$$

$$P(X_n = i, n \geq 1 | X_0 = i) = 1$$

If starting at a recurrent state, the expected time (number of transitions) until the chain returns to the state is finite, then the state is said to be positive recurrent. Otherwise, the recurrent state is called a null recurrent state. In a finite-state, discrete-time Markov chain the recurrent states are also positive recurrent.

A recurrent state is called periodic when the return is possible only in multiples of an integer period d>1. $p_{ii}^{(n)} > 0$ only when $n = d, 2d, 3d, ..$ and $p_{ii}^{(n)} = 0$ otherwise (d is the greatest common divisor of the epochs at which return is possible) (Grimmett & Stirzaker, 2001). If d=1, the state is called aperiodic. Positive recurrent, aperiodic states are called ergodic states, while a chain consisting of ergodic states is called an ergodic chain.

Recurrence, transient-ness, and periodicity are all class properties. If a state has one of these properties, then all the states belonging in the same class will also have that property (Xu, 2008).

In the case when the state space S of the process includes transient states and multiple closed sets, S can be decomposed into disjoint sets. S can be uniquely partitioned as $S = T \cup C_1 \cup C_2 \cup \ldots$, where *T*: the set of transient states and C_i: irreducible closed sets of recurrent states. (Grimmett & Stirzaker, 2001)

4.2.1.3 Steady state behavior

For many applications we are interested in the long run behavior of a Markov chain. Although a Markov chain X_n is inherently dynamic, under certain conditions the distribution of X_n may settle down.

A markov chain $(X_n)_{n\in N}$ is said to have a limiting distribution if the limits

$$\lim_{n\to\infty} P(X_n = j|X_0 = i)$$

exist and

$$\sum_{j\in S} \lim_{n\to\infty} P(X_n = j|X_0 = i) = 1$$

for all $i, j \in S$, S: the state space. (Privault, 2013)

Closely related, but not identical is the concept of stationary probabilities. The vector π is called a stationary distribution of the Markov chain if π has entries $\pi_j, j \in S$ such that:

$\pi_j \geq 0$ for all j and $\sum_j \pi_j = 1$

$\pi P = \pi$, or equivalently $\pi_j = \sum_i \pi_i \cdot p_{ij}$ for all j *(*Grimmett & Stirzaker, 2001).

The distribution $\boldsymbol{\pi}$ is invariant by matrix P, in the sense that if the chain is started in the stationary distribution, then it will remain in that distribution at any subsequent time step (Privault, 2013). The stationary probabilities refer to an equilibrium state of the Markov chain where the mean intensity per time unit of leaving a state is equal to the mean intensity per time unit of arriving at the particular state.

If a Discrete time Markov chain is irreducible and ergodic (positive recurrent and aperiodic), the limiting probabilities exist, they are independent of the initial state, and they coincide with the stationary probabilities. The vector π of the stationary probabilities is uniquely defined by: $\pi P = \pi$ and $\sum_j \pi_j = 1$ (Xu, 2008; Privault, 2013).

If μ_{jj} is the expected number of transitions until the process revisits state j (mean recurrent time,) then:

$$\mu_{jj} = \frac{1}{\pi_j} \quad , j \geq 0$$

4.2.2 Continuous Time Markov Chains

The stochastic process $\{X = X(t): t \geq 0\}$, with countable state space S and indexed by the half line T=[0,∞] is a continuous time Markov chain if it satisfies the Markov property:

$$P\left(X(t_n)=j \mid X(t_1)=i_1,\dots,X(t_{n-1})=i_{n-1}\right)=P\left(X(t_n)=j \mid X(t_{n-1})=i_{n-1}\right)$$

for all $j, i_1, \dots, i_{n-1} \in S$ and any sequence $t_1 < t_2 < \cdots < t_n$ of times (Grimmett & Stirzaker, 2001).

The Markov property means that the future development of the continuous-time Markov chain depends only on its present state and not on its evolution in the past. The conditional probabilities

$$p_{ij}(s,t) = P(X(t) = j|X(s) = i); s < t;\ i,j \in S$$

are the transitional probabilities of the Markov chain. If for all $s,t \in T$ and $i,j \in S$ the transitional probabilities $p_{ij}(s,t)$ depend only on the difference *t-s*, then the Markov chain is characterized as homogeneous: $p_{ij}(s,t) = p_{ij}(0,t-s)$. Since in homogenous chains the transitions depend on only one variable we can denote them as $p_{ij}(t)$.

Similarly to the discrete time case, the transition probabilities can be given in the form of a matrix *P(t)* where probability $p_{ij}(t)$ corresponds to the ith row and jth column element. The family $\{P(t): t \geq 0\}$ is called the transition semigroup of the chain. The entries of *P(t)* are non-negative, while the sum of each row equals 1. *P(t)* depends on t, so different time-values specify different transition matrices. The following also hold:

$P(0) = I$, where I the identity matrix;

$P(s+t) = P(s) \cdot P(t); s,t \geq 0$ (Chapman-Kolmogorov equations, or semigroup property)

The states of a continuous time Markov chain can be classified in a way similar to that of discrete time Markov chains (Beichelt, 2006). The concepts of 4.2.1.2 can also be defined, appropriately modified for continuous time processes.

4.2.2.1 The infinitesimal generator matrix

Given our assumptions, the transition probabilities $p_{ij}(t)$ are differentiable. By differentiating the semi-group property relation we have:

$$P'(t) = \lim_{h\to 0}\frac{P(t+h)-P(t)}{h} = \lim_{h\to 0}\frac{P(t)\cdot P(h)-P(t)}{h} = P(t)\cdot \lim_{h\to 0}\frac{P(h)-P(0)}{h} = P(t)\cdot Q$$

The matrix $Q = P'(0) = \lim_{h\to 0}\frac{P(h)-P(0)}{h}$ is called the infinitesimal generator of the Markov process (Privault, 2013). The matrix *Q,* along with the initial distribution of the process, completely specify the continuous time Markov Chain (Xu, 2008) .

The infinitesimal generator defines a set of transition rates $q_{ij} = \lim_{h\to 0}\frac{p_{ij}(h)}{h}$ and it can be easily proved that the sum of each row of the infinitesimal generator is equal to 0 ($\sum_j q_{ij} = 0$). It follows that $q_{ii} = -\sum_{j\neq i} q_{ij}$.

Transitions probabilities can be computed based on the equations

$$P'(t) = P(t) \cdot Q, \qquad t > 0$$
$$P'(t) = Q \cdot P(t), \qquad t > 0$$

known as the forward and backward Kolmogorov equations respectively.

4.2.2.2 Transition and sojourn times

Starting from $X(t) = i$, in a short time interval (t, t+h):

a) nothing happens with probability $1 + q_{ii} \cdot h + o(h)$

b) the chain jumps to state j≠i with probability $q_{ij} \cdot h + o(h)$

Equivalently we can write that

$$P(X(t+h) = j \mid X(t) = i) = p_{ij}(h) \approx \begin{cases} q_{ij} \cdot h, & i \neq j, h \to 0 \\ 1 - h \cdot \sum_{k \neq i} q_{ik}, & i = j, h \to 0 \end{cases}$$

The time τ_{ij} spent in state i before moving to state j≠i in a single step is an exponentially distributed random variable with parameter q_{ij}:

$$P(\tau_{ij} > t) = e^{-q_{ij}t}, t \in [0, +\infty)$$

$$E[\tau_{ij}] = q_{ij} \int_0^\infty t \cdot e^{-t \cdot q_{ij}}\, dt = \frac{1}{q_{ij}}, \qquad i \neq j$$

The time τ_i spent in state i before the next transition to a different state is an exponentially distributed random variable with parameter $\sum_{j \neq i} q_{ij}$

$$P(\tau_i > t) = exp\left(-t \cdot \sum_{j \neq i} q_{ij}\right) = e^{t \cdot q_{ii}}, t \in [0, +\infty)$$

It follows that

$$\tau_i = \min_{j \in S, j \neq i} \tau_{ij}$$

$$E[\tau_i] = \sum_{k \neq i} q_{ik} \int_0^\infty t \cdot exp\left(-t \sum_{k \neq i} q_{ik}\right) dt = \frac{1}{\sum_{k \neq i} q_{ik}} = -\frac{1}{q_{ii}}$$

The exponentially distributed times are in line with the memoryless property of the Markov chain. (Privault, 2013)

Continuous time Markov chains can be defined as stochastic processes that move from state to state in accordance with a discrete-time Markov chain, but in such a way that the amount of time spent in each state, before proceeding to the next state, is exponentially distributed (Ross, 2010). The continuous Markov process can be described by two sets of parameters, the exponential sojourn time rates q_i and the transitions probabilities $p_{ij}, i \neq j$.

$$p_{ij} = \frac{q_{ij}}{\sum_{k \neq i} q_{ik}}, j \neq i$$

4.2.2.3 Steady state behavior

A probability distribution $\pi = (\pi_i)_{i \in S}$ is said to be a stationary for a Markov process with transition matrix P(t), if

$$\pi \cdot P(t) = \pi \, , t \in [0, +\infty)$$

and

$$\sum_{i \in S} \pi_i = 1$$

Equivalently the following relations can be used:

$$\pi \cdot Q = \mathbf{0}$$
$$\sum_{i \in S} \pi_i = 1$$

If the Markov chain is irreducible and positive recurrent the stationary probabilities exist, they are unique and they coincide with the limiting probabilities.

$$\pi_j = p_j = \lim_{t \to \infty} p_{ij}(t) = \lim_{t \to \infty} p_j, \qquad j \in S$$

The equations resulting from the matrix relations above are known as the balance equations. They represent an equilibrium state of the Markov process where the long run rate at which the process leaves a particular state j is equal to the long run rate at which the process enters state j. (Privault, 2013; Beichelt, 2006; Xu, 2008)

$$q_j \pi_j = \sum_{k \in S, k \neq j} q_{kj} \pi_k \, , \qquad j \in S$$
$$\sum_{i \in S} \pi_i = 1$$

4.2.3 Birth and death processes

A pure birth process is a continuous time Markov chain with state space $S = \{0,1,2, \dots, n\}$ where only transitions from state i to state i+1 are possible. State n is absorbing if $n < \infty$. If we denote $\lambda_i = q_{i,i+1}$ the transition rates of the birth process (Beichelt, 2006):

$$P(X(t+h) - X(t) = 1 | X(t) = i) \cong \lambda_i h, \qquad h \to 0, i \in S$$
$$P(X(t+h) - X(t) = 0 | X(t) = i) \cong 1 - \lambda_i h, \qquad h \to 0, i \in S$$

A pure death process is a continuous Markov chain with state space $S = \{0,1,2, \dots, n\}$ where only transitions from state i to state i-1 are possible. If we denote $\mu_i = q_{i,i-1}$ the transition rates of the death process ($\mu_0 = 0$):

$$P(X(t+h) - X(t) = -1 | X(t) = i) \cong \mu_i h, \qquad h \to 0, i \in S$$
$$P(X(t+h) - X(t) = 0 | X(t) = i) \cong 1 - \mu_i h, \qquad h \to 0, i \in S$$

A continuous time Markov chain $\{X(t), t \geq 0\}$ with state space $S = \{0,1,2,\dots,n\}, n \leq \infty$, is called a birth-and-death process if from any state i only a transition to state i-1 or state i+1 is possible, provided that $i-1 \in S$ and $i+1 \in S$.
$q_{ij} = 0$ for $|i-j| > 1$
$q_{i,i+1} = \lambda_i$ the birth rates
$q_{i,i-1} = \mu_i$ the death rates

A birth-and-death process can be defined as the sum of a pure birth and a pure death process (Privault, 2013). The transition probabilities for state i:

$$P(X(t+h) - X(t) = 1 | X(t) = i) \cong \lambda_i h, \qquad h \to 0, i \in S$$
$$P(X(t+h) - X(t) = -1 | X(t) = i) \cong \mu_i h, \qquad h \to 0, i \in S$$
$$P(X(t+h) - X(t) = 0 | X(t) = i) \cong 1 - (\lambda_i + \mu_i)h, \qquad h \to 0, i \in S$$

By the definition of the process, the time τ_i spent in state i is an exponentially distributed random variable with parameter $(\lambda_i + \mu_i)$ and mean $\frac{1}{\lambda_i+\mu_i}$. Moreover, the probability that a birth occurs before a death, when the process is in state i is $p_{i,i+1} = \frac{\lambda_i}{\lambda_i+\mu_i}$. Similarly the probability that a death occurs before a birth, when the process is in state i, is $p_{i,i-1} = \frac{\mu_i}{\lambda_i+\mu_i}$ (Ibe, 2009).

Birth and death processes offer a useful framework for the analysis of a wide range of systems and they have found practical applications in many fields, including queuing and inventory systems. The characteristic of their infinitesimal generator matrix is that it has a tri-diagonal structure, with non-zero elements only along, immediately above, and immediately below the diagonal.

$$Q = \begin{bmatrix} -\lambda_0 & \lambda_0 & 0 & 0 & 0 & \cdots \\ \mu_1 & -(\lambda_1+\mu_1) & \lambda_1 & 0 & 0 & \cdots \\ 0 & \mu_2 & -(\lambda_2+\mu_2) & \lambda_2 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{bmatrix}$$

4.2.4 Quasi-Birth-and-Death processes

A Quasi-birth-and-death (QBD) process is a generalization of the simple birth-and-death process. QBD processes are infinite state Continuous time Markov chains with a two dimensional state space S. States are grouped into levels and each level l_i consists of m_i (finite of infinite) phases. S can be partitioned as $\bigcup_{n\geq 0} l_n; l_n = \{(n,1),(n,2),\dots,(n,m_n)\}$ for $n \geq 0$, where the term level denotes the whole subset l_n. One step transitions are allowed only within the same level or between adjacent levels: A transition from state (n,j) to state (n',j') is possible only if n'= n, n+1 or n-1.

The first level may be different from the rest as it is the boundary level, while the others are repeating levels that usually have the same transition structure. The second level, sometimes referred to as the border level, may also have some slightly different structure from the other repeating levels. The infinitesimal generator matrix Q of a QBD process has a block tri-diagonal form. In the case of a homogeneous process where $m_i = m$ for all $i \geq 1$ (assuming that l_0 is the boundary level) the matrix Q has the form:

$$Q = \begin{bmatrix} D_0 & D_1 & 0 & 0 & 0 & 0 & \cdots \\ D_2 & A_0 & A_1 & 0 & 0 & 0 & \cdots \\ 0 & A_2 & A_0 & A_1 & 0 & 0 & \cdots \\ 0 & 0 & A_2 & A_0 & A_1 & 0 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{bmatrix}$$

A_0, A_1, and A_2 are mxm matrices, where m is the number of phases in a level that is not the boundary level. A_1 corresponds to transitions from a repeating level to the next higher repeating level, while A_2 deals with the transition rates of transitions from a repeating level to the preceding repeating level. A_0 correspond to intra-level transitions for the repeating levels.

D_0 is a nxn matrix, where n is the number of phases in the boundary level. It corresponds to intra-level transitions for the boundary level.

D_1 is a nxm submatrix that corresponds to transitions from the boundary level to the border level.

D_2 is a mxn submatrix that deals with transitions from the border level to the boundary level.

In general A_0 and D_0 have nonnegative off-diagonal elements and strictly negative diagonal elements, while A_1, A_2, D_1 and D_2 are non-negative matrices (Ibe, 2009)

Quasi-birth-and-death processes find a wide range of applications. One of their main advantages is the fact that the structural properties of their generator matrix allow the employment of algorithmic approaches for the evaluation of the respective systems. Finite QBD process with restricted state space can also be defined (Latouche, 1999), but their analysis is more complicated.

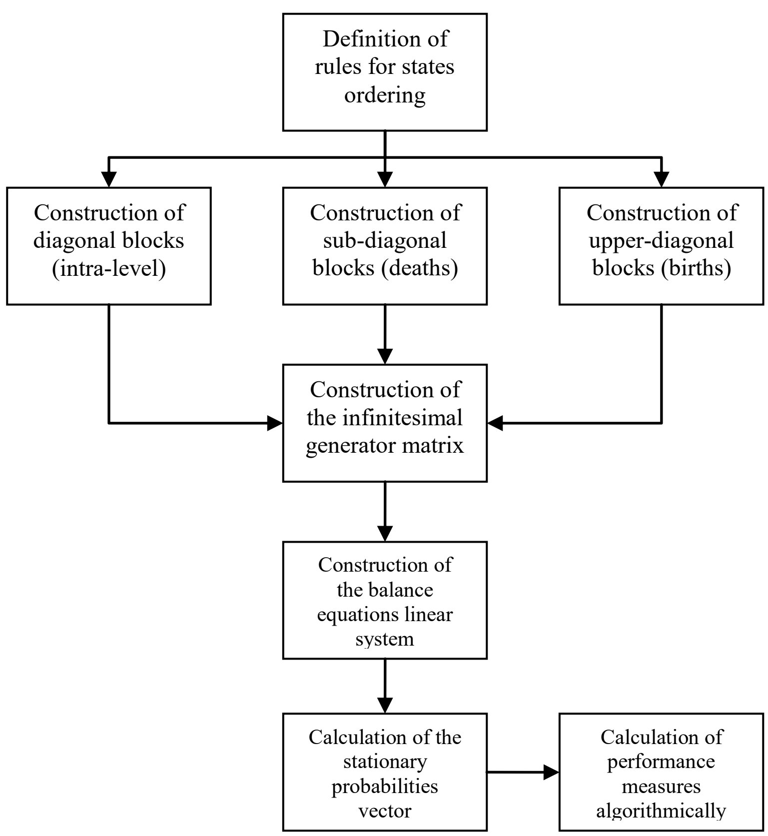
4.3. Methodology

In this thesis various production-inventory systems are analyzed as continuous time, discrete state Markov Chains. The state space of the processes is finite and multi-dimensional, as in all cases under investigation three or more dimensions are necessary to completely describe the respective system. The states are ordered lexicographically, starting from a basic level and proceeding to successive lower sub-levels. The modeling approach is based on the properties of the exponential distribution. However, based on the exponential distribution, more complex distributions such as the Coxian and the compound Poisson are also employed.

The analysis is based on the infinitesimal generator matrix of the process. More precisely, the infinitesimal generator Q is partitioned into recurring blocks of states (sub-matrices), each one of which corresponds to a specific kind of transitions. In general, matrix Q has a three-tier structure. Matrices on the diagonal correspond to transitions within the same basic level, sub-matrices above the diagonal describe "birth" transitions, and sub-matrices below the diagonal correspond to "death" transitions. It must be noted that here the terms "birth" and "death" have a more relaxed meaning than that used for birth-and-death processes. Unlike QBD processes, transitions in non-adjacent basic levels are generally allowed.

Given the generator matrix it is easy to construct the linear system of balance equations from the relations given in paragraph 4.2.2.3. At this stage, the only difficulty for the solution of the model resides with the dimensions of the linear system. Large linear systems can be solved iteratively and several methods are proposed in the literature (Ching, 2006). For our applications we are using LU factorization with partial pivoting from the Matlab toolbox.

Figure 4.1: Outline of methodology



In the last step, system performance measures are computed as functions of stationary probability sums. For numerical results the computation is done algorithmically through an iterative process. The general outline of the methodology is given in figure 4.1

4.4.References

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5. Analysis of a horizontally integrated Push-Pull system

5.1 Research rationale

Production logistics are characterized by the intelligent planning of the processes and the provision of products within systems of a more restricted perspective (Gleisner & Femerling, 2013). In such a context, the working logic of the system is fundamental and bears heavily on its performance. The distinction between push and pull processes is one of the most basic, and it is commonly used in the literature for the classification of production/inventory systems. Although there are no universally accepted definitions (Limperopoulos, 2013), usually the timing of the execution of a process relatively to end customer demand is the base of characterization (Chopra & Meindl, 2007).

In push processes execution is initiated in anticipation of customer orders. Push processes are often based on forecast and offer higher utilization of resources and a better ability to meet customer demand. On the downside, push processes are associated with higher inventory levels which impair the efficiency and the flexibility of the system. Traditional MRP systems fall into this category.

On the other hand, in pull processes execution is initiated in response to end customer demand. Pull processes are constrained by inventory and capacity decisions. Pull systems are more flexible and low inventory levels can be achieved, but often at the price of creating additional stock-out costs and long customer lead times. Systems based on kanban philosophy are typical examples of this category.

A third approach is the combination of push and pull processes. Three modes of integration have been proposed. In vertically integrated systems, one system is super-imposed on another, usually an MRP generating overall production plans for a system working according to Just-in-Time philosophy. In parallel integrated hybrid systems, both push and pull mechanisms coexist and work in parallel or complementarily (Cheikhrouhou, 2009). Finally, in horizontally integrated systems, different parts of the same system are controlled by different mechanisms. In the usual horizontally integrated push-pull system, production at the earlier upstream stations is push-type, while distribution at the later downstream stations is controlled by pull-type policies.

Hybrid systems have been found to perform better than pure push, or pure pull systems, while they are more flexible to address growing product variety, shorter product life cycles and the need of keeping inventory costs as low as possible (Ghrayeb, 2009; Cuypere, 2012). Moreover, they have been suggested for better dynamic performance and the minimization of unwanted phenomena such as the Bullwhip effect (Donner, 2008). However, the analysis of such systems is more complicated.

Here a model based on Markov processes is proposed for the analysis of a linear, horizontally integrated, push-pull system. For more realistic assumptions a lost sales regime has been chosen. Lost sales are common in retail and high competition markets and may even be accepted as part of the business strategy in times of low demand (Kesen et al., 2010), but lost sales models are comparatively rare in the literature as they are more difficult to analyse and solve (Bijvank, 2011). We suggest an algorithm based on the analysis of the infinitesimal generator matrix of the system. An exact numerical solution is offered and performance measures are computed for different combinations of the decision variables. The proposed model can be used to evaluate what if scenarios, to explore the dynamics of the system, or as an evaluative tool in the framework of an optimization model.

5.2 Literature review

In general, four different approaches have been employed for modelling and analyzing supply chains: Continuous time differential equations, discrete time difference equations, discrete event simulation, and classical operational research methods. The latter are the most commonly used and include such techniques as linear programming, queuing theory, Markov chains and dynamic programming (Riddalls et al., 2000). Much of the literature is focused on pure push or pure pull systems, the latter usually in some form of a kanban system. Although of practical importance in production environments, models of hybrid push-pull systems are not so common.

Olhanger and Ostlund (1990) discuss the integration of push and pull strategies in the context of a manufacturing strategy, and examine the linkage between the push-pull boundary and system characteristics such as the customer order point (the point where a product is assigned to a specific customer), bottleneck resources and the product structure. They illustrate the potential benefits of integration through a case study.

Takahashi and Soshiroda (1996) use difference equations to evaluate the performance of multi-stage, horizontally integrated push-pull systems. They test two different integration strategies for a linear production/inventory system, with deterministic production lead times and stochastic demand. A configuration with upstream push-control and downstream pull-control is found to give persistently better results. Deterministic parameters are also used by Donner et al. (2010) who explore the dynamics of a linear supply chain for push, pull, and hybrid push-pull production control strategies. They use a fluid-dynamic input-output model to evaluate the stability of the respective systems. Hybrid systems are found to minimize the bullwhip effect, but under certain conditions they are also found to give rise to linear instabilities.

Lin et al. (2012) assume deterministic times and uniform demand without stock-outs, to study push-pull systems in the context of mass customization. They propose the economic batch production model to determine the best production policies for the push segment upstream the decoupling point.

For more complex systems, simulation is preferred. Discrete Event Simulation is used by Cochran and Kim (1998) to study a horizontally integrated hybrid system comprised of an upstream series of push stations, a safety stock of semi-finished products, and a downstream pull sub-system. A movable junction point between the push and pull sub-systems is assumed. Simulated annealing is employed to find the optimal solution with regard to the safety stock at the junction point and the number of kanbans for the pull segment of the system.

Cochran and Kaylani (2008) and Ghrayeb et al. (2009) combine discrete event simulation with a Genetic Algorithm to evaluate the performance of multi-stage, serial, push-pull systems. Cochran and Kaylani investigate a case with multiple products trying to locate the optimal location of junction points as well as optimal safety stocks for the push and optimal number of kanban cards for the pull part of the system. They report cost efficiency for the hybrid system and they propose locating the junction point between push and pull systems after the bottleneck of the process. Ghrayeb et al. use exponential processing times and a base stock policy for the pull stations of a single product production/inventory system. In most of the cases they test, the push-pull configuration works better than respective pure push or pure pull systems.

Mahapatra et al. (2012) use simulation to evaluate pull and push-pull strategies in an un-capacitated supply chain with one or multiple retailers. Demand uncertainty and lead time variability are taken into account and unmet end-customer demand is assumed to be lost. Pull stations follow a periodic review order-up-to-level inventory control policy, while push-type production is driven by forecasts. The optimal strategy is reported to depend on the specific parameters of the system and the performance measures of concern, but in general the hybrid policy gives better results than the pure pull policy.

Kim et al. (2012) compare different supply chain strategies for a serial, multi-stage, push-pull system with backorders and normally distributed demand. Periodic review policies are assumed and multiple stock points in the pull segment are proposed. The authors formulate a nonlinear, mixed integer programming model with simulated annealing as the search algorithm.

A significant part of the literature on hybrid systems modelling is based on Markov analysis. Hodgson and Wang (1991) investigate a simple assembly production/inventory system in the context of iron and steel processing. For each state in the system, it is possible to use a pure pull, a pure push, or a hybrid push/pull policy, with the objective to determine the optimal control policy with regard to a cost function. The model is based on a Markov Decision Process. The state space of the system, the policy space, the state transition mechanism and the cost structure are defined and a policy iteration procedure is applied. The analysis suggests that a hybrid policy with push upstream stages and pull downstream stages appears to have desirable operational properties. Geraghty and Heavy (2003) further analyze the model proposed by Hodgson and

Wang (1991) and conclude that the optimal push-pull policy is equivalent to a CONWIP/pull policy.

Deleersnyder et al. (1992) study the integration of push and pull policies in multi-stage linear production/inventory systems under production and demand uncertainty. A kanban policy regulates the flow of material between successive stages, but an MRP type information flow is superimposed. A central controller provides production triggers to certain stations, creating hybrid push-pull work centres. The system is modelled as a discrete time Markov process and the analysis is based on the calculation of state probabilities through an iterative process. The hybrid approach is found to combine the benefits of both push and pull control systems. A similar methodology is also employed by Pandey and Khokhajaikiat (1996) for evaluating different combinations of inventory control policies in a four stages assembly system. Using data from a real system, their analysis concludes that when the upstream stages of the line are operated under raw material availability constraint, a hybrid policy with push control upstream stages and pull control downstream stages yield the best results.

Ahn & Kaminsky (2005) analyze a two stage push-pull system, where downstream operations are initiated only to meet outstanding orders. They assume pure Poisson demand and exponentially distributed lead times and they model the problem as a Markov Decision Process. Their objective is the minimization of the long run average cost and they report counter-intuitive optimal policies.

Cheikhrouhou et al. (2009) propose a system with two product classes with different priorities. They examine a single stage production system with exponentially distributed arrival/production times and backorders. Production control of high priority items is pull type based on kanban philosophy, while production of lower priority items is based on a push type MRP plan. The system is modelled as a Markov birth-and-death queue and an exact performance evaluation based on the steady state balance equations is offered.

Takahashi et al. (2011) consider a two echelon, dual-channel supply chain with exponentially distributed setup times, Poisson demand and lost sales. A saturated manufacturer supplies a wholesaler, which in turn supplies a retailer, or sales directly to end customers. Production and delivery take place when the respective inventories fall below a minimum level and up until a maximum level is reached, while several product units are replenished simultaneously. The system is modelled as a Markov chain and an analysis based on flow balance equations provide numerical calculations of performance measures.

Diamantidis et al. (2016) study a two echelon merge push-pull system consisting of multiple suppliers, a semi-finished goods buffer, a downstream station with parallel machines and a finished goods buffer. Processing times are exponentially distributed. Customers arrive according to a Poisson process, while demand that cannot be met from inventory on hand is lost. The system is modelled as a continuous time - discrete space Markov process and the analysis is based on the calculation of the stationary probabilities. The authors propose an algorithm which takes advantage of the transition matrix structure and provides an analytically derived numerical evaluation of performance measures.

Closer to our line of research are the works of Cuypere et al. (2012) and Diamantidis et al. (2017). Cuypere et al. model a push-pull system with decoupling inventory and backorders. Production of semi-finished products starts when a threshold inventory lever is crossed. The system is a three-dimensional continuous-time Markov chain with infinite state space. The Markov process is a homogeneous quasi-birth-and-death process and the Matrix-Geometric technique is applied to provide the vector of stationary probabilities. The model is used to evaluate the performance of the system under different processing time and demand structures.

Diamantidis et al. (2017) study a system where two reliable stations feed a finite buffer which in its turn serves two retailers working according to continuous review (s,S) inventory control policies. Exponentially distributed times, pure Poisson external demand, and lost sales are assumed. A Markov based algorithm for the numerical evaluation of the steady state probabilities of the system is proposed, while the analytical model is verified with simulation results.

In this work, we propose an exact algorithm for the numerical evaluation of performance measures in a serial, push-pull inventory system. With regard to similar models found in the existing literature, our contribution is two-fold. Firstly, in our study transportation has been modelled as a virtual station. This allows us to better study the relation between inventory and transportation processes (Tempelmeier and Bantel, 2015) and permits more realistic assumptions. In most inventory systems analyzed as a Markov chain, transportation is modelled as a phase of the production process. However, as the interruption of a Markov process by an event is statistically equivalent to the restarting the process (Song, 2013), under certain conditions such a modelling approach can result in significant deviations from real practice. We address this drawback by introducing a virtual transportation station, although at the cost of adding more dimensions to our model. Secondly, we model external demand using compound Poisson. By combining random demand size with stochastic customer arrivals we gain significantly in flexibility and we are in a position to better evaluate the effect of the dynamic nature of the demand on system performance.

5.3 Description of the system

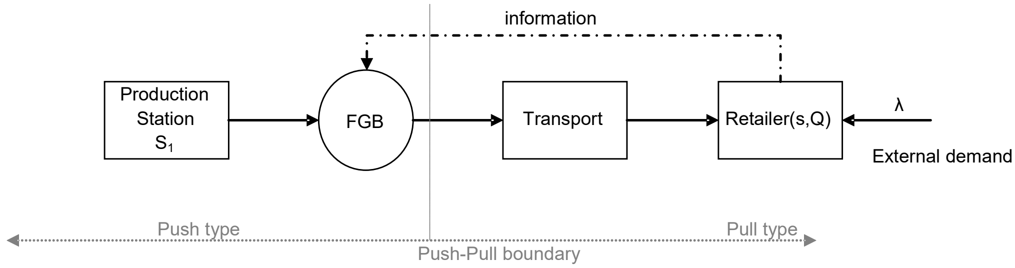


Figure 5.1: System layout

A single product, linear, push-pull supply chain is investigated (Figure 5.1). A reliable station S_1 produces (or administers in the system) product units at a rate μ_1 and exponentially distributed production times. Finished products are stored in a finite Finished Goods Buffer (FGB). Inventory at buffer at time t is denoted by *B(t)*. In the case where S_1 completes processing, but on completion FGB is full, station S_1 blocks (blocking after processing). Station S_1 and the buffer consist the push sub-section of the system. Downstream, the retailer R holds inventory I(t) at time t and faces external demand with compound Poisson characteristics. External customers arrive with exponentially distributed inter-arrival times and each customer's demand is uniformly distributed in the space [1, n].

When the occurring demand is greater than the inventory on hand at the retailer, excess demand is lost. The retailer follows continuous review inventory control policy with parameters (s, Q). When inventory *I(t)* becomes equal to or less than the reorder point s, a replenishment order of *Q* units is placed on the buffer. The actual level of the sent order depends on the available inventory at the buffer. If $B(t) \geq Q$, a full order is dispatched to the retailer. Otherwise, an incomplete order is dispatched. In the case where the FGB is empty, dispatching is suspended until one unit finishes processing at S_1, upon which it is immediately forwarded for transportation to the retailer. Transportation is modelled as a virtual station T. Inventory in transit at time t is denoted by *T(t)*. In the model, transportation is considered independent from both the FGB and the retailer. On transportation initiation, inventory *T(t)* is subtracted from the buffer and remains in the virtual station T until on transportation completion it is added to the inventory of the retailer *I(t)*. Exponentially distributed times for the transportation process are assumed.

To model the system, the following assumptions are also made:

1. At any given time only one order can be in transit from the FGB to the retailer. The one-outstanding-order assumption is common in the literature and it is necessary in order to maintain a tractable level of complexity. In some approaches the assumption is satisfied by assuming Q>s (Bijvank & Vis, 2011). In our approach the assumption is "built into" the transition matrix so that no constraint about *s* and *Q* is necessary.
2. The retailer always orders *Q* units from the FGB.

3. There are always enough raw materials before station S_1 so S_1 never starves.
4. The blocked unit at station S_1 is transferred to the buffer immediately after there is available space and at the same time S_1 resumes production. In the case where Q is greater than buffer capacity, the blocked unit is considered available for transportation to the retailer along with the inventory of the buffer (The blocked unit is considered part of the buffer).
5. Transportation time is independent of the inventory in transit and there are no additional loading and unloading times.
6. All stations are reliable

5.3.1 Model variables

As decision or design variables we denote those parameters of the system whose value a company can usually influence directly in order to achieve the desired outcomes. The determination of these values is part of the company's planning at strategic and tactical level. In the problem under consideration the decision variables concern the capacity of the buffer and the parameters of the inventory policy at the retailer. In detail, the decision variables are:
B: The capacity of the finished goods buffer (FGB).
s: The reorder point at the retailer.
Q: The quantity of the orders requested by the retailer.
All three variables are assumed to be positive integers or zero, with the exception of Q which obviously cannot be zero. Although some scenarios lack physical meaning (for example when $Q > B+1$), for the development of the algorithm no assumptions about the variable values are made.

The other parameters of the model are:
$\boldsymbol{\mu_1}$: The production rate of the Station 1 (exponentially distributed production times).
$\boldsymbol{\mu_2}$: The transfer rate of a replenishment order from the buffer to the retailer (exponentially distributed times).
$\boldsymbol{\lambda}$: The rate of external customers' arrivals (Poisson process).
n: The maximum demand per external customer, assuming a uniform distribution in the space [1,n].

5.4 States definition and state transitions

5.4.1 States definition

The system comprises a 3-dimensional, continuous time - discrete space Markov process $\{B(t), T(t), I(t), t \geq 0\}$. At any given time *t*, the state of the system can be defined by a 3-dimensional vector:

$$\bar{S}_t = (B(t), T(t), I(t))$$

,where:

$B(t)$: The inventory on hand at the FGB at time *t*. $0 \le B(t) \le B+1$, where the case $B(t) = B+1$ corresponds to blocking.
$T(t)$: The inventory in transit from the FGB towards the retailer at time *t*. $0 \le T(t) \le Q$. *T(t)*= 0 means that there is no inventory in transit, while when *T(t)* = *Q* we have a complete order in transit to the retailer. $0 < T(t) < Q$ corresponds to incomplete orders.
$I(t)$: The Inventory on hand at the retailer at time *t*. $0 \le I(t) \le s+Q$.

The state space of the Markov process Ω is comprised of all the possible triplets (B(t), T(t), I(t)) and its dimension depends on B, s and Q. It can be easily proved that for any value of the given parameters, the dimension of the state space is given by:

$$N_{s,Q,B} = (s+1) + (s+2) \cdot Q \cdot (B+2)$$

The states are ordered linearly using the lexicographical ordering (Latouche & Ramaswami, 1999). We take as basic level the subset of all states corresponding to a fixed buffer inventory *B(t)*. Within each level the states are grouped according to the inventory in transit *T(t)*. For fixed basic level and fixed inventory in transit, the states are ordered by inventory at the retailer *I(t)*. To summarize:

State (x, y, z) precedes state (x', y', z') if x<x';
State (x, y, z) precedes state (x, y', z') if y<y';
State (x, y, z) precedes state (x, y, z') if z<z'.

5.4.2 State transitions

The state of the system can be altered instantaneously by three kinds of events. For methodology reasons and without posing any restrictions to our model, it is assumed that no two events can occur at exactly the same time. In infinitesimal time *dt* only one event can occur. The three types of the events are:

1. The completion of processing of one product unit at station S_1. In this case *B(t)* increases by one unit $\left(B(t+dt) = B(t)+1\right)$. In infinitesimal time *dt*, the possibility of the event occurring is $\mu_1 \cdot dt + O(dt)$, where *O(dt)* is an unspecified function such that $\lim_{dt \to 0} \frac{O(dt)}{dt} = 0$. $O(dt)$ stands for the probability that a second event will occur in infinitesimal time *dt*.

2. The arrival of an outstanding order at the retailer. In this case the inventory on hand of the retailer *I(t)* increases by *T(t)* units $\left(I(t+dt) = I(t) + T(t)\right)$. If the value of *I(t+dt)* is not above the reorder point *s*, then a new transfer from FGB is initiated. *T(t+dt)* takes the value of the new order and *B(t+dt)* decreases accordingly. In infinitesimal time *dt*, the possibility of the event occurring is $\mu_2 \cdot dt + O(dt)$.

3. The occurrence of external demand. Each customer may ask for d = 1,2,3,.. or n units. The inventory on hand of the retailer $I(t)$ decreases accordingly $\left(I(t+dt)=\max\left(I(t)-d,0\right)\right)$. If the new inventory is less than or equal to the reorder point s, a replenishment order is given to the FGB. $T(t+dt)$ takes the value of the new inventory in transit and $B(t+dt)$ decreases correspondingly. Each value of d has equal probability of occurrence: $P(d=i)=\frac{1}{n}, 1\le i\le n$. In infinitesimal time dt, the possibility of demand d occurring is $\frac{\lambda}{n}\cdot dt+O(dt)$

5.5 The infinitesimal generator matrix

The infinitesimal generator matrix Q is a matrix such that q_{ij} is the instantaneous transition rate from state i to state j, i≠j, and $q_{ii}=-\sum_{\forall j\neq i}q_{ij}$ (Latouche, 1999). For the system under consideration the infinitesimal generator can be divided into blocks of state transitions, or sub-matrices, which correspond to similar events. In general, there are three tiers of sub-matrices. We denote:

$k=\min(s,Q)$
$f=\max(Q-s,0)$
$h=\min(B+1,Q)$

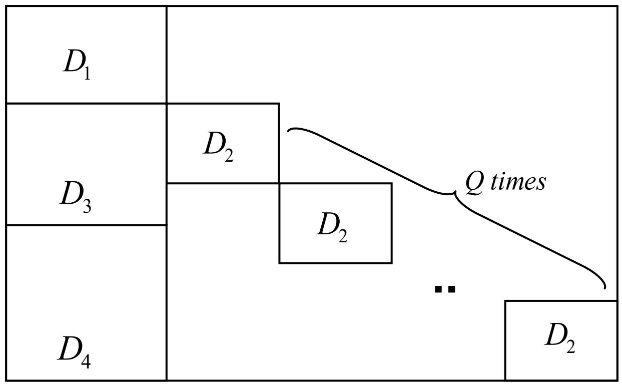
5.5.1 Diagonal sub-matrices

The diagonal tier describes transitions within the same basic level $B(t)$ where no replenishment order towards the retailer is initiated and no product finishes processing at S_1.

The first diagonal sub-matrix D_0 corresponds to the boundary states where there is no inventory in transit (T_t=0), I_t < s and B_t = 0. D_0 is a $(s+1)\times(s+1)$ block. On its diagonal, the first element is ($-\mu_1$) and the rest ($-\mu_1-\lambda$). Above the diagonal all the elements are zero. Bellow the diagonal, in the i^th^ row, there is λ/n from column $max(2,i-n)$ to column $i-1$. In the first column, there are elements from row 2 to row $min(s+1,n+1)$. The element of the i^th^ row is equal to $\frac{n+2-i}{n}\cdot\lambda$.

$$D_0=\begin{bmatrix}-\mu_1 & & & & & & \\ \lambda & -\mu_1-\lambda & & & & & \\ (n-1)\cdot\lambda/n & \lambda/n & -\mu_1-\lambda & & & & \\ (n-2)\cdot\lambda/n & \lambda/n & \lambda/n & -\mu_1-\lambda & & & \\ \cdots & \cdots & \cdots & \cdots & \cdots & & \\ (n-s+2)\cdot\lambda/n & \lambda/n & \lambda/n & \cdots & \lambda/n & -\mu_1-\lambda & \\ (n-s+1)\cdot\lambda/n & \lambda/n & \lambda/n & \lambda/n & \cdots & \lambda/n & -\mu_1-\lambda\end{bmatrix}$$

On the diagonal, the basic repeating block D corresponds to transitions within a given basic level B_t. D is an $(s+2)\cdot Q \times (s+2)\cdot Q$ matrix and it can be further analysed into constituent sub-matrices:



D_1 is a $Q \times Q$ block on the diagonal of D. It corresponds to the occurrence of external demand when no replenishment order is initiated because the inventory on hand at the retailer is higher than the reorder point ($I_{t+dt} > s$). On the diagonal, the elements are equal to ($-\mu_1-\lambda$). Bellow the diagonal, in the i^{th} row there are (λ/n) elements from column *max(1,i-n)* to column *i-1*.

$$D_1 = \begin{bmatrix} -\mu_1-\lambda & & & & & \\ \lambda/n & -\mu_1-\lambda & & & & \\ \lambda/n & \lambda/n & -\mu_1-\lambda & & & \\ \lambda/n & \lambda/n & \lambda/n & \cdots & & \\ \lambda/n & \lambda/n & \lambda/n & \cdots & -\mu_1-\lambda & \\ \lambda/n & \lambda/n & \lambda/n & \cdots & \lambda/n & -\mu_1-\lambda \end{bmatrix}$$

D_2 is a $(s+1)\times(s+1)$ block. It corresponds to the occurrence of external demand when no replenishment order is initiated because there is already one outstanding order ($T_t > 0$). It is repeated Q times on the diagonal of D, each sub-matrix corresponding to a different value of T_t. The first element of the diagonal of D_2 is ($-\mu_1-\mu_2$) and the rest diagonal elements are ($-\mu_1-\mu_2-\lambda$). Above the diagonal all the elements are zero. Bellow the diagonal, in the i^{th} row, there is λ/n from column *max(2,i-n)* to column *i*-1. In the first column, there are elements from row 2 to row *min(s+1,n+1)*. The element of the i^{th} row is equal to $\frac{(n+2-i)\cdot\lambda}{n}$:

$$D_2 = \begin{bmatrix} -\mu_1-\mu_2 & & & & & & \\ \lambda & -\mu_1-\mu_2-\lambda & & & & & \\ (n-1)\cdot\lambda/n & \lambda/n & -\mu_1-\mu_2-\lambda & & & & \\ (n-2)\cdot\lambda/n & \lambda/n & \lambda/n & -\mu_1-\mu_2-\lambda & & & \\ \cdots & \cdots & \cdots & \cdots & \cdots & & \\ (n-s+2)\cdot\lambda/n & \lambda/n & \lambda/n & \cdots & \lambda/n & -\mu_1-\mu_2-\lambda & \\ (n-s+1)\cdot\lambda/n & \lambda/n & \lambda/n & \lambda/n & \cdots & \lambda/n & -\mu_1-\mu_2-\lambda \end{bmatrix}$$

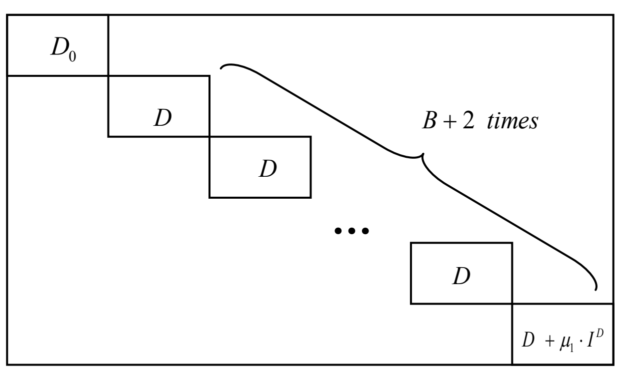
D_3 is a $k\cdot(s+1)\times Q$ block, where k=min(s,Q). It is located just below D_1 and corresponds to the arrival of a replenishment order at the retailer when the new inventory on hand exceeds the reorder point, $I_{t+dt} > s$. D_3 can be divided into k blocks of $(s+1)\times Q$ dimension. The i[th] block consists of (s+1-i) zero lines (corresponding to the arrival of a replenishment order when $I_{t+dt} \leq s$) and a left aligned i×i diagonal matrix of μ_2:

$$D_3 = \left.\begin{bmatrix} \left.\begin{matrix} 0 & & & \\ \cdots & & & \\ & & & \\ \mu_2 & & & \end{matrix}\right\} s+1 \\ \hline \left.\begin{matrix} & & & \\ \cdots & & & \\ \mu_2 & & & \\ & \mu_2 & & \end{matrix}\right\} s+1 \\ \hline \left.\begin{matrix} & & & \\ \cdots & & & \\ \mu_2 & & & \\ & \mu_2 & & \\ & & \mu_2 & \end{matrix}\right\} s+1 \\ \hline \begin{matrix} & \cdots & & \cdots \end{matrix} \\ \hline \left.\begin{matrix} \cdots & & & \\ \left.\begin{matrix} \mu_2 & & & \\ & \mu_2 & \cdots & \\ & & \cdots & \\ & & \mu_2 & \cdots \end{matrix}\right\} k \end{matrix}\right\} s+1 \end{bmatrix}\right\} k\cdot(s+1)$$

D_4 occupies the left down corner of D. It occurs only when Q > s and corresponds to replenishment orders where T_t > s. D_4 is a $(s+1)\cdot f \times Q$ sub-matrix, where $f = \max(Q-s,0)$. It can be divided into Q-s, left aligned diagonal blocks of μ_2 each one corresponding to a different T_t value. Each such sub-block has dimension $(s+1)\times(s+1)$ with the i^th sub-block starting from column i.

$$D_4 = \left[\begin{array}{ccccccc}
\mu_2 & & & & & & \\
 & \mu_2 & & & & & \\
 & & \cdots & & & & \\
 & & & \mu_2 & & & \\
\hline
\underbrace{\quad}_{1} & \mu_2 & & & & & \\
 & & \mu_2 & & & & \\
 & & & \cdots & & & \\
 & & & & \mu_2 & & \\
\hline
 & \cdots & & \cdots & & \cdots & \\
\hline
\underbrace{\qquad\qquad}_{Q-s-1} & & & \mu_2 & & & \\
 & & & & \mu_2 & & \\
 & & & & & \cdots & \\
 & & & & & & \mu_2
\end{array}\right]
\begin{array}{l}
\left.\vphantom{\begin{array}{c}1\\1\\1\\1\end{array}}\right\} s+1 \\
\left.\vphantom{\begin{array}{c}1\\1\\1\\1\end{array}}\right\} s+1 \\
\vphantom{1} \\
\left.\vphantom{\begin{array}{c}1\\1\\1\\1\end{array}}\right\} s+1
\end{array}
\Bigg\} (Q-s)\cdot(s+1)$$

Block D is repeated *B*+2 times in the infinitesimal generator matrix (from B_t=0 to B_t=*B*+1). The last sub-matrix D corresponds to the boundary states where B_t = B +1. In these states the production station S_1 is blocked and therefore there can be no arrivals at the buffer. If I^D the identity matrix of $(s+2)\cdot Q$ dimension, the general structure of the diagonal tier of the infinitesimal generator matrix will be:

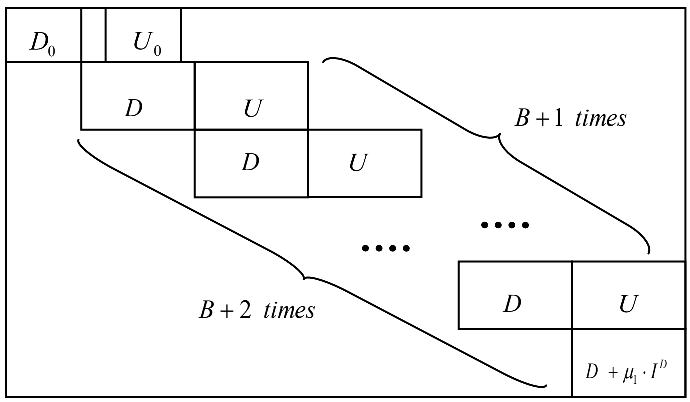


5.5.2 Upper-diagonal blocks

The upper diagonal tier sub-matrices describe arrivals from S_1 to the buffer. Since S_1 processes one product unit at a time, only transitions to adjacent levels occur (skip free to the right). The first upper-diagonal sub-matrix corresponds to the boundary conditions where $T_t = 0$, $I_t < s$ and $B_t = 0$. In these cases, the arriving unit is immediately forwarded for transportation to the retailer. U_0 is a $(s+1)\times(s+1)$ diagonal block of μ_1. The position of its upper left element is at $(1, s+Q+2)$ of the infinitesimal generator matrix.

The repeating block above the diagonal (U) is a $(s+2)\cdot Q$-dimensional diagonal matrix of μ_1. It is repeated $B+1$ times, for buffer inventory from $B_t=0$ to $B_t=B$.

The general structure of the diagonal and upper-diagonal tier of the infinitesimal generator matrix:



5.5.3 Below the diagonal blocks

Sub-diagonal blocks correspond to transitions where there is triggering of a replenishment order from the buffer towards the retailer. The events that can trigger replenishment orders are:

1) The occurrence of external demand such that the new inventory at the retailer does not exceed the reorder point, $I_{t+dt} \le s$.

2) The arrival of a replenishment order when the updated inventory does not exceed the reorder point.

The block below the diagonal L is a $((s+1)\cdot k+Q)\times(s+1)$ matrix and can be divided into two segments. The first Q lines (L_1) correspond to triggering due to external demand and consist of fractions of λ. In the i^{th} row there is λ/n from column *max(*2*, s*+1-*n*+*i)* to column s+1. In the first column, there are non-zero elements in the first *min(Q,n-s)* rows. The first column element of the i^{th} row is equal to $\frac{(n-s-i+1)\cdot\lambda}{n}$:

$$L_1=\begin{bmatrix}
(n-s)\cdot\lambda/n & \lambda/n & \lambda/n & \cdots & \lambda/n & \lambda/n & \lambda/n \\
(n-s-1)\cdot\lambda/n & \lambda/n & \lambda/n & \cdots & \lambda/n & \lambda/n & \lambda/n \\
(n-s-2)\cdot\lambda/n & \lambda/n & \lambda/n & \cdots & \lambda/n & \lambda/n & \lambda/n \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\lambda/n & \lambda/n & \lambda/n & \cdots & \lambda/n & \lambda/n & \lambda/n \\
 & \lambda/n & \lambda/n & \cdots & \lambda/n & \lambda/n & \lambda/n \\
\vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots \\
 & & & \cdots & \lambda/n & \lambda/n & \lambda/n
\end{bmatrix}$$

The rest of the sub-matrix L (L_2) corresponds to triggering due to the arrival of a replenishment order at the retailer when $I_{t+dt} \le s$. The second segment consists of μ_2 elements. L_2 is a $\left((s+1)\cdot k\right)\times(s+1)$ block, $k=min(s,Q)$. It can be divided into k sub-blocks of $(s+1)\times(s+1)$ dimension. The ith sub-block is a right aligned $(s+1-i)\times(s+1-i)$ diagonal matrix of μ_2 followed by i zero lines:

$$L_2=\left[\begin{array}{llllll}
\underbrace{\quad}_{1} & \mu_2 & & & & \\
 & & \mu_2 & & & \\
 & & & \mu_2 & & \\
 & & & & \cdots & \\
 & & & & & \mu_2 \\
 & & & & & \\
\hline
\underbrace{\quad\quad}_{2} & & \mu_2 & & & \\
 & & & \mu_2 & & \\
 & & & & \cdots & \\
 & & & & & \mu_2 \\
 & & & & & \\
\hline
 & & \vdots & \vdots & & \\
\hline
\underbrace{\quad\quad\quad}_{k} & & & \mu_2 & & \\
 & & & & \cdots & \\
 & & & & & \mu_2 \\
 & & & & & \\
 & & & & &
\end{array}\right]
\begin{array}{l}
\left.\vphantom{\begin{array}{l}a\\a\\a\\a\\a\end{array}}\right\}s \\
\left.\vphantom{\begin{array}{l}a\\a\\a\\a\end{array}}\right\}s-1 \\
\vdots \\
\left.\vphantom{\begin{array}{l}a\\a\\a\end{array}}\right\}s+1-k
\end{array}
\quad
\begin{array}{l}
\left.\vphantom{\begin{array}{l}a\\a\\a\\a\\a\\a\end{array}}\right\}s+1 \\
\left.\vphantom{\begin{array}{l}a\\a\\a\\a\\a\end{array}}\right\}s+1 \\
\vdots \\
\left.\vphantom{\begin{array}{l}a\\a\\a\\a\\a\end{array}}\right\}s+1
\end{array}
\quad
\left.\vphantom{\begin{array}{l}a\\a\\a\\a\\a\\a\\a\\a\\a\\a\\a\\a\\a\\a\\a\\a\\a\\a\end{array}}\right\}k\cdot(s+1)$$

$$L=\begin{bmatrix} L_1 \\ L_2 \end{bmatrix}$$

Sub-diagonal blocks may describe transitions between non-adjacent levels and their exact position in the infinitesimal generator matrix depends on the parameters B, s and Q. Sub-matrix L occurs B+2 times. The first occurrence corresponds to the boundary conditions where the buffer is empty. In this case there can be no replenishment order triggering, but the structure of L is the same as in the other instances. The position of the upper left element of the first occurrence of L in the infinitesimal generator is at (s+2,1).

The next $h = min(B+1,Q)$ occurrences correspond to cases where $B_t \leq Q$. Here there is a transition to level 0 (B_t=0). If i the sequencing number of the L matrix ($1 \leq i \leq h$) and defining the position of L by the position of its upper left element (x,y):

$$x = (s+2)\cdot(Q+1) + i\cdot(s+2)\cdot Q$$
$$y = s+2+Q+i\cdot(s+1)$$

The last B+1-h occurrences of L correspond to initiation of complete orders when $B_t > Q$. If i the sequencing number of L blocks for ($h \leq i \leq B$) and (x,y) the position of each upper left element in the infinitesimal generator:

$$x = (s+2)\cdot(Q+1) + i\cdot(s+2)\cdot Q$$
$$y = r + (i-h+1)\cdot(Q+(s+1)\cdot(Q-1)+1) + (i-h)\cdot s$$
$$r = s+2+Q+(h-1)\cdot(s+1)+s$$

5.6 Performance Measures

Our analysis is based on the steady state solution of the system. We denote as X(i) the ith element of the stationary probability vector, which corresponds to the ith state in the hierarchy of states defined according to the rules of paragraph 5.4.1. If **Q** the infinitesimal generator matrix and **X** the vector of the stationary probabilities, then in the steady state:

$$X \cdot Q = 0$$
$$\sum_{i=1}^{N^{s,Q,B}} X(i) = 1$$

From the above a system of linear equations is extracted and the vector X can be computed numerically. Performance measures about the system can be computed algorithmically using the stationary probabilities and taking advantage of the infinitesimal generator matrix structure.

5.6.1 Stock-out probability

Stock-out probability (SO) is the probability of the Retailer having zero inventory on hand.

$$SO = X(1) + \sum_{j=0}^{b+1}\sum_{i=0}^{Q-1} X(r+i\cdot(s+1))$$
$$r = s+Q+2+j\cdot(s+2)\cdot Q$$

5.6.2 Average Inventory in Transit

If T a vector such that T(i) is the probability of inventory in transit being equal to i, then for g=0 to g=Q-1:

$$T(g+1)=\sum_{j=0}^{b+1}\sum_{i=0}^{s}X(r+i)$$

$$r=s+Q+2+(s+1)\cdot g+j\cdot(s+2)\cdot Q$$

, and the average inventory in transit $WIP_{transit}$:

$$WIP_{transit}=\sum_{i=1}^{Q}i\cdot T(i)$$

5.6.3 Utilization of transportation resource

The utilization of transportation resource (u_T) is the probability that there is inventory in transit to the Retailer. It can be easily computed using vector T

$$u_T=\sum_{i=1}^{Q}T(i) \tag{9}$$

5.6.4 Utilization of production station

Blocking is described in the last (s+2)·Q states of the infinitesimal generator matrix. The probability that S_1 is blocked (p_{block})

$$p_{block}=\sum_{i=(s+1)+(s+2)\cdot Q\cdot(b+1)+1}^{(s+1)+(s+2)\cdot Q\cdot(b+2)}X(i)$$

, and the utilization of production station S_1 (u_P)

$$u_P=1-p_{block}$$

5.6.5 Average inventory at the Finished Goods Buffer

Without taking into consideration the cases when S_1 is blocked, if I_{buffer} a vector such that $I_{buffer}(i)$ is the probability of inventory in buffer being equal to i:
For j=0 to B-1:

$$I_{buffer}(j+1)=\sum_{i=r}^{r+(s+2)\cdot Q-1}X(i)$$

$$r=s+1+(s+1)\cdot Q+Q+1+j\cdot(s+2)\cdot Q$$

Taking into consideration the blocking states, the average inventory at buffer (WIP_{buffer}), will be

$$WIP_{buffer}=p_{blocked}\cdot B+\sum_{i=1}^{B}i\cdot I_{buffer}(i)$$

5.6.6 Average inventory at the Retailer

If I_{ro} a Q-dimensional vector recording the probability of different levels of inventory at the retailer while I_t>s, such that I_{ro}(j) is the probability that I_t=s+j, for j=1 to Q:

$$I_{ro}(j) = \sum_{i=0}^{b+1} X(s+1+j+i\cdot(s+2)\cdot Q)$$

In a similar way, if $\mathbf{I_{ru}}$ is an s-dimensional vector recording the probability of different levels of inventory at the retailer while I(t)≤s, such that $I_{ru}(i)$ is the probability that I_t= i, for g=1 to s:

$$I_{ru}(g) = X(g+1) + \sum_{j=0}^{b+1}\sum_{i=0}^{Q-1} X(s+2+g+Q+j\cdot(s+2)\cdot Q+i\cdot(s+1))$$

, and the average inventory at the retailer ($WIP_{retailer}$)

$$WIP_{retailer} = \sum_{i=1}^{s} i\cdot I_{ru}(i) + \sum_{i=1}^{Q}(s+i)\cdot I_{ro}(i)$$

Combining vectors I_{ru} and I_{ro} we construct vector $\mathbf{I_{retailer}}$ such that $I_{retailer}(i)$ corresponds to the probability of inventory on hand at the retailer being i units.

5.6.7 Order Fill Rate

Order Fill Rate (FR) is the percentage of external customers whose demand is fully met by the inventory on hand at the Retailer. If **C** is a n-dimensional vector such that C(i) corresponds to the probability of I_t≥i, then

$$C(i) = \begin{cases} \sum_{j=i}^{s+Q} I_{retailer}(j), i \le s+Q \\ 0, i > s+Q \end{cases}$$

$$FR = \sum_{i=1}^{n} \frac{1}{n}\cdot C(i)$$

5.6.8 Average lost sales

Average lost sales (ALS) are the average lost sales per external order at the retailer.

$$ALS = \sum_{i=1}^{w} \frac{i}{n}\cdot[SO + \sum_{j=1}^{s+Q} I_{retailer}(j)] + \sum_{i=w+1}^{n} \frac{i}{n}\cdot[SO + \sum_{j=1}^{y+w+1-i} I_{retailer}(j)]$$

$$w = \max(0, n-s-Q-1)$$

$$y = \min(n-1, s+Q)$$

Average lost sales per lost order (Lost_Sales) is the average lost sales per order partially met or not met at all from the inventory on hand at the retailer.

$$Lost\_Sales = \frac{ALS}{1-FR}$$

5.6.9 Type II Service Level

Type II Service Level or Service Level (SL_2) is the percentage of total external demand (in terms of product units) that is met from the inventory on hand at the retailer (Brandimarte & Zotteri, 2007). If E the average demand per external customer,

$$SL_2 = \frac{E - ALS}{E} = \frac{\frac{n+1}{2} - ALS}{\frac{n+1}{2}}$$

5.6.10 Replenishment order rate

The replenishment order rate (ROR) is the number of replenishment orders from the buffer to the retailer per unit of time.

$$ROR = \frac{\lambda \cdot (n+1) \cdot SL_2 \cdot u_T}{2 \cdot WIP_{transit}}$$

5.7 Illustrative example

To illustrate the algorithm described above, we present the analysis for a simple example for buffer capacity B = 2, Reorder point s = 1, Order quantity Q = 2, and maximum demand per customer n=3.

5.7.1 States definition and state transitions

5.7.1.1 States definition

At any given time *t*, the state of the system can be defined by a 3-dimensional vector:

$$\bar{S}_t = (B_t, T_t, I_t)$$

B_t is the inventory on hand at the Finished Goods Buffer at time *t*. $B_t = \{0,1,2,3\}$. B_t=3 corresponds to blocking, when there are 2 units in the buffer and one finished unit blocked in station 1.

T_t is the inventory in transit from FGB towards the retailer at time *t*. $T_t = \{0,1,2\}$. T_t= 0 means that there is no inventory in transit, either because the inventory at the retailer exceeds the reorder point, or because there is stock-out at the buffer.

I_t is the inventory on hand at the retailer at time *t*. $I_t = \{0,1,2,3\}$.

The state space Ω of the Markov process is comprised of all permissible $\bar{S}_t$ vectors. In the example under consideration there are 26 possible states. These states are ordered linearly, using a lexicographical ordering and moving from lower to higher values. First the states are ordered according to the basic level B_t, Within each basic level the ordering is done based on inventory in transit T_t. Finally, for given B_t and T_t, the states are order according to the inventory at the retailer I_t. The states definition is independent of demand characteristics (n). The possible states and their respective hierarchy are given in figure 5.2

S/N	State	Station 1	B_t	T_t	I_t
1	000	busy	0	0	0
2	001	busy	0	0	1
3	002	busy	0	0	2
4	003	busy	0	0	3
5	010	busy	0	1	0
6	011	busy	0	1	1
7	020	busy	0	2	0
8	021	busy	0	2	1
9	102	busy	1	0	2
10	103	busy	1	0	3
11	110	busy	1	1	0
12	111	busy	1	1	1
13	120	busy	1	2	0
14	121	busy	1	2	1
15	202	busy	2	0	2
16	203	busy	2	0	3
17	210	busy	2	1	0
18	211	busy	2	1	1
19	220	busy	2	2	0
20	221	busy	2	2	1
21	302	blocked	3	0	2
22	303	blocked	3	0	3
23	310	blocked	3	1	0
24	311	blocked	3	1	1
25	320	blocked	3	2	0
26	321	blocked	3	2	1

Figure 5.2: States definition and hierarchy for B=2, s=1, Q=2

5.7.1.2 State transitions

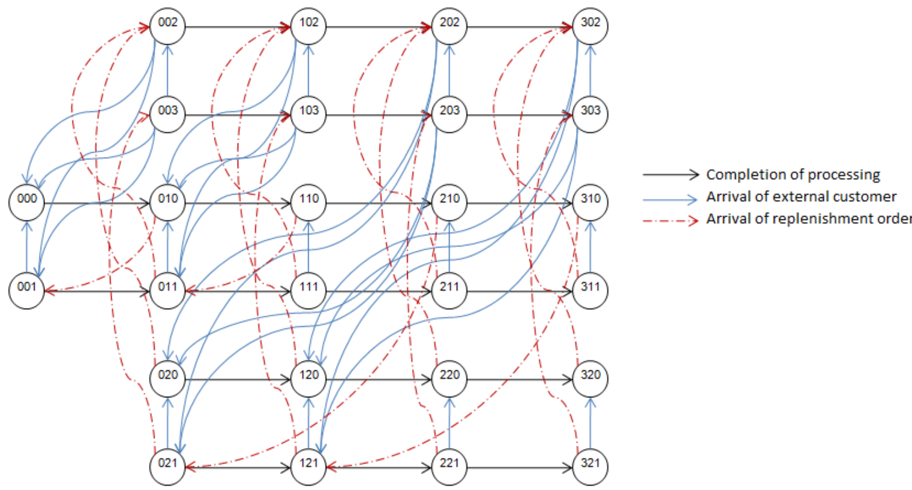
We assume that in infinitesimal time *dt* only one event may occur. The events that instantaneously change the state of the system are:

1. The completion of processing of one product unit at station S_1. Units are processed one at a time, so always the available inventory at the buffer will be increased by one unit. If $B_t=0$ and $I_t \leq 1$, then the finished unit will immediately forwarder for transportation towards the retailer. Otherwise, the finished product remains at the buffer $\left(B_{t+dt} = B_t + 1\right)$. If the buffer is full ($B_t=2$), the finished product after processing remains in the production station which blocks. The instantaneous transition rate of the event is μ_1.
2. The arrival of an outstanding order at the retailer. In this case the inventory on hand of the retailer I_t increases by T_t units $\left(I_{t+dt} = I_t + T_t\right)$. Inventory in transit can be 1 (partial replenishment order) or 2 (full order) product units. If the new inventory at the retailer does

not exceed s=1 and there is inventory at the buffer, a new replenishment order is initiated. The instantaneous transition rate of the event is μ_2.

3. The occurrence of external demand. Each customer may ask for d = 1,2, or 3 units with equal probability of 1/3. The inventory on hand of the retailer I_t decreases accordingly. If $d > I_t$, the external demand is partially met. If $I_{t+dt} \leq 1$, a replenishment order is given to the FGB. T_{t+dt} takes the value of the new inventory in transit and B_{t+dt} decreases correspondingly. The instantaneous transition rate for demand d occurring is $\lambda/3$.

The transition diagram of the system under consideration:



5.7.2 The Infinitesimal Generator Matrix

$k = 1$

$f = 1$

$h = 2$

5.7.2.1 Diagonal sub-matrices

$$D_0 = \begin{bmatrix} -\mu_1 & 0 \\ \lambda & -\mu_1 - \lambda \end{bmatrix}$$

$$D_1 = \begin{bmatrix} -\mu_1 - \lambda & 0 \\ \lambda/3 & -\mu_1 - \lambda \end{bmatrix}$$

$$D_2 = \begin{bmatrix} -\mu_1 - \mu_2 & 0 \\ \lambda & -\mu_1 - \mu_2 - \lambda \end{bmatrix}$$

$$D_3 = \begin{bmatrix} 0 & 0 \\ \mu_2 & 0 \end{bmatrix}$$

$$D_4 = \begin{bmatrix} \mu_2 & 0 \\ 0 & \mu_2 \end{bmatrix}$$

$$D = \begin{bmatrix} -\mu_1 - \lambda & 0 & 0 & 0 & 0 & 0 \\ \lambda/3 & -\mu_1 - \lambda & 0 & 0 & 0 & 0 \\ 0 & 0 & -\mu_1 - \mu_2 & 0 & 0 & 0 \\ \mu_2 & 0 & \lambda & -\mu_1 - \mu_2 - \lambda & 0 & 0 \\ \mu_2 & 0 & 0 & 0 & -\mu_1 - \mu_2 & 0 \\ 0 & \mu_2 & 0 & 0 & \lambda & -\mu_1 - \mu_2 - \lambda \end{bmatrix}$$

Block D is repeated 4 times in the infinitesimal generator matrix. The last sub-matrix D corresponds to the boundary states where the production station is blocked, therefore μ_1 is added to the diagonal elements.

5.7.2.2 Upper-diagonal blocks

U_0 is a 2×2 diagonal block of μ_1. The position of its upper left element is at (1, 5) of the infinitesimal generator matrix.

$$U_0 = \begin{bmatrix} \mu_1 & 0 \\ 0 & \mu_1 \end{bmatrix}$$

U is a 6×6 diagonal matrix of μ_1. It is repeated 3 times.

$$U = \begin{bmatrix} \mu_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & \mu_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & \mu_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \mu_1 & 0 & 0 \\ 0 & 0 & 0 & 0 & \mu_1 & 0 \\ 0 & 0 & 0 & 0 & 0 & \mu_1 \end{bmatrix}$$

5.7.2.3 Below the diagonal blocks

L is a 4×2 matrix and can be divided into two segments. The first 2 lines (L_1) correspond to triggering due to external demand. The rest of the sub-matrix L (L_2) corresponds to triggering due to the arrival of a replenishment order at the retailer when $I_{t+dt} \leq 1$

$$
L = \begin{bmatrix} 2\lambda/3 & \lambda/3 \\ \lambda/3 & \lambda/3 \\ 0 & \mu_2 \\ 0 & 0 \end{bmatrix}
$$

The overall structure of the infinitesimal generator matrix will be:

States	000	001	002	003	010	011	020	021	102	103	110	111	120	121	202	203	210	211	220	221	302	303	310	311	320	321
000	-μ1				μ1																					
001	λ	-μ1-λ				μ1																				
002	2λ/3	λ/3	-μ1-λ						μ1																	
003	λ/3	λ/3	λ/3	-μ1-λ						μ1																
010		μ2			-μ1-μ2						μ1															
011			μ2		λ	-μ1-μ2-λ						μ1														
020			μ2				-μ1-μ2						μ1													
021				μ2			λ	-μ1-μ2-λ						μ1												
102					2λ/3	λ/3			-μ1-λ						μ1											
103					λ/3	λ/3			λ/3	-μ1-λ						μ1										
110						μ2					-μ1-μ2						μ1									
111									μ2		λ	-μ1-μ2-λ						μ1								
120									μ2				-μ1-μ2						μ1							
121										μ2			λ	-μ1-μ2-λ						μ1						
202							2λ/3	λ/3							-μ1-λ						μ1					
203							λ/3	λ/3							λ/3	-μ1-λ						μ1				
210								μ2									-μ1-μ2						μ1			
211															μ2		λ	-μ1-μ2-λ						μ1		
220															μ2				-μ1-μ2						μ1	
221																μ2			λ	-μ1-μ2-λ						μ1
302													2λ/3	λ/3							-λ					
303													λ/3	λ/3							λ/3	-λ				
310														μ2									-μ2			
311																					μ2		λ	-μ2-λ		
320																					μ2				-μ2	
321																						μ2			λ	-μ2-λ

5.7.3 Performance measures

Stock-out probability

$$SO = X(1) + X(5) + X(7) + X(11) + X(13) + X(17) + X(19) + X(23) + X(25)$$

Average Inventory in Transit

The probability that inventory in transit towards the retailer is 1:

$$p(T_t = 1) = X(5) + X(6) + X(11) + X(12) + X(17) + X(18) + X(23) + X(24)$$

The probability that inventory in transit towards the retailer is 2:

$$p(T_t = 2) = X(7) + X(8) + X(13) + X(14) + X(19) + X(20) + X(25) + X(26)$$

The average inventory in transit $\mathrm{WIP}_{transit}$:

$$WIP_{transit} = 1 \cdot p(T_t = 1) + 2 \cdot p(T_t = 2)$$

Utilization of transportation resource

$$u_T = p(T_t = 1) + p(T_t = 2)$$

(9)

Utilization of production station

The probability that S_1 is blocked:

$$p_{block} = \sum_{i=21}^{26} X(i)$$

The utilization of production station S_1:

$$u_P = 1 - p_{block}$$

Average inventory at the Finished Goods Buffer

The probability that inventory at FGB is 1:

$$p(I_{buffer} = 1) = \sum_{i=9}^{14} X(i)$$

Without taking into consideration the states where S_1 is blocked, the probability that inventory at the FGB is 2:

$$p(I_{buffer} = 2) = \sum_{i=15}^{20} X(i)$$

The average inventory at the buffer:

$$WIP_{buffer} = 2 \cdot p_{blocked} + 1 \cdot p(I_{buffer} = 1) + 2 \cdot p(I_{buffer} = 2)$$

Average inventory at the Retailer

The probability that inventory at the retailer is 1:

$$p(I_r = 1) = X(2) + X(6) + X(8) + X(12) + X(14) + X(18) + X(20) + X(24) + X(26)$$

The probability that inventory at the retailer is 2:

$$p(I_r = 2) = X(3) + X(9) + X(15) + X(21)$$

The probability that inventory at the retailer is 3:

$$p(I_r = 3) = X(4) + X(10) + X(16) + X(22)$$

The average inventory at the retailer:

$$WIP_{retailer} = 1 \cdot p(I_r = 1) + 2 \cdot p(I_r = 2) + 3 \cdot p(I_r = 3)$$

Order Fill Rate

Order Fill Rate (FR) is the percentage of external customers whose demand is fully met by the inventory on hand at the Retailer. The probability that the inventory at the retailer is equal to or greater than i, i=1,2,3:

$$p(I_r \geq 1) = p(I_r = 1) + p(I_r = 2) + p(I_r = 3)$$

$$p(I_r \geq 2) = p(I_r = 2) + p(I_r = 3)$$

$$p(I_r \geq 3) = p(I_r = 3)$$

$$FR = \frac{1}{3} \cdot p(I_r \geq 1) + \frac{1}{3} \cdot p(I_r \geq 2) + \frac{1}{3} \cdot p(I_r \geq 3)$$

Average lost sales per external order

$$ALS = 1 \cdot \left(p(d=1 \mid I_r = 0) + p(d=2 \mid I_r = 1) + p(d=3 \mid I_r = 2)\right) + 2 \cdot \left(p(d=2 \mid I_r = 0) + p(d=3 \mid I_r = 1)\right) + 3 \cdot \left(p(d=3 \mid I_r = 0)\right)$$

$$ALS = \frac{1}{3}\cdot\left(SO + p(I_r = 1) + p(I_r = 2)\right) + \frac{2}{3}\cdot\left(SO + p(I_r = 1)\right) + \frac{3}{3}\cdot SO$$

$$SL_2 = \frac{2 - ALS}{2}$$

The rest of the performance measures can be computed from the above through simple relations.

5.7.4 Validation of algorithmic results

The system of linear equations for the system was constructed manually and solved in Mathematica to get the vector of stationary probabilities. Then the performance measures of the system were calculated as described in the section 5.7.3. The results were practically identical to those produced algorithmically. The algorithmic results were also contrasted to simulation results (see section 5.8). Three replications of 2000000 time units each were used. The algorithmic results were within the confidence interval provided by simulation. Arithmetic values for parameters μ_1=0.8, μ_2=0.6 and λ=1, are given below:

Performance measure	Algorithm - Matlab	Manually- Mathematica	Simulation – Arena (95% C.I.)
FR	0.235160	0.235160	0.236 ± 0.001
SL_2	0.296897	0.296897	0.297 ± 0.001
$WIP_{transit}$	0.989656	0.989656	0.990 ± 0.001
WIP_{buffer}	1.112643	1.112643	1.113 ± 0.001
$WIP_{retailer}$	0.705480	0.705479	0.706 ± 0.001
p_{block}	0.257758	0.257758	0.258 ± 0.001

5.8 Validation of the model

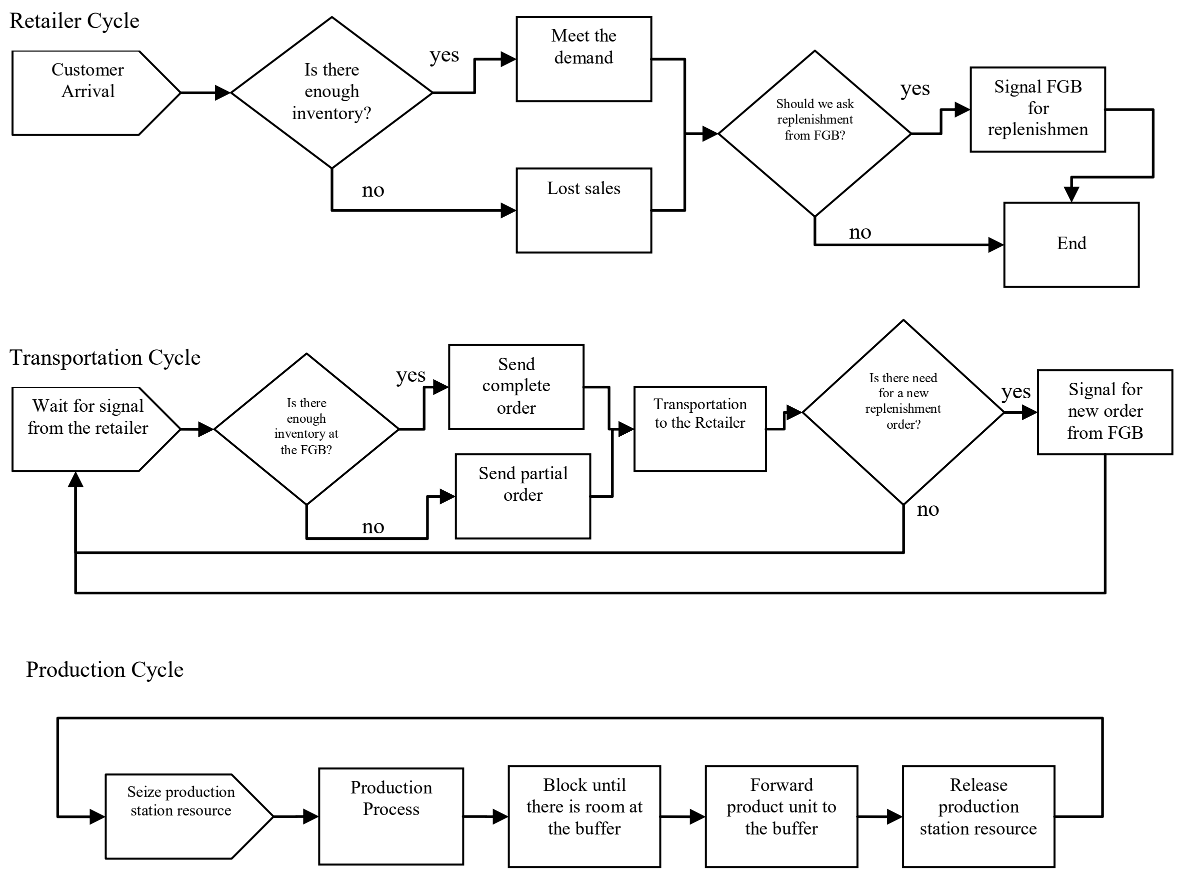
5.8.1 Simulation Model

Simple examples were solved manually using Mathematica and in every case the results were identical to those produced algorithmically. However, for a more rigorous testing of the algorithm such an approach is not practical. To check the validity of the developed algorithm, a simulation model of the system under consideration was developed. The system was modeled as a series of cycles, each cycle describing the interface between successive members of the network. The basic logic of the simulation model is given in figure 5.3.

The simulation model was constructed in Arena simulation package, Version 12.00.00 – CPR 9. Test runs were executed to determine the specific parameters of the simulation that would give statistically rigorous results within a reasonable computation time. A simulation time of 2000000 time units was deemed long enough for our purpose. To eliminate any effects of the initial conditions, a warm-up period of 50000 time units was also selected.

Seven different performance measures were in included in the analysis: Order fill rate (FR), average inventory at the retailer ($WIP_{retailer}$), average inventory in transit towards the retailer ($WIP_{transit}$), average inventory at the buffer (WIP_{buffer}), the percentage of time that the production station is blocked ($p_{blocked}$), the lost sales per lost order (LS), and the service level in terms of product units (SL_2).

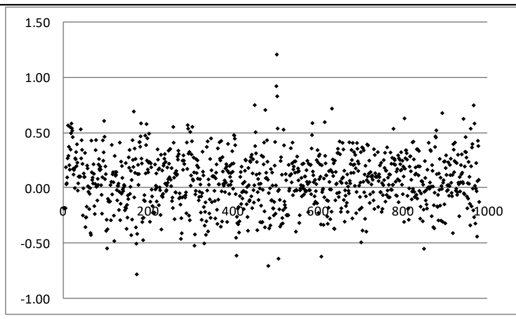
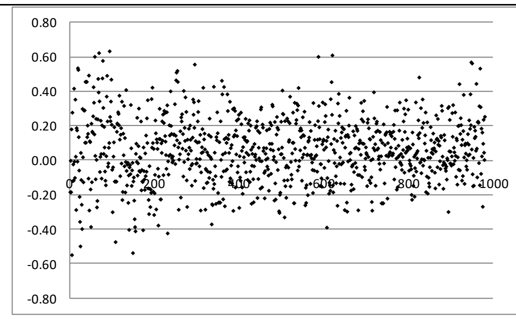
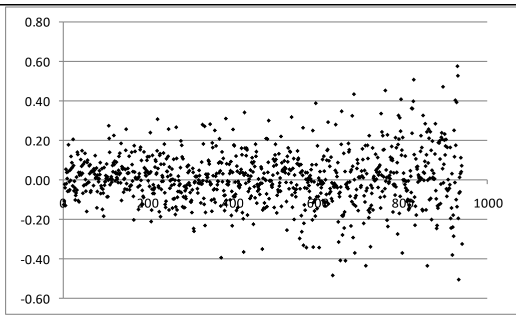
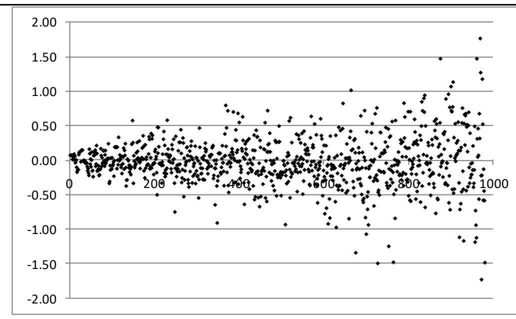
Figure 5.3: Simulation model logic

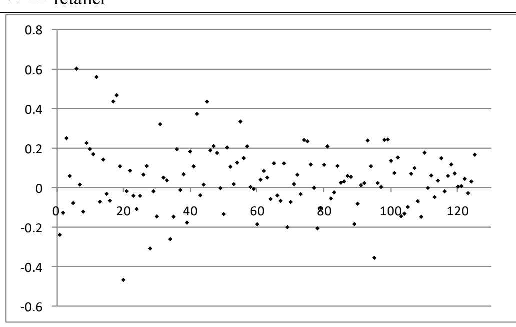
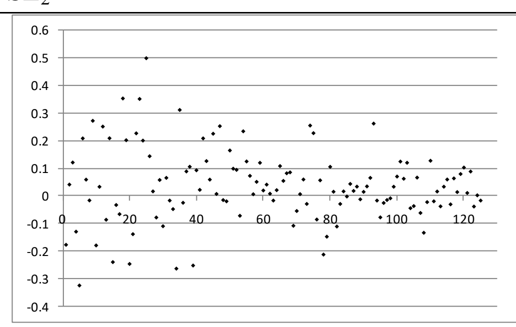
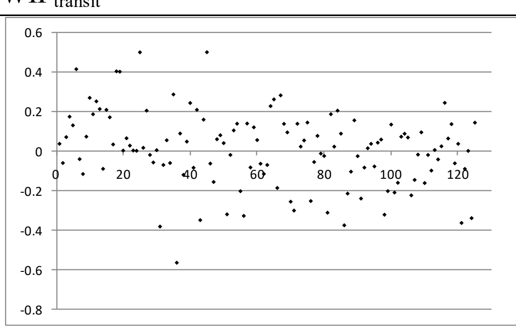
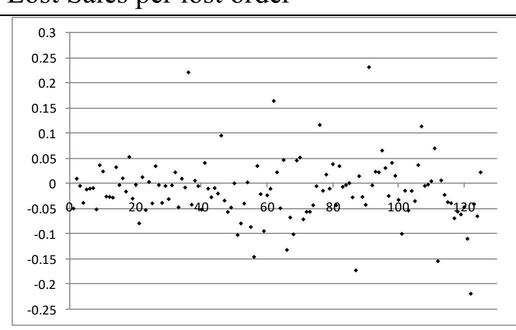


5.8.2 Simulation Results

In all, more than 1360 different scenarios were tested for various combinations of B, s and Q, as well as for different μ_1, μ_2 and λ relations. Simulation results were consistent with the results from the analytic algorithm. For the seven different performance measures that were tested, across all scenarios, the difference between analytic and simulation results in absolute values was typically of 10^{-3} order, which corresponds to the significant digits of the simulation results. The differences observed were well within the limits of the expected variability due to the statistical nature of simulation. Some results are given graphically below, while a sample of the data is given in the Appendix. In the diagrams we give the deviation as a percentage difference between the analytical method and simulation:

$$\% \, deviation = 100 \times \frac{analytic - simulation}{analytic}$$

<table>
<tr><td colspan="2">% Difference of Simulation and Analytic Results. μ_1=0.5, μ_2=0.2, λ=0.5, 0≤B≤6, 0≤s≤6, 1≤Q≤B+1, 1≤n≤5.</td></tr>
<tr><td>$WIP_{retailer}$</td><td>SL_2</td></tr>
<tr><td>
</td><td>
</td></tr>
<tr><td>WIP_{buffer}</td><td>$p_{blocked}$</td></tr>
<tr><td>
</td><td>
</td></tr>
</table>

<table>
<tr><td colspan="2">% Difference of Simulation and Analytic Results. B=5, s=2, Q=3, n=2, 0.5≤ μ_1≤2.5 (step:0.5), 0,5≤ μ_2≤2.5 (step:0.5), 0.5≤ λ ≤2.5 (step:0.5), 1≤n≤5.</td></tr>
<tr><td>$WIP_{retailer}$</td><td>SL_2</td></tr>
<tr><td>
</td><td>
</td></tr>
<tr><td>$WIP_{transit}$</td><td>Lost Sales per lost order</td></tr>
<tr><td>
</td><td>
</td></tr>
</table>

5.9 Model Performance and limitations

The algorithm was programmed in Matlab, version 2018a, 9.4.0.813654. For the runs commented here a computer with Core i-3-4005U CPU at 1.70 GHz processor and 4GB installed RAM was used. Its operating system was Windows 7 – Ultimate, 64-bit.

The proposed algorithm is valid for any combination of the decision variables and for any given system parameters. However, as the systems under consideration become bigger and the dimension of the infinitesimal generator matrix increases, the algorithm becomes computationally demanding. This is a common drawback with models based on Markov analysis (Mehmood and Lu, 2011), and although the problem is alleviated with rising computational power, it still imposes limitations to the application of exact Markov models in real scale systems.

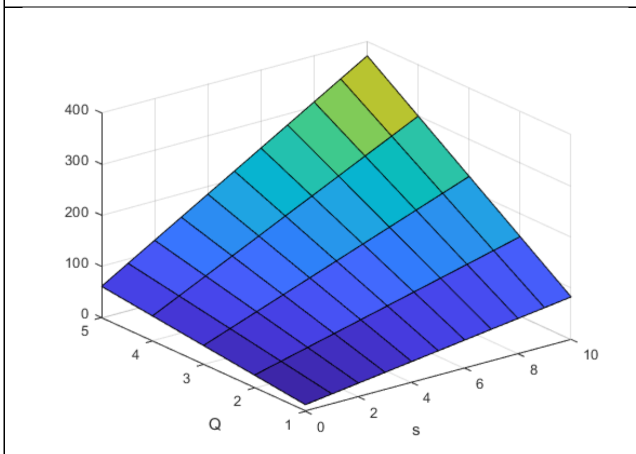
The dimension N of the infinitesimal generator is a function of *B*, *s*, and *Q*:

$$N = (s+1) + (s+2) \cdot Q \cdot (B+2) \,.$$

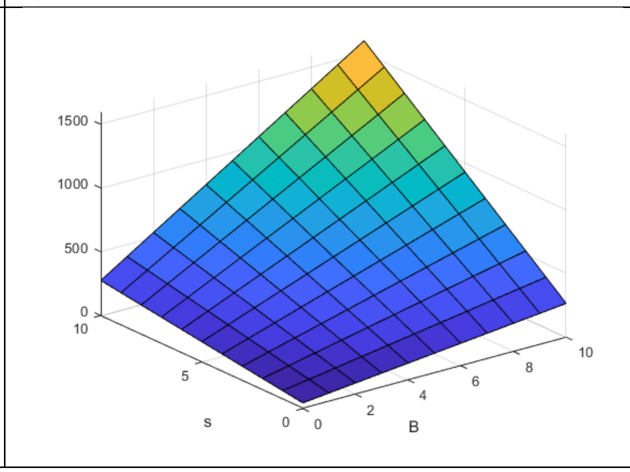
In the figure below some examples are given of the number of states in relation to the decision variables:

Figure 5.4: Number of states as a function of B,s,Q

B=4

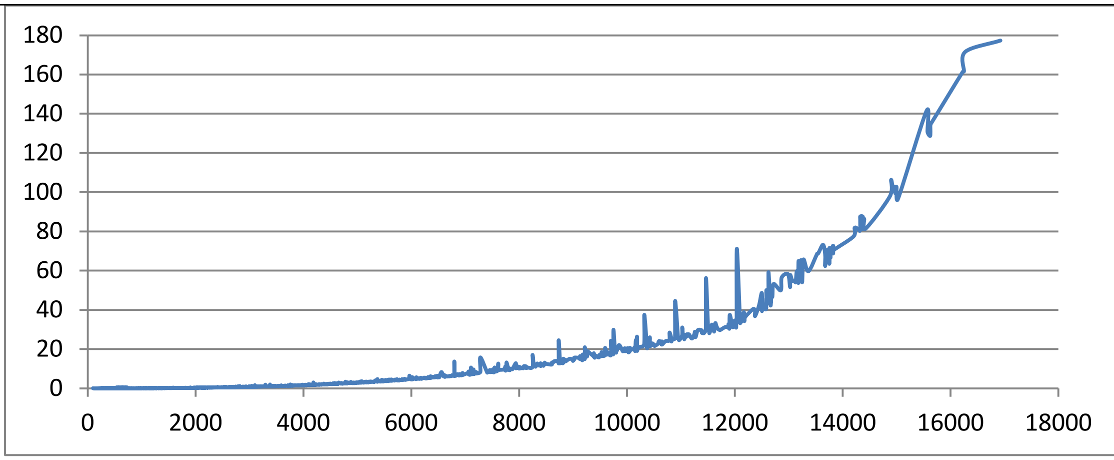


Q=1



Computational time depends mainly on the number of possible states. The exact relation between the decision variables affects the number of steps that are required for the solution of the system, so it also has an effect on computational time. As a general trend the computational time increases with increasing number of states:

Computational time (sec) as a function of the infinitesimal generator matrix dimension



The size of the systems that can be evaluated is restricted by RAM memory requirements, which in its turn depends on the dimensions of the infinitesimal generator matrix. Even in a computer of moderate performance, the developed model offers a satisfactory degree of flexibility as problems with as many as 25229 states were solve in the testing computer. Both algorithm efficiency (number of steps to the solution) and memory consumption can be improved by rephrasing the computer program and by exploiting embedded features of Matlab such as sparse matrices. However, at this stage our priority is the tractability of the computer program in relation to the theory mentioned in the preceding sections. In any case, the problem of increasing system states with increasing decision variable values would persist.

The proposed algorithm offers certain advantages compared to alternative approaches such as simulation. Even for relatively big systems, the exact algorithm is significantly faster than simulation, in most cases the difference in computation time being several orders of magnitude. Moreover, the exact solution poses no limits on precision. This can be especially helpful in cases where low values of the performance measures are concerned, where the specific value may be comparable to the margin of error. Finally, the algorithm can be more easily integrated with other features, as for example, in the context of an optimization model.

5.10 Numerical results

The variables of the system can be characterized as decision variables (B, s and Q) and non-controllable parameters (λ, n, μ_1, μ_2). To better focus on the effects of different policies, we limit our investigation in systems where the non-controllable parameters balance supply and demand. The average total lead time for incoming products to the retailer ($1/\mu_1 + 1/\mu_2$) is equal to the average time between successive external demands for two product units. Customers arrival rate λ and demand variability n are changed simultaneously so that the average external demand in terms of product units remains constant ($\lambda \cdot E$ = const., E: the average demand per external customer). For our analysis $\mu_1=\mu_2=\lambda \cdot (n+1)$. Some data are given in the Appendix.

5.10.1 The effect of the decision variables on the performance measures

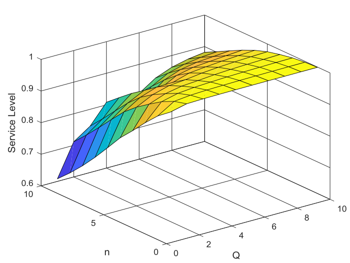
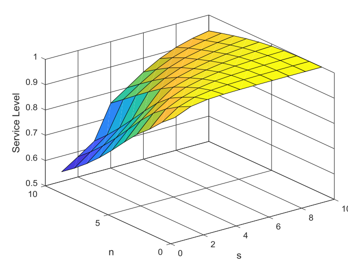
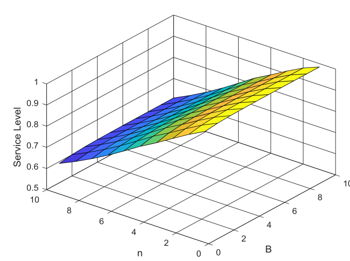
We investigate along two main directions. In 5.10.1.1 we study the joint effect of external demand characteristics (n) with each one of the decision variables. In each case, the other decision variables remain constant. In 5.10.1.2 the joint effect per two decision variables is studied, for given values of the third decision variable and the non-controllable parameters. We analyze scenarios where B, s, Q, and n take values in the range [1, 10]. Some scenarios are given graphically as examples.

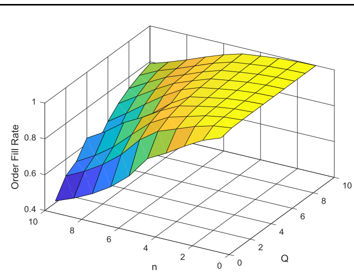
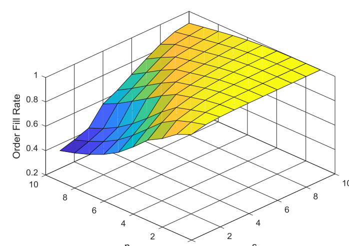
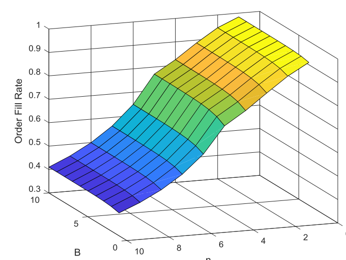
5.10.1.1 Joint effect of demand variability and the decision variables

Our objective is to get insight of the effect of demand characteristics on system performance. We investigate the joint effect of demand variability and each of the decision variables on a range of system performance measures.

Service level and Order Fill Rate

The demand characteristics have a greater impact on Service Level than the decision variables. Increasing the value of B, s and Q tends to offset the effect of increased demand variability, and the importance of each decision variable increases with increasing n. From the decision variables, the reorder point s was found to have the greatest positive impact, followed by reorder quantity Q and then the buffer capacity B.

Joint effect of n and the decision variables on Service Level		
SL_2(n, Q), s=4, B=10	SL_2(n, s), B=5, Q=4	SL_2(n, B), s=4, Q=1
		

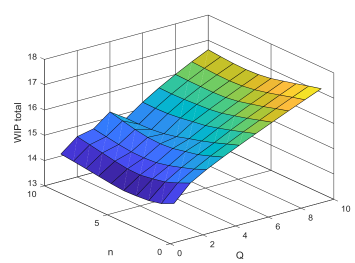
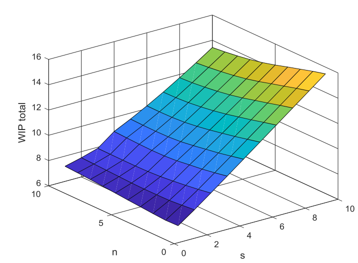
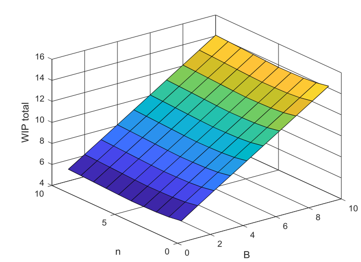
Joint effect of n and the decision variables on Order Fill Rate		
FR(n, Q), s=4, B=10	FR(n,s), B=5, Q=4	FR(n,B), s=4, Q=1
		

Order Fill Rate exhibits a similar behavior to that of Service Level. Again, demand variability n is the most important parameter in determining the value of the performance measure. With regard to the decision variables, reorder point s has the greatest impact, followed by reorder quantity Q and then buffer capacity B. It should be noted that the value of Q is constrained by that of B, with the maximum practical Q value being B+1.

In general, the ability of the system to achieve an appropriate level of customer satisfaction depends heavily on demand characteristics. From a managerial point of view it would be rational for the business to seek selling policies that smooth out demand.

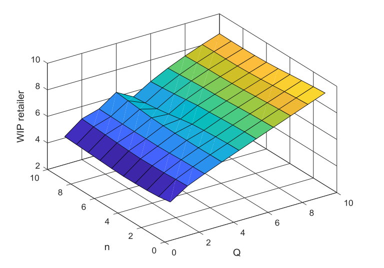
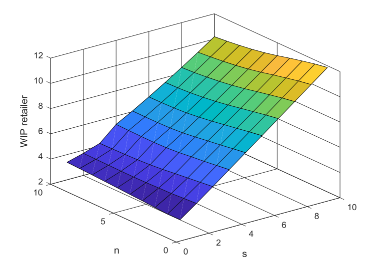
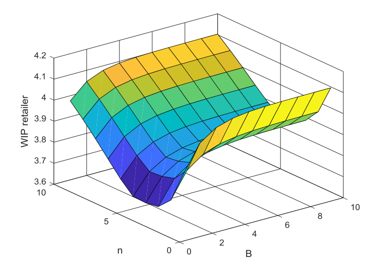
Average total inventory

When only one decision variable is allowed to change, buffer capacity B has the greatest impact on WIP_{total}, followed by reorder point s and then by reorder quantity Q. The effect of B can be described quite accurately with a linear relation (coefficient of determination R^2 above 0.98), with every added slot in the buffer contributing an almost equivalent increment to total inventory. Increased demand variability causes WIP_{total} to decrease, but this is rather a trend and does not hold across all scenarios. In any case, the effect of the decision variables is far more powerful than that of n.

Joint effect of n and the decision variables on WIP_{total}		
WIP_{total} (n,Q), s=4, B=10	WIP_{total} (n,s), B=4, Q=4	WIP_{total} (n,B), s=4, Q=1
		

Average inventory at the retailer

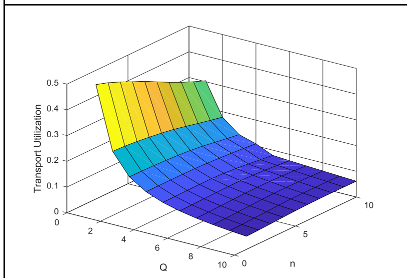
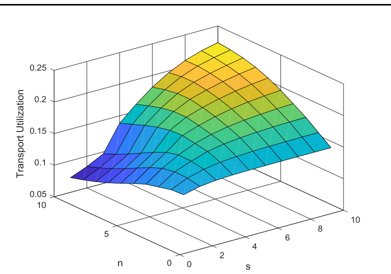
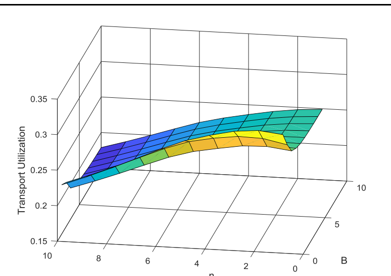
The decision variables B, s, and Q are positively correlated with the average inventory at the retailer. The reorder point s has the greatest impact, while buffer capacity B is the least important. The effect of demand variability is less straightforward due to the dynamic relationship of the parameters of the system. The general trend is for $WIP_{retailer}$ to decrease with increasing n, but for some scenarios a minimum $WIP_{retailer}$ was observed for intermediate n values. In general, the effect of the decision variables is more important than that of demand variability.

<table>
<tr><td colspan="3">Joint effect of n and the decision variables on $WIP_{retailer}$</td></tr>
<tr><td>$WIP_{retailer}$ (n, Q), s=4, B=10</td><td>$WIP_{retailer}$ (n, s), B=4, Q=4</td><td>$WIP_{retailer}$ (n, B), s=4, Q=1</td></tr>
<tr><td>
</td><td>
</td><td>
</td></tr>
</table>

Utilization of Resources

Predictably, transport resource utilization (u_T) is negatively correlated with the reorder quantity Q, as low Q means more often replenishment orders. B is also negatively correlated with transport resource utilization. Higher buffer capacity allows for higher levels of inventory at the buffer and a corresponding decrease in the percentage of lost sales at the buffer (replenishment orders bellow Q) with the result that a given throughput can be achieved with fewer replenishment orders. The effect of n decreases with increasing Q and B values.

With regard to s, there is a positive correlation with u_T, possibly a result of increasing service level. The reorder point also affects the behavior of the performance measure with changing n. For low s values, transport utilization decreases with increasing n, while for higher values the correlation becomes positive.

<table>
<tr><td colspan="3">Joint effect of n and the decision variables on u_t</td></tr>
<tr><td>u_t (n,Q), s=4, B=10</td><td>u_t (n,s), B=4, Q=4</td><td>u_t (n,B), s=4, Q=2</td></tr>
<tr><td>
</td><td>
</td><td>
</td></tr>
</table>

Production station utilization (u_P) is related to service level:

$$u_P = \frac{E \cdot \lambda}{\mu_1} SL_2$$

For our analysis we have chosen parameters such that $E \cdot \lambda$ is constant, hence u_p is in linear relation with SL_2. The two performance measures exhibit similar behaviour with respect to the effect of B, s, Q and n.

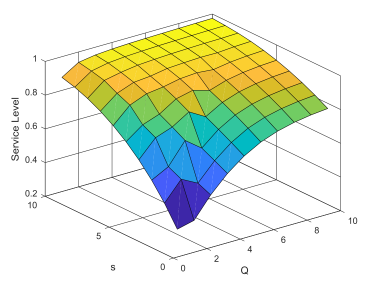
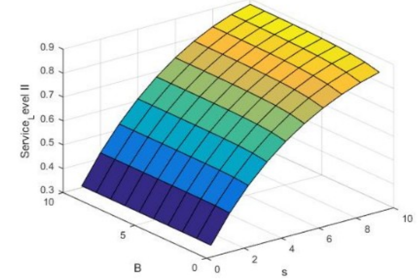
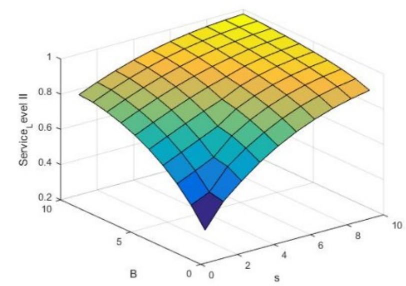
5.10.1.2 The joint effect of the decision variables on the performance measures

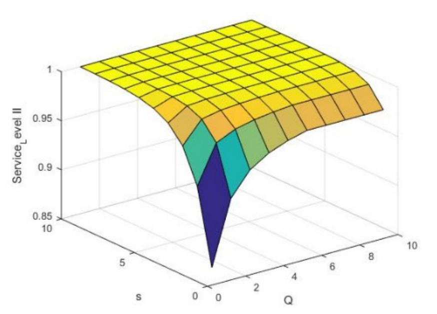
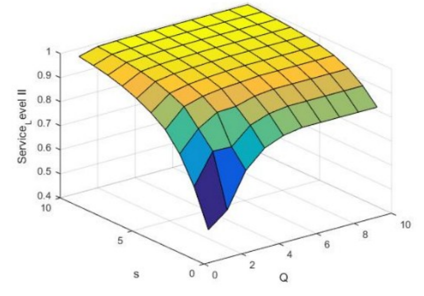
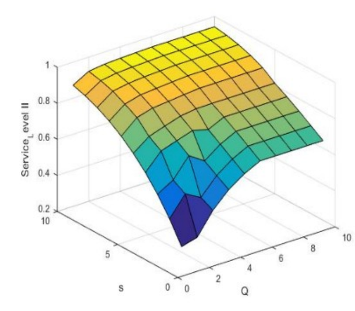
We study the joint effect per two decision variables. In each case, different scenarios with regard to the value of the third decision variable and demand characteristics are investigated. For our analysis we use the elasticity of the performance measures with each decision variable:

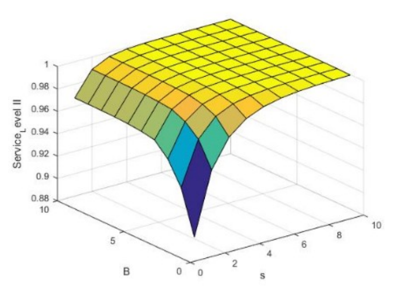
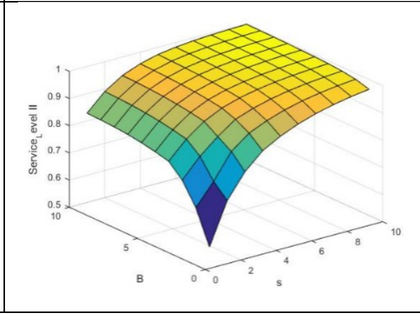
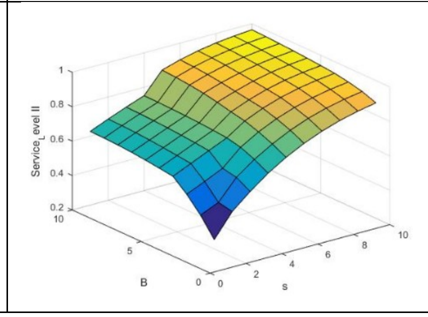
$$elasticisty = \frac{\%\ change\ of\ Performance\ measure}{\%\ change\ of\ Decision\ Variable}$$

Service Level and Order Fill Rate

All three decision variables are positively correlated with SL_2. However, the effect of each parameter decreases as the value of the parameter increases (decreasing elasticities) and beyond a limit the impact becomes negligible. For high decision variable values service level reaches a plateau close to its maximum value. Predictably, the effect of B is important for higher Q values, while for base stock policies (Q=1) its effect is negligible. In general the reorder point s is the most important decision variable followed by Q. However, it should be noted that high Q prerequisites buffer capacity of at least Q-1 slots. For increased demand variability, maximum SL_2 decreases and its plateau is harder to reach, but up to a point the effect of n can be counterbalanced. In any case, demand variability is the most decisive parameter for service level.

Service Level II as a function of (s, Q) and (s, B)		
$SL_2(s,Q)$, B=10, n=10	$SL_2(s,B)$, Q=1, n=10	$SL_2(s,B)$, Q=10, n=10
		

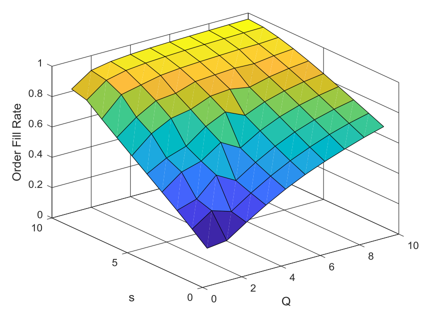
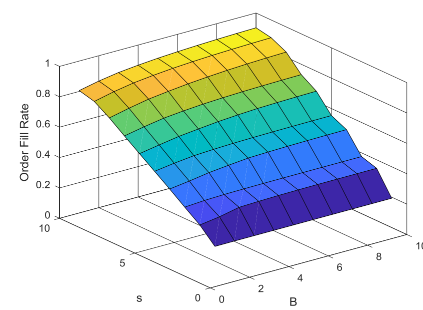
Service Level II as a function of (s, Q), B=5 and n=1, 5, 10		
$SL_2(s,Q)$, n=1, B=5	$SL_2(s,Q)$, n=5, B=5	$SL_2(s,Q)$, n=10, B=5
		

Service Level II as a function of (s, B), Q=5 and n=1, 5, 10		
SL_2(s,B), n=1, Q=5	SL_2(s,Q), n=5, Q=5	SL_2(s,Q), n=10, Q=5
		

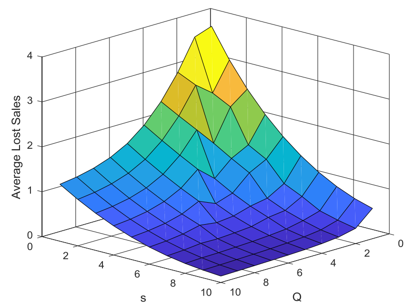
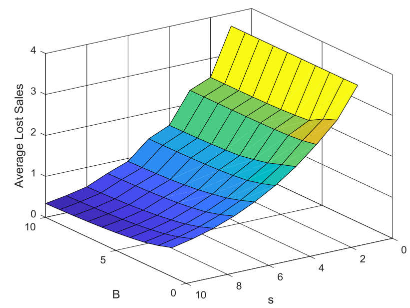
The relation between Order Fill Rate (FR) and type II service level SL_2 is:

$$\frac{1-SL_2}{1-FR}=\frac{Lost\_Sales}{E}$$

As long as lost sales per lost order (Lost_Sales) are lower than the average demand per external order E, SL_2>FR. From the assumptions about the demand structure, in our system $SL_2 \geq$FR with the equality holding for n=1. In the investigated scenarios the two performance measures exhibit a similar behavior with slightly lower values for FR.

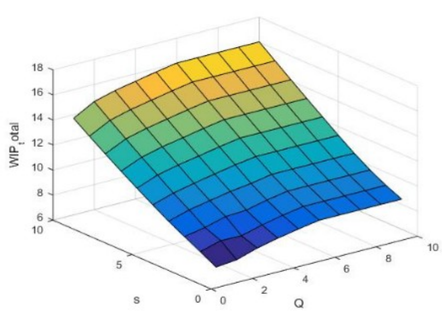
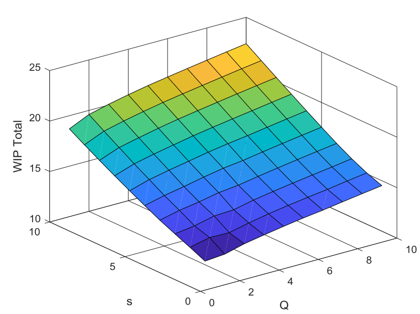
Order Fill Rate as a function of (s, Q) and (s, B)	
FR(s,Q), B=10, n=10	FR(s,B), Q=2, n=10
	

Average lost sales (ALS) are directly related to Service Level (SL_2) and demand variability (n) (paragraph 5.6.9). Demand has the greatest effect and increasing n causes ALS to rise. Increasing s is the most effective way to counter the effects of increased demand variability. Increasing Q (and if necessary B) is an alternative approach, but the effect is less pronounced.

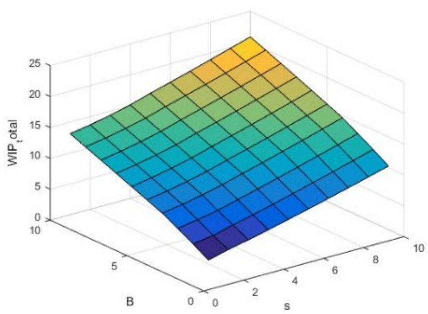
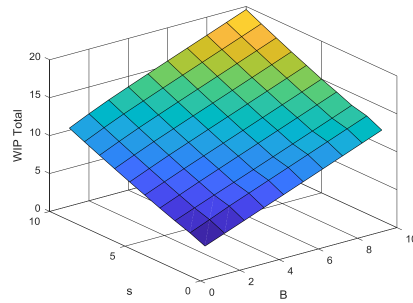
<table>
<tr><td colspan="2">ALS as a function of (s, Q) and (s, B)</td></tr>
<tr><td>ALS(s, Q), B=10, n=10</td><td>ALS(s, B), Q=2, n=10</td></tr>
<tr><td>
</td><td>
</td></tr>
</table>

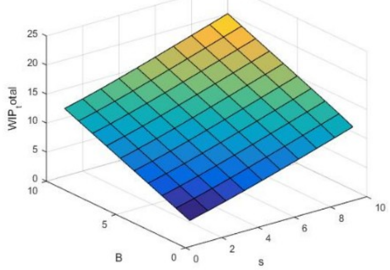
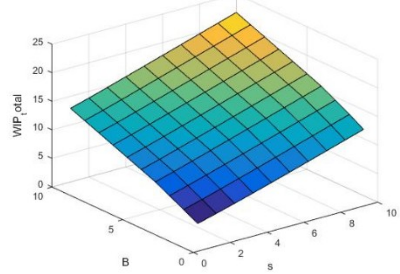
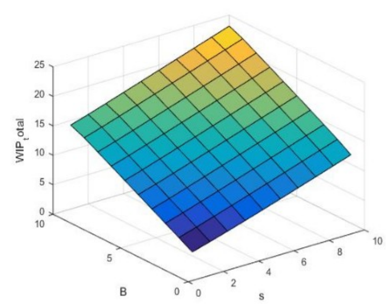
Average Total Inventory

All three decision variables are generally positively correlated with WIP_{total}, but for high n values and for certain scenarios, negative correlation with Q may be observed. For low s and Q, buffer capacity B has the greatest impact, but as the other parameters increase, the effect of B becomes less pronounced and beyond a point s becomes the dominant parameter. As far as the joint effect of s and Q is concerned, the contour lines for given s have a positive inclination, but of a lesser degree than that for given Q. In comparison with reorder quantity Q, the reorder point s plays a more important role in the determination of WIP_{total}.

<table>
<tr><td colspan="2">Average total inventory as a function of (s,Q)</td></tr>
<tr><td>WIP_{total} (s, Q), B=5, n=5</td><td>WIP_{total} (s, Q), B=10, n=5</td></tr>
<tr><td>
</td><td>
</td></tr>
</table>

For n=1 the effect of the decision variables can be described quite accurately using linear relations. With rising demand variability the system becomes more unstable and the effect of the decision variables moves away from linearity. However, the pattern of the relative importance of each decision variable remains the same.

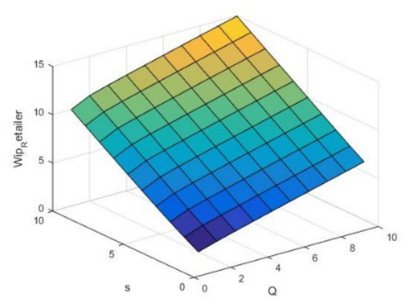
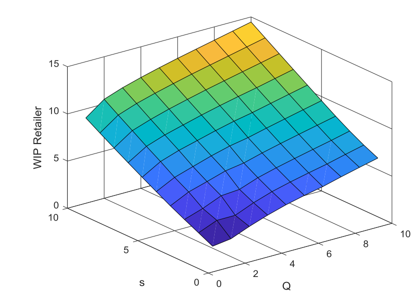
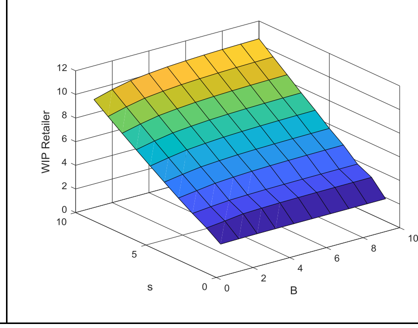
Average total inventory as a function of (s, B)	
WIP_{total} (s, B), Q=5, n=5	WIP_{total} (s, B), Q=2, n=5
	

Average total inventory as a function of (s, B)		
WIP_{total} (s, B), Q=1, n=1	WIP_{total} (s, B), Q=5, n=1	WIP_{total} (s, B), Q=10, n=1
		

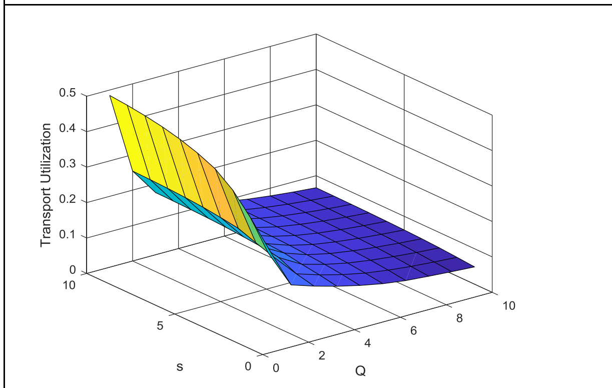
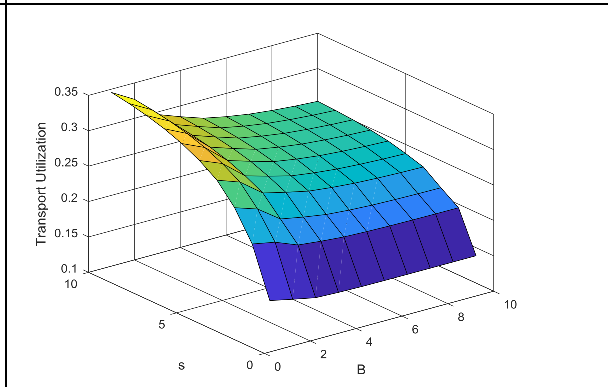
Average inventory at the retailer

B, s and Q are positively correlated with $WIP_{retailer}$. In general, the effect of s is more important than that of Q. The elasticity of $WIP_{retailer}$ with s tends to increase with increasing s values and to decrease with rising Q values. Similarly, the elasticity of $WIP_{retailer}$ with Q tends to increase with increasing Q values and to decrease with increasing s values. In comparison with the other decision variables, the effect of B is negligible.

For low demand variability the effect of s and Q can be approximated with a linear function with good fitting (for n=1, R^2 above 0.99). As external demand variability increases, the effect becomes more dynamic and diverges from linearity.

Average inventory at the retailer as a function of (s,Q) and (s, B)		
$WIP_{retailer}$ (s, Q)		$WIP_{retailer}$ (s, B)
B=10, n=1	B=10, n=5	Q=2, n=5
		

Utilization of transportation resource

Transport utilization as a function of (s, Q) and (s, B)	
u_T(s, Q), B=10, n=5	u_T(s, B), Q=2, n=5
	

The reorder point s affects transport utilization indirectly through the increase of service level and system output. u_T increases with s and the effect is most pronounced for base stock policies (Q=1). With respect to buffer capacity, two counterbalancing trends are manifested. Increased B tends to increase u_T through increased system output and at the same time suppresses reordering rate by allowing more complete orders towards the retailer. Initially transport utilization decreases with increasing B. Beyond a limit the two trends offset each other and in some cases a slight increase of u_T with B was observed.

By far the most important parameter for transport utilization is the reorder quantity Q. With higher Q there are fewer but bigger replenishment orders with the net effect of decreased transport utilization.

5.10.1.3 Synopsis

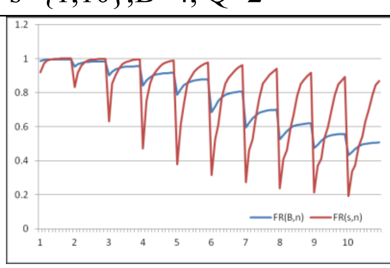
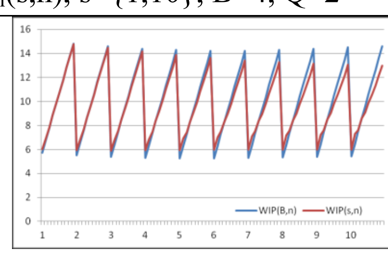
In general, increased demand variability is detrimental to system performance both in terms of customer satisfaction and accumulation of inventory. Moreover, increasing n makes the system more unstable and less predictable. To offset the effect of higher n, higher values of the decision variables are necessary.

Customer satisfaction, expressed either as Order Fill Rate, type II Service Level, or Average Lost Sales is heavily depended on demand variability. As far as the decision variables are concerned, changing reorder point s is the most effective way to enhance FR or SL_2, followed by increasing reorder quantity Q. Buffer capacity is important as long as it is high enough to serve demand Q.

To limit average total inventory (WIP_{total}), decreasing buffer capacity is the most effective way, but it must be kept in mind that B constrains the maximum Q value. To a lesser degree, decreasing s can also decrease WIP_{total}, while Q is the least important variable for total inventory.

With regard to average inventory at the retailer ($WIP_{retailer}$), reorder point s is the most important decision variable, followed by reorder quantity Q. B may have an effect as long as it interferes with Q. It should be noted that every attempt to limit $WIP_{retailer}$ has an adverse effect on customer satisfaction.

Our analysis indicates that, from a managerial point of view, investing in reorder point s is the most efficient way to enhance the performance of the system, increasing order fill rate and service level while keeping total inventory at a lower level. Increased s causes greater levels of average inventory at the retailer, but this is the cost that must be paid for higher customer satisfaction.

Evolution of Order Fill rate and Average total inventory. Each cycle corresponds to different demand variability, n={1,10}	
FR(B,n), B={1,10}, s=4, Q=2 FR(s,n) s={1,10},B=4, Q=2	WIP_{total}(B,n), B={1,10}, s=4, Q=2 WIP_{total}(s,n), s={1,10}, B=4, Q=2
	

5.10.2 Optimal policies under service constraint

5.10.2.1 Minimisation of system inventory (WIP_{total}) for a given Fill Rate

The aim is to determine the policy (B, s, Q) that minimizes average total inventory in the system (WIP_{total}) under the constraint that a minimum Order Fill Rate (FR_{level}) is achieved. The analysis is based on the exhaustive enumeration of systems for $1 \leq B \leq 10$, $1 \leq s \leq 27$, $1 \leq Q \leq 14$, $1 \leq n \leq 10$ and for FR_{level}={0.8, 0.9, 0.95, 0.97, 0.98, 0.99, 0.995}. The results for the optimal policies are given in Table 5.1 and Figure 5.5.

Our analysis indicates that increasing the reorder point *s* is the most efficient way to increase Fill Rate while keeping average inventory as low as possible, irrespectively of the value of n. Practically in all optimal policies high safety stock values, as expressed by s, are used. The importance of Q and B depends on demand variability. For low n, high Q values in relation to *B* are used (keeping in mind that $Q \leq B+1$). However, as n increases and for high Fill rate constraints, B higher than Q is used in the optimal policies. For high customer satisfaction under high demand variability, changes in the retailer's policy are not enough, but an increase in buffer capacity is needed.

Figure 5.5: Optimal (B, Q, s) policies for different levels of n (x-axis) and under different Fill Rate constraints. FR_{level}={0.8, 0.9, 0.95, 0.97, 0.98, 0.99, 0.995}

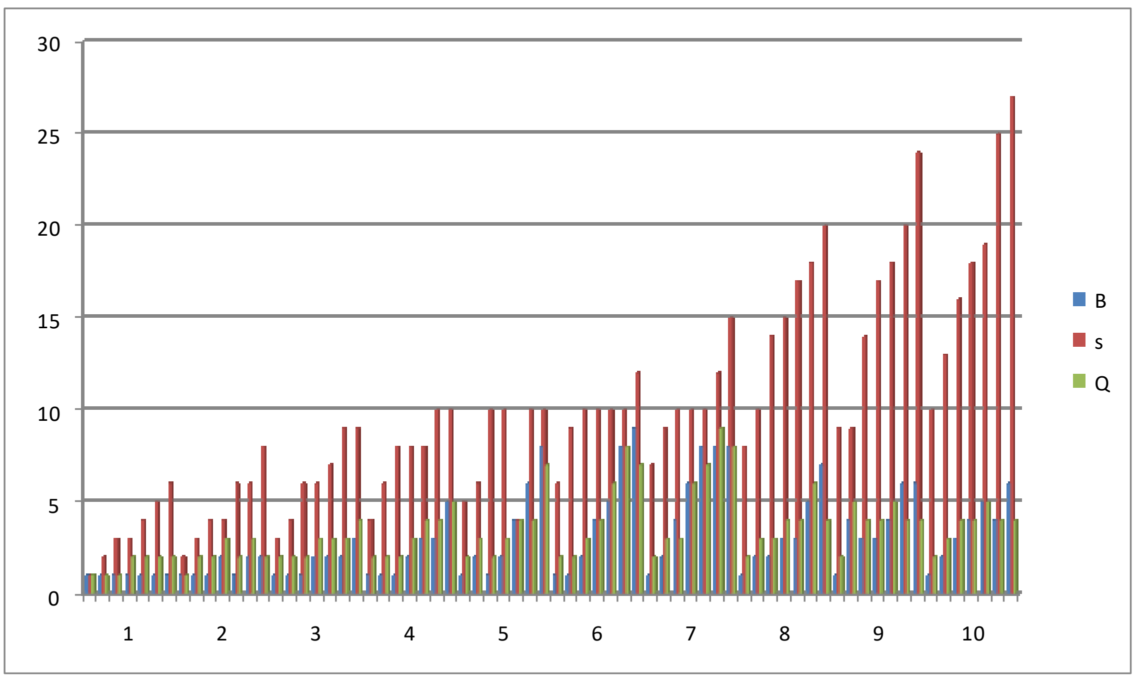


Table 5.1: Minimum inventory policies under fill rate constraint and percentage change of Fill Rate and WIP_{total} between successive steps.

B	s	Q	n	FR_level	FR	WIP total	%FR	% WIP		B	s	Q	n	FR_level	FR	WIP total	%FR	% WIP
1	1	1	1	0.8	0.841	2.6753				1	6	2	6	0.8	0.825	6.7923		
1	2	1	1	0.9	0.916	3.4574	8.9%	29.2%		1	9	2	6	0.9	0.907	9.2661	10.0%	36.4%
1	3	1	1	0.95	0.953	4.2866	4.0%	24.0%		2	10	3	6	0.95	0.951	11.6181	4.8%	25.4%
1	3	2	1	0.97	0.974	4.7945	2.2%	11.8%		4	10	4	6	0.97	0.972	14.0666	2.2%	21.1%
1	4	2	1	0.98	0.986	5.727	1.3%	19.4%		5	10	6	6	0.98	0.981	15.7664	0.9%	12.1%
1	5	2	1	0.99	0.993	6.6821	0.6%	16.7%		8	10	8	6	0.99	0.99	19.4997	1.0%	23.7%
1	6	2	1	0.995	0.996	7.653	0.3%	14.5%		9	12	7	6	0.995	0.995	22.2035	0.5%	13.9%
1	2	1	2	0.8	0.814	3.4033				1	7	2	7	0.8	0.819	7.5516		
1	3	2	2	0.9	0.925	4.6344	13.6%	36.2%		2	9	3	7	0.9	0.91	10.5876	11.2%	40.2%
1	4	2	2	0.95	0.952	5.5103	3.0%	18.9%		4	10	3	7	0.95	0.95	13.598	4.4%	28.4%
2	4	3	2	0.97	0.973	6.9189	2.2%	25.6%		6	10	6	7	0.97	0.972	16.5622	2.3%	21.8%
1	6	2	2	0.98	0.98	7.3355	0.7%	6.0%		8	10	7	7	0.98	0.981	18.9792	0.9%	14.6%
2	6	3	2	0.99	0.991	8.8203	1.0%	20.2%		8	12	9	7	0.99	0.99	21.4766	1.0%	13.2%
2	8	2	2	0.995	0.995	10.4032	0.5%	17.9%		8	15	8	7	0.995	0.995	24.1234	0.5%	12.3%
1	3	2	3	0.8	0.855	4.5305				1	8	2	8	0.8	0.814	8.3125		
1	4	2	3	0.9	0.901	5.3595	5.3%	18.3%		2	10	3	8	0.9	0.901	11.3482	10.7%	36.5%
1	6	2	3	0.95	0.952	7.0911	5.7%	32.3%		2	14	3	8	0.95	0.953	14.9055	5.8%	31.3%
2	6	3	3	0.97	0.973	8.578	2.2%	21.0%		3	15	4	8	0.97	0.971	17.2261	1.9%	15.6%
2	7	3	3	0.98	0.982	9.5181	0.9%	11.0%		3	17	4	8	0.98	0.98	19.1215	0.9%	11.0%
2	9	3	3	0.99	0.991	11.4402	1.0%	20.2%		5	18	6	8	0.99	0.99	22.8733	1.1%	19.6%
3	9	4	3	0.995	0.995	12.8644	0.4%	12.4%		7	20	4	8	0.995	0.995	26.4392	0.5%	15.6%
1	4	2	4	0.8	0.843	5.2828				1	9	2	9	0.8	0.809	9.0747		
1	6	2	4	0.9	0.914	6.9403	8.4%	31.4%		4	9	5	9	0.9	0.902	12.8982	11.5%	42.1%
1	8	2	4	0.95	0.951	8.6854	4.1%	25.1%		3	14	4	9	0.95	0.95	16.0996	5.3%	24.8%
2	8	3	4	0.97	0.971	10.1956	2.1%	17.4%		3	17	4	9	0.97	0.971	18.8837	2.2%	17.3%
3	8	4	4	0.98	0.982	11.6268	1.1%	14.0%		4	18	5	9	0.98	0.981	21.183	1.0%	12.2%
3	10	4	4	0.99	0.991	13.5519	0.9%	16.6%		6	20	4	9	0.99	0.99	25.0437	1.0%	18.2%
5	10	5	4	0.995	0.995	15.9974	0.5%	18.0%		6	24	4	9	0.995	0.995	28.9633	0.5%	15.7%
1	5	2	5	0.8	0.833	6.0352				1	10	2	10	0.8	0.806	9.8377		
2	6	3	5	0.9	0.909	8.2131	9.1%	36.1%		2	13	3	10	0.9	0.902	13.7453	12.0%	39.7%
1	10	2	5	0.95	0.951	10.2826	4.6%	25.2%		3	16	4	10	0.95	0.952	17.5883	5.5%	28.0%
2	10	3	5	0.97	0.97	11.8209	2.1%	15.0%		4	18	4	10	0.97	0.971	20.6896	2.0%	17.6%
4	4	4	5	0.98	0.981	13.703	1.2%	15.9%		5	19	5	10	0.98	0.98	22.9468	0.9%	10.9%
6	10	4	5	0.99	0.99	16.455	0.9%	20.1%		4	25	4	10	0.99	0.99	27.4111	1.0%	19.5%
8	10	7	5	0.995	0.995	19.455	0.5%	18.2%		6	27	4	10	0.995	0.995	31.5944	0.5%	15.3%

As n increases all three decision variables should be increased and the system has to carry more inventory in order to achieve the same service level. The rise of optimal s is more dramatic and in practice the proposed policies may not be tenable. From a managerial point of view a hierarchy of steps seems appropriate if the enhancement of system performance in terms of Fill rate–WIP_{total} is sought. Firstly changes in *s* should be considered, and then a parallel increase in Q and B taking into consideration demand variability.

The dynamic nature of the system should be noted. Although general trends in the effect of the decision variables can be identified, there is no monotony and the sensitivity of the performance measures to the decision variables varies.

5.10.2.2 Minimisation of total cost for a given Fill Rate

As the inventory holding cost along the supply chain is not constant, the use of WIP_{total} as a base for decision making could be problematic. To address this, we introduce a cost function that allows as to allocate different weights to inventory held at different stages of the network:

$$TC(B,s,Q,n) = h_1 \cdot WIP_{buffer} + h_2 \cdot WIP_{retailer} + h_3 \cdot WIP_{\text{int}ransit} + h_4 \cdot (1-OFR) \cdot \lambda \cdot Lost\_Sales$$

$TC(B,s,Q,n)$: Total cost per time unit as a function of the decision variables and demand variability.

h_1: Inventory holding cost per unit at buffer per time unit.

h_2: Inventory holding cost per unit on hand at the retailer per time unit.

h_3: Inventory holding cost per unit in transit to the retailer per time unit.
h_4: Cost incurred because of lost sales, per unit of lost sales.

Policies that keep inventory holding costs at a minimum tend to yield low service level. Such policies tend to incur high lost sales costs, especially in the cases where there is increased demand variability. From a managerial point of view, a balance must be stricken between the two opposing aims of low inventory and low lost sales. Our objective is to determine the (B, s, Q) policy that minimizes total cost under the constraint of a given minimum Order Fill Rate (FR_{level}). We investigate scenarios where $1 \leq B \leq 10$, $1 \leq s \leq 10$, $1 \leq Q \leq 10$, $1 \leq n \leq 5$, and $FR_{level}=\{0.8, 0.9, 0.95, 0.97, 0.99\}$.

Effect of the decision variables on total cost

The cost parameters were $h_1 = 1$, $h_2 = 2$, $h_3=\{10, 7, 4, 1\}$ and $h_4=15$. Our analysis is based on an exhaustive enumeration of all possible policies. The results for $h_3 = 4$ are given in table 5.2 and graphically in figure 5.6.

Figure 5.6: Cost minimizing policies for different Fill Rate constraints. $1 \leq n \leq 5$ (x-axis)

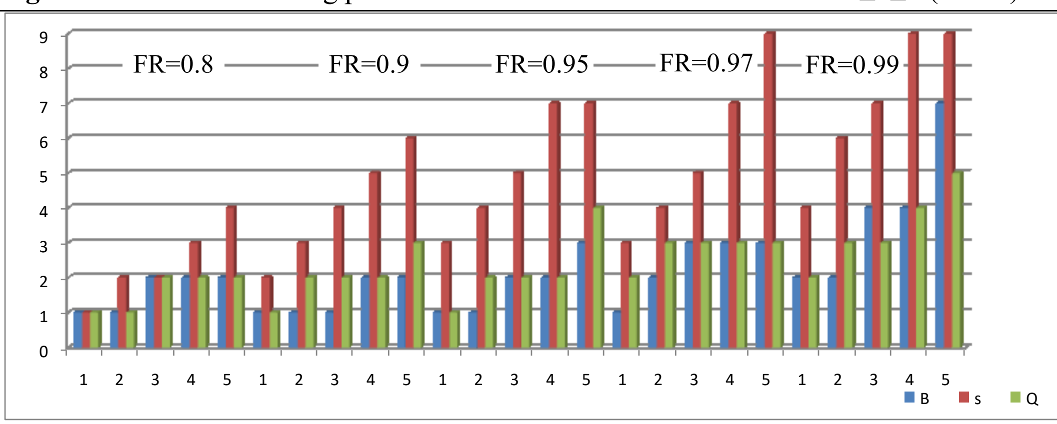


Table 5.2: Cost minimizing policies under Fill Rate constraint. $h_1 = 1$, $h_2 = 2$, $h_3=4$, $h_4=15$

B	s	Q	n	FR_Level	Total Cost
1	1	1	1		7.7078
1	2	1	2		9.1884
2	2	2	3	0.8	10.9455
2	3	2	4		12.3345
2	4	2	5		13.8549
1	2	1	1		8.2619
1	3	2	2		10.4875
1	4	2	3	0.9	12.1159
2	5	2	4		14.7937
2	6	3	5		16.9497
1	3	1	1		9.4256
1	4	2	2		11.9092
2	5	2	3	0.95	14.6477
2	7	2	4		17.8892
3	7	4	5		20.3144
1	3	2	1		10.2566
2	4	3	2		13.7473
3	5	3	3	0.97	16.5127
3	7	3	4		19.8196
3	9	3	5		23.1379
2	4	2	1		13.1482
2	6	3	2		17.3593
4	7	3	3	0.99	21.3353
4	9	4	4		25.5117
7	9	5	5		29.4308

In general, increasing the reorder point s is the most efficient way to address both increasing demand variability and the need for a higher Fill Rate. Simultaneous increases in buffer capacity and order quantity could also be proposed, but in the optimal policies B and Q levels are consistently well below s.

The superiority of s over B and Q can be interpreted as the optimality of accumulating system inventory downstream and closer to external demand. Higher reorder point s creates a safety stock for the system, protecting it from excessive lost sales costs. A better service level is achieved, though at the price of increased $WIP_{retailer}$. Similarly, increased Q causes an indirect decrease in WIP_{buffer} and a downstream shift of inventory. However, Q is constrained by buffer capacity ($Q \leq B+1$) and as no blocking costs were taken into consideration here, minimum buffer capacity is preferred. In practical cases where blocking causes disruptions in the production process, or the inventory capacity of the retailer is constrained, higher B and Q values may have to be chosen.

Effect of cost parameters on the optimal policies

Cost parameters must take into consideration vaguely defined terms, such as loss of customer goodwill, and unpredictable events, such as stock obsolescence. As a result, they are characterized by high uncertainty. We investigate the sensitivity of optimal (B, s, Q) policies with changing cost parameters.

With regard to the effect of lost sales cost h_4, we study scenarios where h_1 =1, h_2 =2, h_3=10 και h_4={15, 10, 7, 3}. The analysis is based on the exhaustive enumeration of all possible policies within the designated bounds. The optimal policies for h_4=3 are given in table 5.3.

Table 5.3: Optimal policies for given minimum fill rate and cost parameters h_1 =1, h_2 =2, h_3=10 and h_4=3

B	s	Q	n	FR_Level	Total Cost
1	1	1	1		8.3236
1	2	1	2		9.8209
2	2	2	3	0,8	11.8095
2	3	2	4		13.2809
2	4	2	5		14.8747
1	2	1	1		10.0022
1	3	2	2		12.5007
1	4	2	3	0,9	13.9843
2	5	2	4		16.7746
2	6	3	5		18.9836
1	3	1	1		11.7178
1	4	2	2		14.2893
2	5	2	3	0,95	17.0588
2	7	2	4		20.3654
3	7	4	5		22.8007
1	3	2	1		12.8623
2	4	3	2		16.4038
3	5	3	3	0,97	19.1530
3	7	3	4		22.4903
3	9	3	5		25.2060
2	4	2	1		16.0248
2	6	3	2		20.2368
4	7	3	3	0,99	24.2241
4	9	4	4		28.4027
7	9	5	5		32.3265

Changing h_4 in the area between 3 and 15 does not change the optimal policy (B, s, Q) that minimizes total cost for a given minimum Fill rate.

To study the effect of inventory holding cost, we investigate scenarios where $h_1=\{1, 2, 3, 5\}$, $h_2=2\cdot h_1$, $h_3=10$ και $h_4=15$. Through exhaustive enumeration of all possible policies it was found that the optimal policy is robust to changes of inventory holding cost at the buffer and the retailer.

Predictably, the in-transit cost h_3 was found to have no effect on the optimal (B, s, Q) policy. Scenarios were $h_1=1$, $h_2=2$, $h_3=\{10, 7, 4, 1\}$ and $h_4=15$ were studied. Under a Fill Rate constraint, system throughput is constrained, and thus $WIP_{intransit}$ is not allowed to vary considerably. Changing in-transit cost from 10 to 7, 4, and 1 does not have any effect on the cost minimization policies. For more realistic modeling, fixed reorder costs should also be included in the cost function.

Our analysis indicates that under our assumptions, and as far as cost minimization is concerned, the optimal policy of the system depends only on external demand variability (n) and the desired Fill Rate level. The cost minimizing policy was found to be robust to cost parameters changes within a wide range of values. It should be noted that the total cost do change with changing cost parameters, so from a managerial point of view they are still important for efficient systems.

5.11 Conclusions

We have presented an exact algorithm based on Markov processes for the numerical analysis of a simple, single product, serial, push-pull production-inventory system. The proposed model captures relationships between variables, offers insight on key features of the system at hand, and can be used as a design tool for the evaluation of appropriate systems and the determination of optimal parameter values.

The model was used to investigate the effect of different policies in balanced systems. Our analysis was based on an exhaustive enumeration and evaluation of all the possible policies (B, s, Q) within prescribed bounds. Under the assumptions of our analysis, increasing the reorder point s was found to be the most efficient and effective way to enhance system performance for most scenarios. High s values decrease unnecessary stock at intermediate stages and “transfer” inventory downstream, closer to the external customers. When the analysis was based on a cost function, the optimal policies were found to be robust for a wide range of cost parameters values.

Under certain conditions the effect of the decision variables on some performance measures can be described with good accuracy by simple relations. However, in general the system has a dynamic behavior, especially as external demand variability increases. External demand was a major source of uncertainty. Increasing demand variability makes the system more unstable and less predictable, causes disruptions across the network and impairs performance. To offset the effect of demand uncertainty, high values for the decision variables were necessary.

In a further step, three directions of research are proposed: The application of phase type distributions (Erlang, Coxian) to model times would allow for more realistic modelling and greater flexibility; the investigation of different topologies would offer a better insight in the behaviour of push-pull systems as a general class; finally, the development of a more elaborated cost function and its application for the analysis of real life case studies would add value to the developed model and underline the practical significance of such modelling approaches.

5.12 References

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5.13 Appendix

5.13.1 Matlab algorithm

We present the computer code for the described model. The computer algorithm is given for Mathworks' Matlab, version R2018a (9.4.0.813654). The computer code essentially follows the lines of the presented analysis. Comments start with the symbol "%".

Important note: Some lines of the algorithm have been omitted on purpose. Their position is denoted with […]

```
% ------------ Input -----------
%{
% m1: production rate at the manufacturer; m2 transportation rate from the
buffer to the retailer; n:maximum demand per external customer (uniform
distribution is assumed); l: external customer % arrival rate; b: buffer
capacity; s: reorder point at the retailer; q: reorder quantity at the
retailer
%}
m1=.8;
m2=0.6;
l=1;
b=2;
s=1;
q=2;
n=3;
k=min(s,q);
t=max(q-s,0);
h=min(b+1,q);
% ---------- Sub-matrices -------------
% Sub-matrix D0
D0=zeros(s+1,s+1);
D0(1,1)=-m1;
for i=2:s+1
    D0(i,i)=-m1-l;
end
for i=3:s+1
    u=max(2,i-n);
    for j=u:i-1
        D0(i,j)=l/n;
    end
end
v=min(s+1,n+1);
for i=2:v
    D0(i,1)=((n+2-i)*l)/n;
end
% Sub-matrix D1
D1=zeros(q,q);
for i=1:q
```

```
    D1(i,i)=-m1-l;
end
for i=1:q-1
    D1(i+1,i)=l;
end
for i=2:q
    u=max(1,i-n);
    for j=u:i-1
        D1(i,j)=l/n;
    end
end
% Sub-matrix D2
D2=zeros(s+1,s+1);
D2(1,1)=-m1-m2;
for i=2:s+1
    D2(i,i)=-m1-m2-l;
end
for i=1:s
    D2(i+1,i)=l;
end
for i=3:s+1
    u=max(2,i-n);
    for j=u:i-1
        D2(i,j)=l/n;
    end
end
v=min(s+1,n+1);
for i=2:v
    D2(i,1)=((n+2-i)*l)/n;
end
% Sub-matrix D3
[...]
% Sub-matrix D4
D4=zeros((s+1)*t,q);
for z=1:t
    for i=1:s+1
        D4((s+1)*(z-1)+i,z-1+i)=m2;
    end
end
% Sub-matrix D
D=zeros((s+2)*q,(s+2)*q);
D(1:q,1:q)=D1;
for i=0:q-1
    D(q+1+i*(s+1):q+((i+1)*(s+1)),q+1+i*(s+1):q+((i+1)*(s+1)))=D2;
end
D(q+1:q+k*(s+1),1:q)=D3;
D(q+k*(s+1)+1:q+(k+t)*(s+1),1:q)=D4;
% Sub-matrix L
L=zeros((s+1)*k+q,s+1);
% Block L1
for i=1:q
    u=max(2,s+1-n+i);
    for j=u:s+1
        L(i,j)=l/n;
    end
end
v=min(q,n-s);
```

```
for j=1:v
    L(j,1)=((n-s-j+1)*l)/n;
end
 % Block L2
[...]
% Sub-matrix U0
U0=zeros(s+1);
for i=1:s+1
    U0(i,i)=m1;
end
% Sub-matrix U
U=zeros((s+2)*q,(s+2)*q);
for i=1:(s+2)*q
    U(i,i)=m1;
end
% Infinitesimal Generator Matrix
P=zeros((s+2)*q*(b+2)+s+1);
P(1:s+1,1:s+1)=D0;
for i=0:b+1
    P(s+2+i*(s+2)*q:s+1+(i+1)*(s+2)*q, s+2+i*(s+2)*q:s+1+(i+1)*(s+2)*q)=D;
end
% Boundary states - Blocking
for i=(s+2)*q*(b+1)+s+2:(s+2)*q*(b+2)+s+1
    P(i,i)=P(i,i)+m1;
end
P(1:s+1,s+q+2:s+q+s+2)=U0;
for i=0:b

P(s+2+i*(s+2)*q:s+1+(i+1)*(s+2)*q,(s+2)*(q+1)+i*(s+2)*q:(s+2)*(q+1)+(i+1)*(s+
2)*q-1)=U;
end
P(s+2:s+1+(s+1)*k+q,1:s+1)=L;
for i=0:h-1
    P((s+2)*(q+1)+i*(s+2)*q:(s+2)*(q+1)+i*(s+2)*q+(s+1)*k+q-1,
s+2+q+i*(s+1):s+2+q+i*(s+1)+s)=L;
end
r=s+2+q+(h-1)*(s+1)+s;
for i=h:b
    P((s+2)*(q+1)+i*(s+2)*q:(s+2)*(q+1)+i*(s+2)*q+(s+1)*k+q-1,r+(i-
h+1)*(q+(s+1)*(q-1)+1)+
(i-h)*s:r+(i-h+1)*(q+(s+1)*(q-1)+1)+(i-h+1)*s)=L;
end
% Calculation of the stationary probabilities vector X
Q=P';
for i=1:(s+2)*q*(b+2)+s+1
    Q((s+2)*q*(b+2)+s+1,i)=1;
end
Y=zeros((s+2)*q*(b+2)+s+1,1);
Y((s+2)*q*(b+2)+s+1,1)=1;
X=linsolve(Q,Y);
% -------   Calculation of performance measures  -------------
% Stock-out probability
SO=X(1);
for j=0:b+1
    r=s+q+2+j*((s+1)*q+q);
    for i=0:q-1
        SO=SO+X(r+i*(s+1));
```

```
    end
end
% Inventory in transit
T=zeros(q,1);
for g=0:q-1
    for j=0:b+1
        r=s+q+2+(s+1)*g+j*((s+1)*q+q);
        for i=0:s;
            T(g+1)=T(g+1)+X(r+i);
        end
    end
end
intransit=0;
for i=1:q
    intransit=intransit+i*T(i);
end
% Utilization of transportation resource
utilization=0;
for i=1:q
    utilization=utilization+T(i);
end
idle=1-utilization;
% Probability of blocking
blocked=0;
for i=(s+1)+(s+2)*q*(b+1)+1:(s+1)+(s+2)*q*(b+2)
    blocked=blocked+X(i);
end
% Average buffer inventory
inbuffer=zeros(b,1);
for j=0:b-1
    r=s+1+(s+1)*q+q+1+j*((s+2)*q);
    for i=r:r+(s+2)*q-1
        inbuffer(j+1)=inbuffer(j+1)+X(i);
    end
end
WIPbuffer=0;
for i=1:b
    WIPbuffer=WIPbuffer+inbuffer(i)*i;
end
WIPbuffer=WIPbuffer+blocked*b;
% Average inventory at the retailer
overs=zeros(q,1);
for j=1:q
    for i=0:b+1
        r=s+1+j+i*(s+2)*q;
        overs(j)=overs(j)+X(r);
    end
end
unders=zeros(s,1);
for g=1:s
    for j=0:b+1
        r=s+2+g+q+j*(s+2)*q;
        for i=0:q-1
            unders(g)=unders(g)+X(r+i*(s+1));
        end
    end
end
```

```
for i=1:s
    unders(i)=unders(i)+X(i+1);
end
Inventory=[unders;overs];
WIPretailer=0;
for i=1:s+q
    WIPretailer=WIPretailer+i*Inventory(i);
end
% Order Fill Rate
CI=zeros(n,1);
k=min(n,s+q);
for i=1:k
    for j=i:s+q
        CI(i)=CI(i)+Inventory(j);
    end
end
OFR=0;
for i=1:n
    OFR=OFR+((1/n)*CI(i));
end
% Average lost sales
ALS=0;
w=max(0,n-s-q-1);
for i=1:w
    athr=SO;
    for j=1:s+q
        athr=athr+Inventory(j);
    end
    ALS=ALS+(i*athr/n);
end
for i=w+1:n
    athr=SO;
    y=min(n-1,s+q);
    for j=1:y+w+1-i
        athr=athr+Inventory(j);
    end
    ALS=ALS+(i*athr/n);
end
% Lost sales per lost order
Lost_Sales=ALS/(1-OFR);
% Service Level II
E=0;
for i=1:n
    E=E+(i/n);
end
SL2=(E-ALS)/E;
% Replenishment ordering rate
ROR=(l*E*SL2*utilization)/intransit;
```

5.13.2 Arithmetic Data

5.13.2.1 Validation data

$\mu_1 = 0.5$, $\mu_2 = 0.2$, $\lambda = 0.5$. Simulation parameters: Run time=1x2000000, Warm-up period=50000.

	Input				Matlab							Arena						
A/A	b	s	q	n	Order Fill Rate	WIP Retailer	WIP in transit	WIP buffer	% blocked	Average Lost sales	Service level ii	Order Fill Rate	WIP Retailer	WIP in transit	WIP buffer	% blocked	Average Lost sales	Service level ii
1	0	0	1	1	0.27451	0.27451	0.68627	0	0.72549	1	0.27451	0.274	0.274	0.686	0.000	0.726	1.000	0.274
2	0	0	1	2	0.137255	0.27451	0.68627	0	0.72549	1.420455	0.18301	0.137	0.274	0.686	0.000	0.726	1.420	0.183
3	0	0	1	3	0.091503	0.27451	0.68627	0	0.72549	1.899281	0.13725	0.091	0.274	0.686	0.000	0.726	1.898	0.137
4	0	0	1	4	0.068627	0.27451	0.68627	0	0.72549	2.389474	0.1098	0.068	0.274	0.686	0.000	0.726	2.388	0.110
5	0	0	1	5	0.054902	0.27451	0.68627	0	0.72549	2.883817	0.0915	0.055	0.274	0.686	0.000	0.726	2.882	0.091
6	0	1	1	1	0.337072	0.423209	0.84268	0	0.662928	1	0.33707	0.338	0.424	0.842	0.000	0.663	1.000	0.338
7	0	1	1	2	0.18894	0.37788	0.84853	0	0.660589	1.430954	0.22627	0.189	0.378	0.849	0.000	0.661	1.431	0.226
8	0	1	1	3	0.121616	0.364848	0.85021	0	0.659917	1.889739	0.17004	0.122	0.365	0.850	0.000	0.660	1.890	0.170
9	0	1	1	4	0.089666	0.358662	0.85101	0	0.659597	2.372312	0.13616	0.090	0.359	0.851	0.000	0.660	2.372	0.136
10	0	1	1	5	0.07101	0.35505	0.85147	0	0.659411	2.862691	0.11353	0.071	0.356	0.851	0.000	0.660	2.863	0.114
11	0	2	1	1	0.353029	0.484257	0.88257	0	0.646971	1	0.35303	0.353	0.487	0.883	0.000	0.647	1.000	0.353
12	0	2	1	2	0.198833	0.415795	0.88591	0	0.645635	1.429958	0.23624	0.199	0.417	0.886	0.000	0.646	1.429	0.236
13	0	2	1	3	0.130846	0.392537	0.88689	0	0.645244	1.892925	0.17738	0.131	0.394	0.887	0.000	0.645	1.893	0.178
14	0	2	1	4	0.095554	0.382215	0.88732	0	0.645072	2.371696	0.14197	0.096	0.383	0.887	0.000	0.645	2.372	0.142
15	0	2	1	5	0.075276	0.376382	0.88756	0	0.644975	2.860287	0.11834	0.076	0.377	0.888	0.000	0.645	2.860	0.118
16	0	3	1	1	0.357337	0.507202	0.89334	0	0.642663	1	0.35734	0.358	0.510	0.893	0.000	0.642	1.000	0.358
17	0	3	1	2	0.201349	0.427513	0.89471	0	0.642118	1.430059	0.23859	0.202	0.429	0.895	0.000	0.641	1.430	0.239
18	0	3	1	3	0.132433	0.401012	0.89507	0	0.64197	1.892614	0.17901	0.133	0.403	0.895	0.000	0.641	1.892	0.179
19	0	3	1	4	0.097182	0.38873	0.89523	0	0.641907	2.372469	0.14324	0.098	0.391	0.895	0.000	0.641	2.371	0.144
20	0	3	1	5	0.076387	0.381937	0.89532	0	0.641872	2.860368	0.11938	0.077	0.384	0.895	0.000	0.641	2.859	0.120
21	0	4	1	1	0.358521	0.515324	0.8963	0	0.641479	1	0.35852	0.359	0.518	0.897	0.000	0.642	1.000	0.359
22	0	4	1	2	0.201925	0.431014	0.89679	0	0.641285	1.430047	0.23914	0.202	0.433	0.897	0.000	0.642	1.429	0.239
23	0	4	1	3	0.132818	0.403308	0.89691	0	0.641237	1.89261	0.17938	0.133	0.404	0.898	0.000	0.642	1.892	0.179
24	0	4	1	4	0.097444	0.390532	0.89696	0	0.641218	2.372394	0.14351	0.097	0.391	0.898	0.000	0.642	2.372	0.143
25	0	4	1	5	0.076676	0.38338	0.89698	0	0.641207	2.860542	0.1196	0.077	0.384	0.898	0.000	0.642	2.860	0.119
26	0	5	1	1	0.358849	0.518076	0.89712	0	0.641151	1	0.35885	0.359	0.519	0.897	0.000	0.641	1.000	0.359
27	0	5	1	2	0.202064	0.432013	0.89728	0	0.641087	1.430048	0.23928	0.202	0.432	0.898	0.000	0.642	1.430	0.239
28	0	5	1	3	0.132906	0.403919	0.89732	0	0.641073	1.892613	0.17946	0.133	0.405	0.898	0.000	0.642	1.893	0.179
29	0	5	1	4	0.097506	0.390986	0.89733	0	0.641067	2.372388	0.14357	0.098	0.392	0.898	0.000	0.642	2.373	0.143
30	0	5	1	5	0.07672	0.383755	0.89734	0	0.641065	2.860525	0.11965	0.077	0.385	0.898	0.000	0.642	2.861	0.120
31	0	6	1	1	0.35894	0.518979	0.89735	0	0.64106	1	0.35894	0.359	0.520	0.898	0.000	0.641	1.000	0.359
32	0	6	1	2	0.202096	0.432291	0.8974	0	0.64104	1.430048	0.23931	0.202	0.434	0.898	0.000	0.641	1.430	0.239
33	0	6	1	3	0.132925	0.404079	0.89741	0	0.641036	1.892612	0.17948	0.133	0.405	0.898	0.000	0.641	1.893	0.180
34	0	6	1	4	0.097519	0.3911	0.89741	0	0.641035	2.372389	0.14359	0.098	0.391	0.898	0.000	0.641	2.372	0.144

35	0	6	1	5	0.076731	0.383846	0.89742	0	0.641034	2.860524	0.11966	0.077	0.384	0.898	0.000	0.641	2.860	0.120
36	1	0	1	1	0.283713	0.283713	0.70928	0.95096	0.716287	1	0.28371	0.284	0.284	0.709	0.951	0.716	1.000	0.284
37	1	0	1	2	0.141856	0.283713	0.70928	0.95096	0.716287	1.417347	0.18914	0.142	0.284	0.709	0.951	0.716	1.416	0.190
38	1	0	1	3	0.094571	0.283713	0.70928	0.95096	0.716287	1.895551	0.14186	0.095	0.284	0.709	0.951	0.716	1.895	0.142
39	1	0	1	4	0.070928	0.283713	0.70928	0.95096	0.716287	2.385485	0.11349	0.071	0.284	0.709	0.951	0.716	2.384	0.114
40	1	0	1	5	0.056743	0.283713	0.70928	0.95096	0.716287	2.879688	0.09457	0.057	0.284	0.709	0.951	0.716	2.878	0.095
41	1	0	2	1	0.407457	0.585885	1.01864	0.77097	0.592543	1	0.40746	0.408	0.589	1.015	0.770	0.592	1.000	0.408
42	1	0	2	2	0.262799	0.525597	1.08532	0.74881	0.56587	1.445833	0.28942	0.264	0.527	1.084	0.748	0.565	1.446	0.290
43	1	0	2	3	0.168273	0.504819	1.10831	0.74117	0.556677	1.87162	0.22166	0.168	0.507	1.107	0.741	0.556	1.872	0.222
44	1	0	2	4	0.123574	0.494297	1.11994	0.7373	0.552022	2.341353	0.17919	0.124	0.496	1.117	0.737	0.551	2.340	0.179
45	1	0	2	5	0.097588	0.48794	1.12698	0.73496	0.54921	2.824885	0.15026	0.098	0.489	1.125	0.734	0.548	2.824	0.151
46	1	1	1	1	0.353029	0.451135	0.88257	0.90851	0.646971	1	0.35303	0.352	0.450	0.883	0.909	0.648	1.000	0.352
47	1	1	1	2	0.200009	0.400019	0.89152	0.90317	0.643391	1.429256	0.23774	0.200	0.400	0.892	0.903	0.644	1.429	0.238
48	1	1	1	3	0.128499	0.385497	0.89406	0.90166	0.642374	1.884535	0.17881	0.128	0.386	0.894	0.902	0.642	1.884	0.179
49	1	1	1	4	0.094658	0.378631	0.89527	0.90094	0.641893	2.365838	0.14324	0.095	0.379	0.895	0.901	0.642	2.365	0.143
50	1	1	1	5	0.074926	0.37463	0.89597	0.90052	0.641613	2.855569	0.11946	0.075	0.375	0.896	0.900	0.641	2.854	0.119
51	1	1	2	1	0.470007	0.790495	1.17502	0.7217	0.529993	1	0.47001	0.471	0.793	1.174	0.722	0.530	1.000	0.471
52	1	1	2	2	0.296026	0.627244	1.20153	0.70873	0.519387	1.448046	0.32041	0.296	0.625	1.203	0.710	0.520	1.447	0.320
53	1	1	2	3	0.190849	0.572547	1.20801	0.7054	0.516798	1.874555	0.2416	0.191	0.571	1.207	0.705	0.516	1.873	0.242
54	1	1	2	4	0.137118	0.548471	1.21057	0.70406	0.515771	2.33609	0.19369	0.137	0.547	1.210	0.704	0.515	2.333	0.194
55	1	1	2	5	0.106984	0.534919	1.21192	0.70335	0.515232	2.816558	0.16159	0.107	0.534	1.212	0.703	0.514	2.813	0.162
56	1	2	1	1	0.374417	0.536918	0.93604	0.87756	0.625583	1	0.37442	0.376	0.538	0.936	0.878	0.626	1.000	0.376
57	1	2	1	2	0.213331	0.453201	0.94186	0.87402	0.623254	1.427861	0.25116	0.214	0.454	0.942	0.874	0.624	1.428	0.252
58	1	2	1	3	0.141313	0.42394	0.94359	0.87296	0.622563	1.889587	0.18872	0.142	0.424	0.944	0.874	0.623	1.889	0.189
59	1	2	1	4	0.102788	0.411151	0.94434	0.87249	0.622262	2.365396	0.1511	0.103	0.411	0.945	0.873	0.623	2.363	0.152
60	1	2	1	5	0.080796	0.40398	0.94476	0.87224	0.622094	2.852571	0.12597	0.081	0.404	0.945	0.873	0.623	2.850	0.126
61	1	2	2	1	0.515571	1.036008	1.28893	0.68257	0.484429	1	0.51557	0.515	1.034	1.289	0.683	0.484	1.000	0.515
62	1	2	2	2	0.325536	0.785434	1.31685	0.66821	0.473261	1.443013	0.35116	0.326	0.785	1.317	0.668	0.473	1.442	0.352
63	1	2	2	3	0.220068	0.69002	1.32498	0.66363	0.470009	1.88479	0.265	0.220	0.690	1.326	0.665	0.471	1.884	0.265
64	1	2	2	4	0.160913	0.643652	1.32864	0.66152	0.468546	2.346057	0.21258	0.161	0.641	1.330	0.662	0.469	2.346	0.212
65	1	2	2	5	0.123731	0.618656	1.33064	0.66037	0.467742	2.816193	1.77E-01	0.124	0.616	1.330	0.660	0.467	2.816	0.177
66	1	3	1	1	0.381099	0.574536	0.95275	0.86706	0.618901	1	0.3811	0.382	0.577	0.953	0.867	0.618	1.000	0.382
67	1	3	1	2	0.217269	0.472338	0.9555	0.86535	0.6178	1.428076	0.2548	0.218	0.473	0.956	0.865	0.617	1.427	0.256
68	1	3	1	3	0.143791	0.437691	0.95625	0.86489	0.617501	1.889142	0.19125	0.144	0.438	0.956	0.864	0.617	1.888	0.192
69	1	3	1	4	0.105403	0.421611	0.95657	0.86469	0.617373	2.366846	0.15305	0.106	0.422	0.957	0.864	0.616	2.364	0.154
70	1	3	1	5	0.082568	0.412838	0.95674	0.86458	0.617305	2.852859	1.28E-01	0.083	0.413	0.957	0.864	0.616	2.850	0.128
71	1	3	2	1	0.529361	1.153536	1.3234	0.66291	0.470639	1	0.52936	0.529	1.156	1.323	0.663	0.470	1.000	0.529
72	1	3	2	2	0.331372	0.831946	1.33849	0.65434	0.464602	1.44266	0.35693	0.332	0.833	1.338	0.654	0.464	1.442	0.357
73	1	3	2	3	0.223501	0.718199	1.34193	0.65208	0.463227	1.884389	0.26839	0.224	0.717	1.345	0.653	0.464	1.882	0.269
74	1	3	2	4	0.164692	0.66258	1.34327	0.65116	0.462692	2.349664	0.21492	0.165	0.661	1.344	0.652	0.463	2.348	0.215
75	1	3	2	5	0.126556	0.632778	1.34393	0.65069	0.462426	2.819213	0.17919	0.127	0.633	1.344	0.650	0.462	2.817	0.179
76	1	4	1	1	0.383228	0.589979	0.95807	0.86368	0.616772	1	0.38323	0.384	0.592	0.958	0.864	0.617	1.000	0.384
77	1	4	1	2	0.218297	0.478923	0.9592	0.86297	0.616318	1.42806	0.25579	0.219	0.481	0.959	0.862	0.616	1.428	0.257
78	1	4	1	3	0.14448	0.441963	0.95948	0.86279	0.616208	1.889153	0.1919	0.145	0.442	0.960	0.863	0.616	1.888	0.193
79	1	4	1	4	0.10587	0.424942	0.95959	0.86272	0.616162	2.366728	0.15354	0.106	0.425	0.960	0.863	0.617	2.364	0.154

80	1	4	1	5	0.083096	0.415481	0.95965	0.86268	0.616139	2.853231	0.12795	0.083	0.415	0.960	0.863	0.617	2.851	0.128
81	1	4	2	1	0.535969	1.232004	1.33992	0.65133	0.464031	1	0.53597	0.537	1.233	1.343	0.652	0.464	1.000	0.537
82	1	4	2	2	0.334016	0.863776	1.34901	0.64584	0.460397	1.442072	0.35974	0.334	0.864	1.350	0.646	0.460	1.441	0.360
83	1	4	2	3	0.225558	0.738439	1.35089	0.64455	0.459645	1.884771	0.27018	0.225	0.740	1.353	0.645	0.460	1.884	0.270
84	1	4	2	4	0.166085	0.678225	1.35161	0.64404	0.459357	2.349588	0.21626	0.166	0.680	1.354	0.645	0.460	2.347	0.216
85	1	4	2	5	0.128577	0.644898	1.35197	0.64379	0.459213	2.822065	0.18026	0.129	0.646	1.353	0.644	0.459	2.819	0.180
86	1	5	1	1	0.383911	0.596051	0.95978	0.86259	0.616089	1	0.38391	0.384	0.597	0.960	0.862	0.616	1.000	0.384
87	1	5	1	2	0.218582	0.481079	0.96021	0.86231	0.615915	1.428063	0.25606	0.219	0.482	0.961	0.863	0.616	1.428	0.257
88	1	5	1	3	0.144661	0.443264	0.96031	0.86225	0.615877	1.889165	0.19206	0.146	0.444	0.961	0.862	0.616	1.889	0.193
89	1	5	1	4	0.105995	0.4259	0.96034	0.86223	0.615862	2.366723	0.15366	0.107	0.426	0.961	0.862	0.616	2.365	0.154
90	1	5	1	5	0.083186	0.416267	0.96036	0.86222	0.615855	2.853202	0.12805	0.084	0.416	0.961	0.863	0.616	2.852	0.128
91	1	5	2	1	0.539097	1.277006	1.34774	0.64613	0.460903	1	0.5391	0.539	1.274	1.350	0.647	0.462	1.000	0.539
92	1	5	2	2	0.33502	0.877157	1.35278	0.64299	0.458889	1.441983	0.36074	0.335	0.878	1.354	0.643	0.459	1.441	0.361
93	1	5	2	3	0.226212	0.745783	1.35365	0.64237	0.45854	1.884934	0.27073	0.227	0.749	1.355	0.642	0.457	1.884	0.271
94	1	5	2	4	0.166521	0.683319	1.35396	0.64214	0.458416	2.349687	0.21663	0.167	0.685	1.353	0.642	0.457	2.347	0.218
95	1	5	2	5	0.128888	0.648909	1.35411	0.64203	0.458357	2.82209	0.18055	0.129	0.651	1.353	0.641	0.458	2.819	0.181
96	1	6	1	1	0.384132	0.598365	0.96033	0.86224	0.615868	1	0.38413	0.385	0.602	0.961	0.863	0.615	1.000	0.385
97	1	6	1	2	0.218659	0.481766	0.96049	0.86214	0.615805	1.428063	0.25613	0.220	0.484	0.961	0.863	0.616	1.428	0.257
98	1	6	1	3	0.144705	0.443651	0.96052	0.86212	0.615793	1.889164	0.1921	0.145	0.444	0.961	0.862	0.615	1.888	0.193
99	1	6	1	4	0.106027	0.426173	0.96053	0.86211	0.615788	2.366725	0.15368	0.107	0.427	0.961	0.862	0.615	2.364	0.154
100	1	6	1	5	0.08321	0.416483	0.96054	0.86211	0.615786	2.8532	0.12807	0.084	0.417	0.961	0.862	0.616	2.851	0.128
101	1	6	2	1	0.54063	1.303109	1.35157	0.64354	0.45937	1	0.54063	0.539	1.298	1.352	0.644	0.459	1.000	0.539
102	1	6	2	2	0.335427	0.883572	1.35431	0.64179	0.458275	1.441942	0.36115	0.335	0.881	1.356	0.642	0.458	1.442	0.361
103	1	6	2	3	0.226445	0.74913	1.35472	0.6415	0.458113	1.88495	0.27094	0.226	0.745	1.355	0.642	0.458	1.885	0.271
104	1	6	2	4	0.166694	0.685585	1.35485	0.6414	0.458059	2.349748	0.21678	0.167	0.683	1.355	0.641	0.457	2.348	0.217
105	1	6	2	5	0.129017	0.650657	1.35491	0.64136	0.458034	2.822137	0.18066	0.129	0.650	1.356	0.641	0.458	2.819	0.181
106	2	0	1	1	0.285354	0.285354	0.71339	1.94016	0.714646	1	0.28535	0.285	0.285	0.713	1.941	0.716	1.000	0.285
107	2	0	1	2	0.142677	0.285354	0.71339	1.94016	0.714646	1.416789	0.19024	0.142	0.285	0.713	1.941	0.716	1.416	0.190
108	2	0	1	3	0.095118	0.285354	0.71339	1.94016	0.714646	1.894883	0.14268	0.094	0.285	0.713	1.941	0.716	1.894	0.142
109	2	0	1	4	0.071339	0.285354	0.71339	1.94016	0.714646	2.384772	0.11414	0.071	0.285	0.713	1.941	0.716	2.384	0.114
110	2	0	1	5	0.057071	0.285354	0.71339	1.94016	0.714646	2.87895	0.09512	0.057	0.285	0.713	1.941	0.716	2.878	0.095
111	2	0	2	1	0.427297	0.627847	1.06824	1.68849	0.572703	1	0.4273	0.428	0.628	1.068	1.688	0.572	1.000	0.428
112	2	0	2	2	0.284841	0.569682	1.16003	1.63008	0.535987	1.448611	0.30934	0.285	0.568	1.161	1.631	0.537	1.448	0.310
113	2	0	2	3	0.182902	0.548705	1.19252	1.60923	0.52299	1.863901	0.2385	0.183	0.549	1.192	1.610	0.523	1.863	0.239
114	2	0	2	4	0.134476	0.537904	1.20914	1.59854	0.516343	2.329621	1.93E-01	0.135	0.540	1.208	1.599	0.516	2.327	0.194
115	2	0	2	5	0.106264	0.531322	1.21923	1.59203	0.512306	2.811017	0.16256	0.107	0.531	1.221	1.594	0.514	2.809	0.163
116	2	0	3	1	0.500985	0.947836	1.25246	1.44488	0.499015	1	0.50098	0.502	0.949	1.253	1.445	0.499	1.000	0.502
117	2	0	3	2	0.333916	0.81185	1.35992	1.37582	0.456033	1.435304	0.36264	0.335	0.813	1.358	1.376	0.456	1.435	0.364
118	2	0	3	3	0.246543	0.739629	1.41831	1.3342	0.432676	1.901469	0.28366	0.247	0.740	1.421	1.335	0.433	1.900	0.284
119	2	0	3	4	0.176196	0.704783	1.44509	1.31487	0.421965	2.333036	0.23121	0.176	0.703	1.445	1.313	0.422	2.333	0.231
120	2	0	3	5	0.136863	0.684316	1.46035	1.30377	0.415861	2.798932	0.19471	0.137	0.681	1.461	1.303	0.416	2.798	0.195
121	2	1	1	1	0.357337	0.458672	0.89334	1.87754	0.642663	1	0.35734	0.358	0.460	0.893	1.878	0.642	1.000	0.358
122	2	1	1	2	0.203212	0.406423	0.90392	1.86791	0.638431	1.428774	0.24105	0.203	0.407	0.904	1.868	0.638	1.428	0.241

5.13.2.2 Numerical results data

$\mu_1=\mu_2=\lambda\cdot(n+1)$

A/A	B	s	Q	n	Order Fill Rate	WIP Retailer	WIP in transit	WIP buffer	% blocked	Average Lost sales	Service level ii
1	1	1	1	1	0.841048437	1.386725935	0.420524218	0.8680258	0.5794758	1	0.841048437
2	1	1	1	2	0.706554375	1.41310875	0.371487325	0.873713	0.6285127	1.313831227	0.74297465
3	1	1	1	3	0.498349533	1.495048599	0.318926076	0.8945446	0.6810739	1.443825418	0.637852152
4	1	1	1	4	0.390734113	1.562936451	0.277315293	0.9110777	0.7226847	1.82748379	0.554630587
5	1	1	1	5	0.323267801	1.616339003	0.244654434	0.9235944	0.7553456	2.263928625	0.489308868
6	1	1	2	1	0.892838491	1.840906124	0.446419245	0.7379227	0.5535808	1	0.892838491
7	1	1	2	2	0.788236709	1.804552031	0.405511915	0.7587011	0.5944881	1.338590153	0.811023829
8	1	1	2	3	0.600360673	1.801082018	0.35525867	0.7911437	0.6447413	1.448719583	0.71051734
9	1	1	2	4	0.456867538	1.827470151	0.308958668	0.821487	0.6910413	1.758699263	0.617917336
10	1	1	2	5	0.370514338	1.85257169	0.27128087	0.8456471	0.7287191	2.180057248	0.54256174
11	1	2	1	1	0.916021601	2.170752657	0.458010801	0.8286455	0.5419892	1	0.916021601
12	1	2	1	2	0.813509113	2.154308615	0.421081494	0.8279098	0.5789185	1.269528611	0.842162989
13	1	2	1	3	0.721922391	2.165767174	0.387269456	0.8352381	0.6127305	1.62156952	0.774538913
14	1	2	1	4	0.559309849	2.237239397	0.34943765	0.8515165	0.6505624	1.708256355	0.6988753
15	1	2	1	5	0.461759967	2.308799834	0.316079998	0.8667487	0.68392	2.05023771	0.632159996
16	1	2	2	1	0.948689993	2.710350014	0.474344996	0.7129736	0.525655	1	0.948689993
17	1	2	2	2	0.873276938	2.643853113	0.445674909	0.7229783	0.5543251	1.28607429	0.891349819
18	1	2	2	3	0.793440794	2.59452986	0.415948467	0.7357987	0.5840515	1.627650185	0.831896935
19	1	2	2	4	0.655263066	2.621052266	0.381866937	0.7554236	0.6181331	1.713379847	0.763733874
20	1	2	2	5	0.534555478	2.672777389	0.347896628	0.7770977	0.6521034	1.960749757	0.695793255
21	1	3	1	1	0.952810646	3.00202156	0.476405323	0.8081349	0.5235947	1	0.952810646
22	1	3	1	2	0.882904907	2.918438165	0.449672015	0.7997581	0.550328	1.289413165	0.89934403
23	1	3	1	3	0.799167909	2.90762712	0.421260722	0.8042669	0.5787393	1.568260876	0.842521444
24	1	3	1	4	0.728405359	2.913621438	0.395253562	0.8118116	0.6047464	1.928359804	0.790507125
25	1	3	1	5	0.595230628	2.976153139	0.366021545	0.8250578	0.6339785	1.985996921	0.732043091
26	1	3	2	1	0.973714174	3.607264725	0.486857087	0.7003446	0.5131429	1	0.973714174
27	1	3	2	2	0.924538424	3.465028144	0.467106654	0.7022546	0.5328933	1.307685866	0.934213309
28	1	3	2	3	0.854940542	3.377138626	0.44277371	0.710612	0.5572263	1.578009204	0.88554742
29	1	3	2	4	0.790366527	3.334417741	0.419336422	0.7209525	0.5806636	1.923919327	0.838672844
30	1	3	2	5	0.670681054	3.353405269	0.391931016	0.7373064	0.608069	1.968954143	0.783862032
31	1	4	1	1	0.972604201	3.87471993	0.4863021	0.7970037	0.5136979	1	0.972604201
32	1	4	1	2	0.921653397	3.722975001	0.466489044	0.78301	0.533511	1.283180935	0.932978089
33	1	4	1	3	0.856763484	3.665002443	0.444026211	0.7832549	0.5559738	1.563115058	0.888052422
34	1	4	1	4	0.789349878	3.655521376	0.421183375	0.7890929	0.5788166	1.87079466	0.842366751
35	1	4	1	5	0.731755805	3.658779025	0.400012362	0.7963684	0.5999876	2.236491369	0.800024724
36	1	4	2	1	0.986123794	4.540198431	0.493061897	0.6937275	0.5069381	1	0.986123794
37	1	4	2	2	0.952142381	4.341066458	0.479338025	0.6898478	0.520662	1.295215415	0.95867605
38	1	4	2	3	0.900631987	4.205570034	0.461080506	0.6928347	0.5389195	1.566680978	0.922161012
39	1	4	2	4	0.842968288	4.141170052	0.441155307	0.7004395	0.5588447	1.873656366	0.882310613
40	1	4	2	5	0.788306598	4.104104848	0.421780537	0.7099118	0.5782195	2.216964596	0.843561074

41	1	5	1	1	0.98380132	4.781285086	0.49190066	0.7906992	0.5080993	1	0.98380132
42	1	5	1	2	0.947000677	4.559313539	0.477268413	0.7722537	0.5227316	1.286710006	0.954536827
43	1	5	1	3	0.897227896	4.450591519	0.459375684	0.7690527	0.5406243	1.581141734	0.918751368
44	1	5	1	4	0.838296523	4.409334438	0.440097636	0.7724744	0.5599024	1.852228697	0.880195272
45	1	5	1	5	0.782232612	4.400761541	0.421016288	0.7785955	0.5789837	2.176185694	0.842032575
46	1	5	2	1	0.992531771	5.495449649	0.496265885	0.6903374	0.5037341	1	0.992531771
47	1	5	2	2	0.969600841	5.241743768	0.486829607	0.6823121	0.5131704	1.299745797	0.973659214
48	1	5	2	3	0.932308925	5.05685704	0.473026316	0.6813098	0.5269737	1.593928552	0.946052632
49	1	5	2	4	0.883333049	4.957935225	0.456714595	0.6858877	0.5432854	1.855084278	0.91342919
50	1	5	2	5	0.832859316	4.902000119	0.439701564	0.6935256	0.5602984	2.1645874	0.879403127
51	2	1	1	1	0.851755617	1.414564605	0.425877809	1.8150637	0.5741222	1	0.851755617
52	2	1	1	2	0.718285845	1.43657169	0.376716488	1.8251863	0.6232835	1.312857478	0.753432976
53	2	1	1	3	0.504426002	1.513278006	0.322443934	1.8601973	0.6775561	1.433134645	0.644887867
54	2	1	1	4	0.394323148	1.577292593	0.279687022	1.8862749	0.720313	1.818733677	0.559374045
55	2	1	1	5	0.325572638	1.627863188	0.246305052	1.905033	0.7536949	2.256980919	0.492610104
56	2	1	2	1	0.909937158	1.924775312	0.454968579	1.6426803	0.5450314	1	0.909937158
57	2	1	2	2	0.812731009	1.881964928	0.416215777	1.6614751	0.5837842	1.342201217	0.832431555
58	2	1	2	3	0.617427843	1.852283529	0.363669645	1.7150815	0.6363304	1.425408019	0.72733929
59	2	1	2	4	0.465820935	1.863283738	0.314529463	1.7655733	0.6854705	1.736033376	0.629058926
60	2	1	2	5	0.375628909	1.878144543	0.274915875	1.8038244	0.7250841	2.162984112	0.54983175
61	2	1	3	1	0.931595312	2.375877705	0.465797656	1.483006	0.5342023	1	0.931595312
62	2	1	3	2	0.854924643	2.363470911	0.435856563	1.5063153	0.5641434	1.326416252	0.871713126
63	2	1	3	3	0.74490893	2.398466614	0.403133168	1.5406166	0.5968668	1.518937242	0.806266336
64	2	1	3	4	0.607765025	2.431060099	0.366874179	1.5842286	0.6331258	1.697016189	0.733748359
65	2	1	3	5	0.495765744	2.478828722	0.331943728	1.62749	0.6680563	1.999740444	0.663887455
66	2	2	1	1	0.926627931	2.229503894	0.463313966	1.7512917	0.536686	1	0.926627931
67	2	2	1	2	0.827411493	2.206438407	0.427331359	1.7481334	0.5726686	1.263154346	0.854662718
68	2	2	1	3	0.737684008	2.213052025	0.393757174	1.7569031	0.6062428	1.620073943	0.787514348
69	2	2	1	4	0.569770027	2.279080108	0.354846627	1.7839983	0.6451534	1.68692772	0.709693253
70	2	2	1	5	0.469193653	2.345968267	0.320461805	1.8091409	0.6795382	2.02942029	0.64092361
71	2	2	2	1	0.961897224	2.843897793	0.480948612	1.5975526	0.5190514	1	0.961897224
72	2	2	2	2	0.897531125	2.7957983	0.456191909	1.590216	0.5438081	1.282577477	0.912383819
73	2	2	2	3	0.824588138	2.749679346	0.428800769	1.5949416	0.5711992	1.623589884	0.857601538
74	2	2	2	4	0.695610329	2.782441317	0.396646881	1.6171003	0.6033531	1.697710679	0.793293762
75	2	2	2	5	0.567824645	2.839123227	0.362864709	1.6495672	0.6371353	1.903884004	0.725729418
76	2	2	3	1	0.970573885	3.283911525	0.485286942	1.4507224	0.5147131	1	0.970573885
77	2	2	3	2	0.915026667	3.179451422	0.463806718	1.4596187	0.5361933	1.277810843	0.927613436
78	2	2	3	3	0.854166868	3.09207964	0.43978372	1.4797075	0.5602163	1.651648814	0.87956744
79	2	2	3	4	0.723269212	3.024290304	0.406924023	1.5190737	0.593076	1.681706213	0.813848046
80	2	2	3	5	0.598943893	2.994719465	0.371414254	1.5647113	0.6285857	1.92370709	0.742828508
81	2	3	1	1	0.961593025	3.095285673	0.480796512	1.7152769	0.5192035	1	0.961593025
82	2	3	1	2	0.897054206	3.004925216	0.455882185	1.6957801	0.5441178	1.285661525	0.911764369
83	2	3	1	3	0.815244741	2.987232539	0.428087677	1.6989078	0.5719123	1.556920734	0.856175353
84	2	3	1	4	0.746613195	2.986452781	0.402403721	1.7077312	0.5975963	1.925835862	0.804807442
85	2	3	1	5	0.608573673	3.042868365	0.37248105	1.7294992	0.6275189	1.954681244	0.7449621

86	2	3	2	1	0.982605128	3.775815479	0.491302564	1.5763738	0.5086974	1	0.982605128
87	2	3	2	2	0.943581004	3.649753748	0.475364064	1.5530892	0.5246359	1.309980898	0.950728129
88	2	3	2	3	0.880466724	3.556230843	0.453403846	1.5483692	0.5465962	1.559269696	0.906807693
89	2	3	2	4	0.822031184	3.501421156	0.431547443	1.5518862	0.5684526	1.923161548	0.863094887
90	2	3	2	5	0.699978537	3.499892685	0.404035168	1.5740806	0.5959648	1.919159333	0.808070336
91	2	3	3	1	0.986848681	4.24037272	0.49342434	1.4354069	0.5065757	1	0.986848681
92	2	3	3	2	0.954877769	4.096635544	0.480357954	1.4275922	0.519642	1.305922515	0.960715909
93	2	3	3	3	0.904193253	3.998473611	0.462213766	1.4368978	0.5377862	1.577602229	0.924427531
94	2	3	3	4	0.851547034	3.898916706	0.442623362	1.4495678	0.5573766	1.932485392	0.885246725
95	2	3	3	5	0.745458267	3.863383956	0.417914846	1.4747465	0.5820852	1.934892628	0.835829693
96	2	4	1	1	0.979265518	4.000126583	0.489632759	1.6959811	0.5103672	1	0.979265518
97	2	4	1	2	0.934233297	3.844723089	0.471985793	1.6642917	0.5280142	1.277890133	0.943971586
98	2	4	1	3	0.872814921	3.778389793	0.450689622	1.658042	0.5493104	1.55082272	0.901379245
99	2	4	1	4	0.807188544	3.760712412	0.428338627	1.6634745	0.5716614	1.858327668	0.856677255
100	2	4	1	5	0.751183905	3.755919526	0.407480066	1.672252	0.5925199	2.231043803	0.814960131
101	2	4	2	1	0.991775633	4.73789544	0.495887817	1.5654721	0.5041122	1	0.991775633
102	2	4	2	2	0.96663537	4.568590698	0.485611758	1.5300626	0.5143882	1.29372708	0.971223516
103	2	4	2	3	0.92289389	4.433665189	0.470139852	1.5129331	0.5298601	1.549041849	0.940279705
104	2	4	2	4	0.870755199	4.36133789	0.452025823	1.5105575	0.5479742	1.855942251	0.904051645
105	2	4	2	5	0.820793908	4.311592326	0.433994903	1.5155566	0.5660051	2.209916954	0.867989807
106	2	4	3	1	0.993837107	5.208396136	0.496918553	1.4283195	0.5030814	1	0.993837107
107	2	4	3	2	0.973403607	5.019221027	0.488551913	1.4111623	0.5114481	1.291312806	0.977103825
108	2	4	3	3	0.93792775	4.870496538	0.475656213	1.4103612	0.5243438	1.568738783	0.951312427
109	2	4	3	4	0.89124763	4.741602827	0.459357229	1.4171195	0.5406428	1.868592444	0.918714457
110	2	4	3	5	0.845012376	4.649783808	0.442714206	1.4270357	0.5572858	2.21769169	0.885428411
111	2	5	1	1	0.98860585	4.93466623	0.494302925	1.6855944	0.5056971	1	0.98860585
112	2	5	1	2	0.957662574	4.715569083	0.481902821	1.6444492	0.5180972	1.282353234	0.963805643
113	2	5	1	3	0.912237292	4.598638236	0.46549466	1.6305691	0.5345053	1.57266523	0.930989321
114	2	5	1	4	0.855567411	4.547635425	0.447010145	1.6307153	0.5529899	1.834414787	0.894020289
115	2	5	1	5	0.801108295	4.529676839	0.428345617	1.637186	0.5716544	2.16161001	0.856691233
116	2	5	2	1	0.995998807	5.714347751	0.497999403	1.5604892	0.5020006	1	0.995998807
117	2	5	2	2	0.980272153	5.503398902	0.491446472	1.517182	0.5085535	1.300729088	0.982892944
118	2	5	2	3	0.950478448	5.322926387	0.480370928	1.4915037	0.5196291	1.585497329	0.960741856
119	2	5	2	4	0.907145842	5.214027497	0.466031	1.4824346	0.533969	1.829158775	0.932062001
120	2	5	2	5	0.861288887	5.143179339	0.45038508	1.4836699	0.5496149	2.146111533	0.90077016
121	2	5	3	1	0.997026085	6.190550727	0.498513043	1.4248791	0.501487	1	0.997026085
122	2	5	3	2	0.984372268	5.965263753	0.493237312	1.4014447	0.5067627	1.298209024	0.986474625
123	2	5	3	3	0.960137036	5.774841628	0.484042416	1.3934206	0.5159576	1.601244052	0.968084833
124	2	5	3	4	0.921825464	5.610342355	0.471289976	1.393572	0.52871	1.836277219	0.942579952
125	2	5	3	5	0.880523635	5.496200768	0.456978688	1.3994971	0.5430213	2.160493181	0.913957376
126	3	1	1	1	0.855326504	1.42384891	0.427663252	2.7940589	0.5723367	1	0.855326504
127	3	1	1	2	0.722197009	1.444394018	0.378449039	2.8061454	0.621551	1.312631234	0.756898078
128	3	1	1	3	0.506272384	1.518817152	0.323501091	2.8481553	0.6764989	1.429929406	0.647002182
129	3	1	1	4	0.395325724	1.581302896	0.280340959	2.8783975	0.719659	1.81634187	0.560681918
130	3	1	1	5	0.326172561	1.630862804	0.246728834	2.8996003	0.7532712	2.255216849	0.493457669

131	3	1	2	1	0.917971875	1.972044803	0.458985938	2.5808406	0.5410141	1	0.917971875
132	3	1	2	2	0.826442496	1.933102875	0.422074988	2.5913478	0.577925	1.346960113	0.844149977
133	3	1	2	3	0.627454291	1.882362874	0.368211515	2.6636398	0.6317885	1.41500473	0.73642303
134	3	1	2	4	0.470471965	1.881887861	0.317293469	2.7308183	0.6827065	1.725182795	0.634586938
135	3	1	2	5	0.377954406	1.889772031	0.276540362	2.7796922	0.7234596	2.155401214	0.553080724
136	3	1	3	1	0.939885518	2.446627972	0.469942759	2.4095771	0.5300572	1	0.939885518
137	3	1	3	2	0.870198529	2.447850362	0.442584204	2.4184726	0.5574158	1.327006442	0.885168408
138	3	1	3	3	0.76792899	2.48554407	0.411963953	2.4510777	0.588036	1.517398438	0.823927906
139	3	1	3	4	0.626923333	2.507693331	0.375519607	2.5062999	0.6244804	1.668295077	0.751039213
140	3	1	3	5	0.509506095	2.547530477	0.339433357	2.5634898	0.6605666	1.964142362	0.678866715
141	3	1	4	1	0.951662669	2.902873425	0.475831334	2.2320897	0.5241687	1	0.951662669
142	3	1	4	2	0.892187357	2.877139181	0.452058064	2.2558749	0.5479419	1.334034708	0.904116128
143	3	1	4	3	0.79327004	2.930873269	0.423697944	2.2963344	0.5763021	1.476361837	0.847395888
144	3	1	4	4	0.716813538	2.994696476	0.398663251	2.3339703	0.6013367	1.78922304	0.797326503
145	3	1	4	5	0.610060374	3.050301869	0.371160121	2.3811866	0.6288399	1.982458882	0.742320242
146	3	2	1	1	0.930698226	2.252061815	0.465349113	2.7161919	0.5346509	1	0.930698226
147	3	2	1	2	0.832985491	2.22754004	0.429842454	2.7102117	0.5701575	1.26020571	0.859684908
148	3	2	1	3	0.743879711	2.231639133	0.396339204	2.7206429	0.6036608	1.618939239	0.792678407
149	3	2	1	4	0.573603504	2.294414017	0.356867373	2.7551972	0.6431326	1.678398255	0.713734746
150	3	2	1	5	0.471729267	2.358646335	0.321989263	2.7865724	0.6780107	2.021812593	0.643978527
151	3	2	2	1	0.968190317	2.919308935	0.484095158	2.519896	0.5159048	1	0.968190317
152	3	2	2	2	0.912128603	2.904581941	0.462365725	2.4853444	0.5376343	1.284864331	0.92473145
153	3	2	2	3	0.845019854	2.887551277	0.437489782	2.4733192	0.5625102	1.613373566	0.874979564
154	3	2	2	4	0.73795226	2.951809038	0.409632692	2.4835944	0.5903673	1.724252759	0.819265384
155	3	2	2	5	0.606420114	3.03210057	0.378243254	2.518538	0.6217567	1.856142808	0.756486508
156	3	2	3	1	0.976373589	3.379482941	0.488186795	2.3657098	0.5118132	1	0.976373589
157	3	2	3	2	0.927724869	3.288977626	0.469310319	2.3536813	0.5306897	1.273868943	0.938620637
158	3	2	3	3	0.872225215	3.202514368	0.447217814	2.3665066	0.5527822	1.6523506	0.894435628
159	3	2	3	4	0.741677089	3.11163252	0.414500082	2.4160028	0.5854999	1.654903884	0.829000165
160	3	2	3	5	0.61233565	3.061678248	0.377787546	2.4780473	0.6222125	1.891519624	0.755575091
161	3	2	4	1	0.980846108	3.828023959	0.490423054	2.1980734	0.5095769	1	0.980846108
162	3	2	4	2	0.941195296	3.744335398	0.474932649	2.1999227	0.5250674	1.278844159	0.949865298
163	3	2	4	3	0.893030789	3.66767714	0.456276293	2.2167197	0.5437237	1.635001562	0.912552587
164	3	2	4	4	0.810411499	3.643321418	0.433449258	2.2500852	0.5665507	1.755136556	0.866898517
165	3	2	4	5	0.703090303	3.620592448	0.406362384	2.2976473	0.5936376	1.892244352	0.812724768
166	3	3	1	1	0.965145345	3.133375351	0.482572673	2.6703208	0.5174273	1	0.965145345
167	3	3	1	2	0.903279843	3.043021495	0.458613937	2.6413864	0.5413861	1.283684749	0.917227873
168	3	3	1	3	0.822482808	3.022143795	0.431141054	2.6438879	0.5688589	1.551600612	0.862282108
169	3	3	1	4	0.754653878	3.018615512	0.405594098	2.6535422	0.5944059	1.923933029	0.811188197
170	3	3	1	5	0.614308679	3.071543395	0.375279321	2.6816088	0.6247207	1.940214973	0.750558641
171	3	3	2	1	0.986727785	3.868963395	0.493363893	2.4931753	0.5066361	1	0.986727785
172	3	3	2	2	0.954384682	3.774652267	0.480011587	2.4381222	0.5199884	1.31458556	0.960023174
173	3	3	2	3	0.897310455	3.703242303	0.460348339	2.4124936	0.5396517	1.544525734	0.920696678
174	3	3	2	4	0.844132437	3.650124241	0.440083914	2.4064944	0.5599161	1.922019094	0.880167827
175	3	3	2	5	0.723841993	3.619209965	0.412767421	2.4352498	0.5872326	1.895275397	0.825534841

176	3	3	3	1	0.990453852	4.354993039	0.495226926	2.3454586	0.5047731	1	0.990453852
177	3	3	3	2	0.964755006	4.243433925	0.48463793	2.3060944	0.5153621	1.307595876	0.969275861
178	3	3	3	3	0.920527074	4.165522937	0.468848981	2.2998217	0.531151	1.567880841	0.937697961
179	3	3	3	4	0.873198824	4.074919178	0.451157885	2.3010301	0.5488421	1.925933049	0.90231577
180	3	3	3	5	0.775250409	4.045877875	0.428207198	2.3229786	0.5717928	1.91660776	0.856414397
181	3	3	4	1	0.992173798	4.799353545	0.496086899	2.1832211	0.5039131	1	0.992173798
182	3	3	4	2	0.970066978	4.656874133	0.48693564	2.1677055	0.5130644	1.309359261	0.97387128
183	3	3	4	3	0.929826364	4.529744746	0.472520449	2.1733823	0.5274796	1.56637465	0.945040898
184	3	3	4	4	0.888481892	4.414228917	0.456501612	2.1891802	0.5434984	1.95028362	0.913003224
185	3	3	4	5	0.792744899	4.30741856	0.433864633	2.2264282	0.5661354	1.914607639	0.867729267
186	3	4	1	1	0.981996011	4.05324698	0.490998005	2.6448712	0.509002	1	0.981996011
187	3	4	1	2	0.939990488	3.901344098	0.474504061	2.5981752	0.5254959	1.2745949	0.949008121
188	3	4	1	3	0.880464749	3.832270813	0.453862023	2.5867761	0.546138	1.543912012	0.907724045
189	3	4	1	4	0.815663623	3.811242953	0.43178318	2.5915866	0.5682168	1.850335273	0.86356636
190	3	4	1	5	0.76063446	3.8031723	0.411094139	2.6008229	0.5889059	2.228537861	0.822188277
191	3	4	2	1	0.994284675	4.845539725	0.497142338	2.4792709	0.5028577	1	0.994284675
192	3	4	2	2	0.974664756	4.722977701	0.489063589	2.405612	0.5109364	1.295003667	0.978127177
193	3	4	2	3	0.937492904	4.622355978	0.475966158	2.359836	0.5240338	1.537991255	0.951932317
194	3	4	2	4	0.890481704	4.565078788	0.459757109	2.343183	0.5402429	1.837267935	0.919514219
195	3	4	2	5	0.845129571	4.512426345	0.443110075	2.3404742	0.5568899	2.204033099	0.88622015
196	3	4	3	1	0.995938113	5.334099365	0.497969056	2.3370947	0.5020309	1	0.995938113
197	3	4	3	2	0.980429539	5.185107323	0.491590655	2.2847284	0.5084093	1.289087349	0.983181311
198	3	4	3	3	0.95086195	5.055332346	0.480848612	2.2635879	0.5191514	1.558986392	0.961697225
199	3	4	3	4	0.908846772	4.92987579	0.466305193	2.2559631	0.5336948	1.848250891	0.932610386
200	3	4	3	5	0.867493347	4.834943584	0.451092941	2.2547309	0.5489071	2.214548079	0.902185882
201	3	4	4	1	0.996672412	5.785868417	0.498336206	2.176576	0.5016638	1	0.996672412
202	3	4	4	2	0.983481018	5.620625084	0.492894089	2.1492247	0.5071059	1.290499173	0.985788179
203	3	4	4	3	0.957373689	5.46918312	0.483394616	2.1392385	0.5166054	1.558228615	0.966789231
204	3	4	4	4	0.92149371	5.347471676	0.470720938	2.1452777	0.5292791	1.864758988	0.941441876
205	3	4	4	5	0.88355598	5.216039879	0.456754267	2.1549399	0.5432457	2.228318804	0.913508534
206	3	5	1	1	0.990553366	5.000419821	0.495276683	2.6312003	0.5047233	1	0.990553366
207	3	5	1	2	0.962614844	4.790183998	0.484055247	2.5705459	0.5159448	1.27949874	0.968110494
208	3	5	1	3	0.919620336	4.671996344	0.468499526	2.5474734	0.5315005	1.567584258	0.936999052
209	3	5	1	4	0.864246697	4.617539216	0.450487115	2.5442215	0.5495129	1.823634628	0.900974231
210	3	5	1	5	0.810619277	4.595893259	0.432066307	2.5501193	0.5679337	2.152289592	0.864132614
211	3	5	2	1	0.997455042	5.831008237	0.498727521	2.4735588	0.5012725	1	0.997455042
212	3	5	2	2	0.985910778	5.674008577	0.493876189	2.3897088	0.5061238	1.303935202	0.987752378
213	3	5	2	3	0.961605217	5.530246638	0.484858189	2.3317206	0.5151418	1.577486328	0.969716377
214	3	5	2	4	0.923510146	5.429359934	0.47231846	2.3051646	0.5276815	1.809490957	0.944636921
215	3	5	2	5	0.881227418	5.350443717	0.457904302	2.2967584	0.5420957	2.126536128	0.915808604
216	3	5	3	1	0.998207052	6.323960099	0.499103526	2.3331568	0.5008965	1	0.998207052
217	3	5	3	2	0.989212054	6.147443507	0.495329489	2.2723907	0.5046705	1.298813705	0.990658979
218	3	5	3	3	0.970006583	5.980213969	0.488009848	2.2405619	0.5119902	1.599037809	0.976019696
219	3	5	3	4	0.936541834	5.819701612	0.476985164	2.2222748	0.5230148	1.813386463	0.953970329
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221	3	5	4	1	0.998524303	6.776451879	0.499262152	2.1737405	0.5007378	1	0.998524303
222	3	5	4	2	0.990862209	6.58601834	0.49604075	2.1394331	0.5039592	1.299849085	0.992081501
223	3	5	4	3	0.973924126	6.400975137	0.489589355	2.1200253	0.5104106	1.596977305	0.979178711
224	3	5	4	4	0.945386944	6.245490578	0.479985471	2.1168362	0.5200145	1.832394126	0.959970942
225	3	5	4	5	0.911373201	6.091671444	0.468163965	2.1203641	0.531836	2.155287228	0.936327931
226	4	1	1	1	0.856529065	1.426975568	0.428264532	3.7858888	0.5717355	1	0.856529065
227	4	1	1	2	0.723491113	1.446982227	0.379023419	3.7988889	0.6209766	1.312542783	0.758046838
228	4	1	1	3	0.506812997	1.520438992	0.323812025	3.8441482	0.676188	1.428975005	0.64762405
229	4	1	1	4	0.395587237	1.582348946	0.280512563	3.8760912	0.7194874	1.815708158	0.561025126
230	4	1	1	5	0.326313656	1.631568279	0.246829186	3.8981815	0.7531708	2.254795423	0.493658372
231	4	1	2	1	0.920916839	1.987863343	0.460458419	3.5546111	0.5395416	1	0.920916839
232	4	1	2	2	0.8315905	1.950934479	0.424302603	3.5592574	0.5756974	1.348452383	0.848605206
233	4	1	2	3	0.630796436	1.892389307	0.369797035	3.6422498	0.630203	1.410636057	0.73959407
234	4	1	2	4	0.471890275	1.8875611	0.318159149	3.7182906	0.6818409	1.721619983	0.636318298
235	4	1	2	5	0.378614252	1.89307126	0.277006311	3.7720928	0.7229937	2.153190894	0.554012622
236	4	1	3	1	0.944093017	2.490404024	0.472046508	3.358779	0.5279535	1	0.944093017
237	4	1	3	2	0.879360044	2.509698641	0.446634808	3.3477228	0.5533652	1.327052685	0.893269615
238	4	1	3	3	0.785196097	2.555404769	0.418147227	3.3723148	0.5818528	1.524232503	0.836294455
239	4	1	3	4	0.642463643	2.569854574	0.382016564	3.4368213	0.6179834	1.649950196	0.764033127
240	4	1	3	5	0.520735508	2.603677542	0.345237083	3.5067664	0.6547629	1.937505316	0.690474167
241	4	1	4	1	0.955841802	2.963071264	0.477920901	3.1747432	0.5220791	1	0.955841802
242	4	1	4	2	0.901123129	2.949905415	0.455995145	3.1824502	0.5440049	1.335140999	0.911990291
243	4	1	4	3	0.808016176	3.013523778	0.429340186	3.2153501	0.5706598	1.472203484	0.858680372
244	4	1	4	4	0.735929552	3.083531234	0.405971806	3.2491843	0.5940282	1.780361922	0.811943612
245	4	1	4	5	0.627224611	3.136123054	0.378796145	3.3026447	0.6212039	1.950834599	0.757592291
246	4	1	5	1	0.963194225	3.425451351	0.481597113	2.9841744	0.5184029	1	0.963194225
247	4	1	5	2	0.916223461	3.40900359	0.462831225	3.0035109	0.5371688	1.330997025	0.925662451
248	4	1	5	3	0.831263796	3.424893437	0.437773237	3.0505944	0.5622268	1.475125363	0.875546473
249	4	1	5	4	0.754328915	3.513411871	0.414171335	3.0956872	0.5858287	1.746820649	0.82834267
250	4	1	5	5	0.697790158	3.593691112	0.394458619	3.1335191	0.6055414	2.095392659	0.788917239
251	4	2	1	1	0.932293186	2.260901715	0.466146593	3.7005342	0.5338534	1	0.932293186
252	4	2	1	2	0.835259057	2.236124224	0.430866096	3.6926274	0.5691339	1.258956687	0.861732192
253	4	2	1	3	0.746419474	2.239258421	0.397392172	3.7038077	0.6026078	1.618544286	0.794784344
254	4	2	1	4	0.575110258	2.300441032	0.357654936	3.7425039	0.6423451	1.675082382	0.715309872
255	4	2	1	5	0.472681992	2.363409962	0.322557433	3.7771456	0.6774426	2.019000649	0.645114867

6. Analysis of a Vendor Managed Inventory system with Coxian-2 transportation times and Compound Poisson external demand

6.1 Research rationale

In traditional inventory systems decisions are taken locally, with each member of the system seeking to maximize its benefit, often to the detriment of the overall efficiency of the supply chain. On the other hand, under the Vendor Managed Inventory (VMI) logic the downstream member (buyer or retailer) delegates the decisions concerning its inventory control policy to the upstream partner (vendor). Depending on the details of the arrangement, a number of possible variations can be found. However, two basic characteristics are common in all VMI systems: information sharing, and the transfer of decision making responsibility to the upstream partner (Choudhary & Shankar, 2015).

Since the first succesful implementations of vendor managed inventory systems in the late 80s, VMI has been extensively investigated as a promising alternative to the traditional supply chains. The potential benefits of VMI are well documented. Amongst others, they include higher customer service level, cost savings due to economies of scale, lower transportation costs, improved coordination and forecasting, reduced uncertainty, and a significant reduction in the bullwip effect (Kannan et al., 2013; Claassen et al, 2008). However, it is not always easy to apply VMI philosophy, nor are all VMI initiatives succesful (Kuk, 2004; Niranjan et al., 2012). Among the problems cited are high administrative costs, high inventory costs for the vendor, as well as excessive stock at the buyer.

Successful implementation of VMI policies requires close cooperation between vendors and buyers, and a sharing of costs and benefits in the quest for the global optimization of the supply chain. In this framework, theoretical results and quantitative models can be effective tools to promote understanding between different partners in the supply chain and to facilitate the design of effective and viable VMI policies (Guan & Zhao, 2010).

6.2 Literature review

There is a growing body of literature concerning modeling and applications of VMI systems (Govindan, 2013). Most works focus on the optimal costs and the potential benefits of VMI policies in comparison with more traditional schemes such as retailer managed inventory (RMI).

Due to the complexity of the related models, some authors assume deterministic parameters. Yao et al. (2007) use analytical models to investigate the benefits of VMI application in a two stages inventory system with deterministic characteristics and negligible lead times. Their analysis illustrates the importance of the cost parameters as well as the unequal distribution of benefits between the vendor and the retailer. Bookbinder et al. (2010) also assume deterministic parameters and zero lead times and investigate the benefits of VMI for a one manufacturer - one retailer system. Torres et al. (2014) analyze the performance of a manufacturer-buyer system with synchronized replenishment cycles. They assume deterministic demand and periodic replenishments and offer closed form solutions for the optimal parameters. Choudhary et al. (2014) assume deterministic demand that varies with time and they employ integer linear programming models to compare VMI policies with other schemes of manufacturer-retailer cooperation for a two echelon chain. Tat et al. (2015) investigate a two stage supply network of a deteriorating product in a deterministic setting. They study two scenarios, one system with no shortages and one with backorders. They analyze the problem based on EOQ models and report on the superiority of VMI logic over the traditional approach in terms of coordination and total cost. Rahim et al. (2016) are concerned with more complex topologies and propose a deterministic model for a system with one warehouse and multiple retailers.

Deterministic parameters are rarely realistic, so most authors include stochastic elements in their models. Both periodic review and continuous review inventory policies can be found.

Lee et al. (2000) investigate the benefits of information sharing in a two echelon system with auto-correlated external demand. Excess demand at the retailer is back-logged, while the supplier always finds inventory to meet retailers demand. Periodic review and order-up-to policies with changing parameters are assumed. The authors develop analytic models for their analysis and conclude on the importance of the lead times and the characteristics of the demand process for the potential benefits of information sharing. Yao and Dresner (2008) and Kalpakam et al. (2014) expand the model of Lee et al. (2000). Yao and Dresner (2008) also report on the significance of the demand process and logistics parameters on the potential benefits of VMI and discuss the allocation of these benefits. Kalpakam et al. (2014) analyze multi-stages serial systems and propose that the benefits of information sharing increase as the number of installations in the supply chain increases.

Song and Dinwoodie (2008) consider Integrated Inventory Management and VMI policies in a three stages, serial supply chain, with exponentially distributed lead times, Poisson demand and backorders. Order-up-to and base-stock policies are investigated. The authors employ stochastic dynamic programming to evaluate the respective systems with regard to their cost effectiveness.

Kiesmuller and Broekmeulen (2010) analyze the benefits of VMI strategies in a two echelon periodic review system with multiple products. They assume stochastic demand (Bernouli or Poisson) with backorders and constant lead times. In the case of

VMI strategies they use simulation to obtain estimates for the expectations of average cost.

Savaşaneril and Erkip (2010) investigate a system consisting of a manufacturer with finite manufacturing capacity and a retailer with service level constraints and periodic review inventory control policy. They assume stochastic demand with backordering and they model the system as a Markov Decision Process.

Choudhary et al. (2016) study the benefit of implementing full VMI instead of simple information sharing in a two echelon serial system. They assume periodic review policies with a normally distributed dynamic external demand. They define the problem as a shortest path problem and they develop a mixed integer programming formulation.

With regard to continuous review systems, Ching and Tai (2005) investigate a system where a vendor supplies multiple retailers facing external demand with Poisson characteristics and zero lead times. They propose a mixed time and quantity based dispatching strategy. They base their analysis on renewal theory and offer a closed form solution for the optimal dispatching policy.

Bichescu and Fry (2009) examine a system with deterministic production rate, positive lead time and normally distributed demand during lead time. Customer demand that cannot be satisfied from the available inventory at the retailer is backordered, while the manufacturer in the case of insufficient inventory on hand can procure extra inventory at a cost penalty. The authors analyze the costs for traditional and VMI policies and employ a game theoretic approach to investigate the effect of different channel power scenarios.

Guan and Zhao (2010) employ a cost function to find the best contract for a serial supply chain with deterministic lead times, stochastic demand and backordering. Razmi et al. (2010) develop an iterative procedure based also on cost to evaluate a two echelon supply chain with lead times varying with lot size, stochastic demand, and continuous review inventory control policy. They assume backorders for the buyer and no shortages for the vendor.

Salzarulo and Jacobs (2014) investigate the performance of VMI, MTO and MTS policies in a two stage, serial supply chain with stochastic demand, constant lead times and back-ordering of excessive demand. They develop probability models based on renewal theory and conclude that VMI systems consistently provide results superior, or at least equal, to those of MTO and MTS systems.

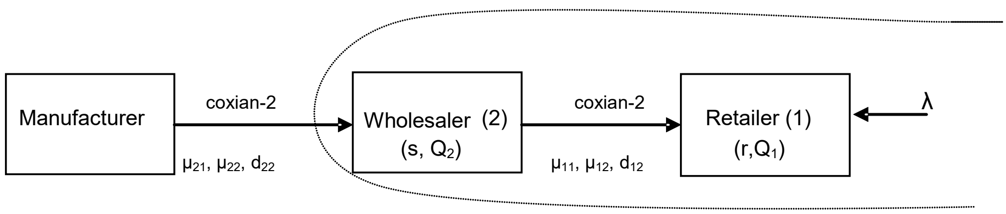
Lee and Cho (2014) consider a two installations continuous review system where the manufacturer does not hold inventory but follows a consignment stock policy. They compare traditional and VMI policies under specific contracts and examine both deterministic and stochastic (Poisson) demand with backorders. Based on the EOQ model with allowed shortages they develop a model to compute the optimal inventory policies as well as the optimal contract parameters and compare the average costs under Retailer Managed Inventory and VMI.

Yu et al. (2015) propose an EOQ based model for the evaluation of VMI benefits in a one supplier - one retailer system with stochastic demand at the retailer and exchange rate uncertainty for the vendor. They assume a continuous review inventory policy with backorders. They find that VMI is not always preferable to traditional policies and report on the demand fluctuation effect.

In this section we present an analytic model for the exact numerical evaluation of a two echelon, continuous review, inventory system working according to VMI logic. The system is modeled as a continuous time Markov Process and our analysis is based on the Infinitesimal Generator Matrix characteristic structure. Compared to the existing literature the contribution of our work is threefold. Firstly, our model makes more generic assumptions for both supply and demand uncertainty. Lead times are modeled using Coxian distribution with two phases, while the external demand is described as a compound Poisson process. These assumptions allow us greater modeling flexibility and permit us to describe more accurately realistic situations. Secondly, unlike most of the literature models, we assume that demand that cannot be met from inventory on hand is lost. Lost sales are common for commodity products and in highly competitive markets, so despite their complexity, lost sales models are important from a practical point of view. Finally, our investigation is focused not so much on the optimal policies, but on the general behavior of the system. Supply chains are dynamic systems with complex interrelations and they operate in a fluid environment where optimal policies may cannot be defined or followed, or may vary with time. Our purpose is to investigate and understand the underlying relationships between the decision variables and the performance measures so that a better understanding of the overall system can be achieved.

6.3 Description of the system

Figure 6.1: System Lay-out

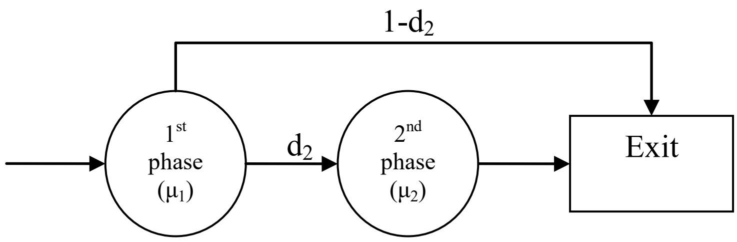


We investigate a three stages, single product, serial inventory system working under Vendor-Managed-Inventory (VMI) logic (Figure 6.1). The wholesaler has real time information about and controls the inventory of the downstream retailer. The retailer holds inventory $I_1(t)$ and faces external demand with compound Poisson characteristics. Customers arrive according to a Poisson process with rate λ, and the demand of each customer follows an empirical distribution ($dm_1, dm_2, ..., dm_{md}$), where dm_i is the probability that the customer will ask for exactly *i* product units. The demand that cannot be met from the inventory on hand at the retailer is lost. The wholesaler continuously reviews the inventory at the retailer and whenever $I_1(t)$ reaches the reorder point r, a replenishment order of Q_1 product units is dispatched.

The wholesaler also holds inventory $I_2(t)$ and decides its replenishment orders based on echelon information. Echelon inventory of stage i is the sum of the inventory between stage i and the final customer (Chopra and Meindl, 2007). Whenever the echelon inventory (the inventory on hand at the wholesaler plus the inventory in transit to the retailer plus the inventory on hand at the retailer) reaches the echelon reorder point s, a replenishment order of Q_2 units is asked upstream from the manufacturer.

Transportation times are independent of the amount of the replenishment order and are modeled using Coxian-2 distribution (Fig. 6.2). For transportation towards node i there is a first phase with exponentially distributed processing times and transition rate μ_{i1}. Then, with probability d_{i2}, follows a second phase with exponentially distributed processing times and rate μ_{i2}. The transportation is accomplished after one phase with probability $1-d_{i2}$. $d_{i2}=1$ corresponds to Erlang distribution with two phases, while $d_{i2}=0$ corresponds to a simple exponential distribution. Practically any empirical distribution can be approximated by a Coxian distribution with the appropriate number of phases. In our model, the use of a phase type distribution offers greater flexibility and allows more realistic modelling of practical situations. Moreover it enables us to better capture variability, allowing for a wide range for the Coefficient of Variance.

Figure 6.2: The general logic of the Coxian-2 distribution



The following assumptions are also made about the system:

1. The manufacturer has always enough inventory on hand to cover the vendor's demand. All replenishment orders towards the wholesaler are of Q_2 units.
2. Only complete orders of Q_1 are sent from the vendor to the retailer. If the inventory on hand at the vendor is not enough ($I_2(t) < Q_1$), dispatching is deferred until a replenishment order is received from the manufacturer.

3. At any given moment there can be only one outstanding order in transit from the vendor to the retailer. Similarly, there can be only one outstanding order in transit from the manufacturer to the vendor. The one outstanding order assumption is common in supply chain modeling and is necessary to keep a tractable level of complexity (Bijvank & Vis, 2011).
4. Q_2 is a multiple of Q_1 such that $Q_2=nQ_1$, where n a positive integer. It is a logical assumption about the inventory control policies given assumption 2. For our modeling approach, this assumption is necessary to ensure a unique class of recurrent states. However, this assumption is common in modeling VMI systems, while it has been found that under certain conditions such policies are optimal (Chen, 2000).
5. Inventory in transit belongs to the upstream node. This assumption is made to facilitate the definition of system states and does not pose any restrictions to the model.
6. All stations are reliable
7. Lead time for the information flow is zero.
8. At any time t, there can be at most one event changing the state of the system. This assumption is necessary for reasons of methodology, so that we can model the system as a Markov chain and do not pose any restrictions to our model.

6.3.1 Model variables

We denote as decision or design variables those parameters of the system whose value a company can usually influence directly in order to achieve the desired outcomes. The determination of these values is part of the company's planning at strategic and tactical level. For the system under consideration the decision variables concern the parameters of the inventory policies at the vendor and the retailer. More precisely, the decision variables are:
r: the reorder point at the retailer.
Q_1: the quantity of the replenishment orders towards the retailer.
s: the echelon reorder point at the vendor.
n$=Q_2/Q_1$: the parameter defining the quantity of the replenishment orders towards the vendor. We have assumed $Q_2 \geq Q_1$,or $n \geq 1$.
These parameters also define the dimension and structure of the infinitesimal generator matrix that corresponds to the system (section 5).

The other parameters of the system include:
μ_{ij}: the transition rate of the j_{th} phase during transportation towards the i_{th} node. i,j={1,2}
d_{ij}: the probability that during transportation to the i_{th} node there will be exactly j phases. i,j={1,2}
λ: the arrival rate of external customers (exponentially distributed inter-arrival times)
dm(i):The probability that an external customer will ask exactly i product units from the retailer. To simplify our notation we assume $dm(i) = 0, \forall i > md$.

md: The maximum possible demand of an external customer

From the description of the system it follows that the maximum number of product units in the system is $s+Q_2$. $s+Q_2 \geq r+Q_1$, or $s+Q_2-Q_1 \geq r$. When $r>s$ we use the equivalent r^* value where $r^*=s+Q_2-Q_1$. In any case, it would be an extreme scenario to have a VMI system where the reorder point of the retailer is greater than the echelon reorder point of the vendor.

$$r = \min(r, s + Q_2 - Q_1)$$

6.4 States definition and state transitions

6.4.1 States definition

The whole system is a 4-dimensional continuous time Markov chain:

$$\{I_2(t), p_2(t), I_1(t),\ p_1(t), t \geq 0\}$$

At any time t, the state of the system can be defined by a four dimensional vector

$$\bar{S}_t = \left(I_2(t), p_2(t), I_1(t),\ p_1(t)\right)$$

, where:

$I_2(t)$: The vendor's inventory at time t. $I_2(t)$ includes inventory on hand at the vendor plus any inventory in transit to the retailer. $0 \leq I_2(t) \leq s + Q_2$. By the definition of the system, $I_2(t)$ is a multiple of Q_1.

$p_2(t)$: The phase of the replenishment order towards the vendor. If there is no inventory in transit towards the vendor, $p_2(t)=0$. In general, the permissible values for $p_2(t)$ are 0,1, or 2.

$I_1(t)$: The inventory on hand at the retailer at time t. $0 \leq I_1(t) \leq r + Q_1$

$p_1(t)$: The phase of the replenishment order towards the retailer. If there is no inventory in transit towards the retailer, $p_1(t)=0$. In general, the permissible values for $p_2(t)$ are 0,1, or 2.

The state space of the Markov process Ω is comprised of all the possible vectors $\bar{S}_t$ and its dimension can be computed as a function of the design variables through an iterative process. The states are ordered lexicographically. As basic level we define the set of all the states corresponding to a fixed vendor inventory I_2. Within each basic level, states are ordered according to the transition phase towards node 2 (p_2). For fixed I_2 and p_2, the states are ordered by retailer inventory I_1, and finally for fixed I_2, p_2 and I_1, states are ordered by p_1. To summarize:

State (w, x, y, z) precedes state (w', x', y', z') if w<w';

State (w, x, y, z) precedes state (w, x', y', z') if x<x';

State (w, x, y, z) precedes state (w, x, y', z') if y<y'.

State (w, x, y, z) precedes state (w, x, y, z') if z<z'.

6.4.2 State transitions

The state of the system can be altered instantaneously by five kinds of events. As we already mentioned, for methodology reasons and without posing any restrictions to our model, it is assumed that no two events can occur at exactly the same time. In infinitesimal time *dt* only one event can occur. The five classes of the events are:

1. The transition from the first phase to the second phase of the Coxian for an order in transit towards the vendor $(p_2(t) = 1 \rightarrow p_2(t + dt) = 2)$. The probability of such a transition happening in infinitesimal time dt is $d_{22} \cdot \mu_{21} \cdot dt + O(dt)$, where $O(dt)$ is an unspecified function such that $\lim_{dt \rightarrow 0} \frac{O(dt)}{dt} = 0$. $O(dt)$ stands for the probability that a second event will occur in infinitesimal time *dt*.
2. The arrival of an outstanding order at the vendor. The incoming order increases the inventory at the vendor ($I_2(t+dt) = I_2(t) + n \cdot Q_1$). If $I_2(t) < Q_1$ and $I_1(t) \le r$, on replenishment order arrival, Q_1 units are immediately forwarded for transportation towards the retailer ($p_1(t+dt) = 1$). The arrival of the replenishment order can be from the first or the second Coxian phase. In the first case, the probability of the event happening in infinitesimal time dt is $d_{21} \cdot \mu_{21} \cdot dt + O(dt)$. In the second case, the respective probability is $\mu_{22} \cdot dt + O(dt)$.
3. The transition from the first phase to the second phase of the Coxian for an order in transit towards the retailer $(p_1(t) = 1 \rightarrow p_1(t + dt) = 2)$. The probability of such a transition happening in infinitesimal time *dt* is $d_{12} \cdot \mu_{11} \cdot dt + O(dt)$.
4. The arrival of an outstanding order at the retailer. Q_1 units possessed by the vendor (assumption 5, section 6.3) are transferred to the retailer. The inventory on hand at the retailer increases by Q_1 units ($I_1(t+dt) = I_1(t) + Q_1$), while the inventory of the vendor decreases correspondingly ($I_2(t+dt) = I_2(t) - Q_1$). If $I_1(t+dt) \le r$, the vendor initiates a new order towards the retailer. The arrival of the replenishment order can be from the first or the second Coxian phase. In the first case, the probability of the event happening in infinitesimal time dt is $d_{11} \cdot \mu_{11} \cdot dt + O(dt)$. In the second case, the respective probability is $\mu_{12} \cdot dt + O(dt)$.
5. The occurrence of external demand at the retailer. The demand of each external customer follows an empirical distribution which can be described with a vector $dm = (dm_1, dm_2, ..., dm_{md})$, $\sum dm_i = 1$, where dm_i is the probability that an external customer will ask for exactly i product units. The inventory on hand at the retailer decreases correspondingly, while any excessive demand is lost. If $I_1(t+dt) \le r$ and $p_1(t) = 0$, a new replenishment order is sent from the vendor $(p_1(t+dt) = 1)$. If $I_1(t+dt) + I_2(t+dt) \le s$ and $p_2(t) = 0$, a new replenishment order is asked from the manufacturer $(p_2(t+dt) = 1)$. The probability that in infinitesimal time dt external demand of i product units will occur is $dm_i \cdot \lambda \cdot dt + O(dt)$.

6.5 The infinitesimal generator matrix

The infinitesimal generator matrix Q is a matrix such that q_{ij} is the instantaneous transition rate from state i to state j, i≠j, and $q_{ii} = -\sum_{\forall j \neq i} q_{ij}$. Similar transitions or events correspond to similar patterns or sub-matrices. These blocks are divided into three categories, those along the diagonal, those above the diagonal and those below the diagonal.

To facilitate our analysis we define the following characteristic values of the system:

f: it is the lowest basic level where the system reaches its maximum capacity. The maximum number of product units in the system is $s+Q_2$.

$f \cdot Q_1 + r + Q_1 > s + Q_2$ =>

$f \cdot Q_1 > s + n \cdot Q_1 - Q_1 - r$ =>

$$f > \frac{(s-r)}{Q_1} + n - 1$$

Since f is an integer:

$$f = \text{int}\left(\frac{s-r}{Q_1}\right) + n$$

(s-r) could be described as the vendor's "safety stock".

k: is the lowest basic level where the inventory at the vendor exceeds the echelon reorder point s. The structure of certain sub-matrices changes at this level. Moreover, from basic level k and above there can be no replenishment order towards the wholesaler.

$k \cdot Q_1 > s$

$$k = \text{int}\left(\frac{s}{Q_1} + 1\right)$$

h: is the lowest basic level for which the echelon inventory may exceed the echelon reorder point (when $s>r+Q_1$). For level below h, there is a replenishment order towards the wholesaler for any value of the retailer's inventory.

$h \cdot Q_1 + r + Q_1 > s$ =>

$$h > \frac{s-r}{Q_1} - 1$$

Since h is an integer:

$$h = \text{int}\left(\frac{s-r}{Q_1}\right)$$

Nl: It is the highest basic level.

$$Nl = \text{int}\left(\frac{s + Q_2}{Q_1}\right)$$

As we start with level "0", the number of basic levels will be Nl+1. At basic level Nl, the inventory at the vendor will have its maximum value $I_2(t) = Nl \cdot Q_1$.

We also define a vector *rd* such that the i[th] element of the vector rd_i is equal to the probability that the demand of an external customer will be equal to or more than i. If *md* is the maximum possible demand of an external customer:

$$rd_i = \sum_{j=i}^{md} dm_j$$

To simplify our notation we assume that $rd_i = 0, \forall i > md$.

Finally we denote:

$d_{11}=1-d_{12}$: the probability that during transportation of a replenishment order towards the retailer there will be only one phase.

$d_{21}=1-d_{22}$: the probability that during transportation of a replenishment order towards the vendor there will be only one phase.

6.5.1 Diagonal blocks

Sub-matrices along the diagonal are square blocks that describe transitions within the same basic level ($I_2(t)$=const). They correspond to transitions in the transportation phase or the occurrence of external demand. Each diagonal sub-matrix can be further analyzed into smaller blocks describing transitions between the same or different sub-levels. The general structure a diagonal block D:

$$\begin{array}{|c|c|c|}
\hline
D_0 & U_0 & \\
\hline
 & D_1 & U_1 \\
\hline
 & & D_2 \\
\hline
\end{array}$$

6.5.1.1 Dimension of sub-blocks

It is convenient for each basic Level L (L>0) to denote:

Z_1^L: the dimension of the sub-matrix of level L that corresponds to $p_2 = 0$ and $I_1 \leq r$

Z_2^L: the dimension of the sub-matrix of level L that corresponds to $p_2 = 0$ and $I_1 > r$

O_1^L: the dimension of the sub-matrix of level L that corresponds to $p_2 = 1$ and $I_1 \leq r$

O_2^L: the dimension of the sub-matrix of level L that corresponds to $p_2 = 1$ and $I_1 > r$

Z_1 states

With increasing basic level L, the number of Z_1 states increases, as the echelon inventory $L\cdot Q_1+I_1$ exceeds the echelon reorder point s for lower values of I_1. With every basic level, Z_1 increases by $2\cdot Q_1$ states until it takes its maximum value $2\cdot(r+1)$ for L=k.

Beyond level L=f, the number of permissible Z_1 states decreases as for higher values of I_1, the echelon inventory exceeds $s+Q_2$. For every level beyond L=f+1, the number of the permissible states decreases by $2\cdot(r-n\cdot Q_1-s+L\cdot Q_1)$ states.

If we define an index in_f indicating if L>f and an index in_h indicating if L>h

$$in_f=\begin{cases}0, & L\le f\\ 1, & L>f\end{cases}$$

$$in_h=\begin{cases}0, & L\le h\\ 1, & L>h\end{cases}$$

,then the number of permissible Z_1 states for level L is

$$Z_1=\left(\min\left(2\cdot(r-s)+2\cdot L\cdot Q_1,2\cdot(r+1)\right)-in_f\cdot 2\cdot(r-n\cdot Q_1-s+L\cdot Q_1)\right)\cdot in_h$$

The first part of above equation corresponds to the increase in the number of Z_1 states with increasing L until L=k. The second part of the equation corresponds to the decrease in the number of states for L>f. If L≤h, then $p_2>0$ and $Z_1=0$.

Z_2 states

Z_2 states correspond to $I_1>r$ when $p_2=0$. For basic level L<h, the echelon inventory $I_{ech}<s$ for every possible value of I_1, so $p_2>0$ and $Z_2=0$.

For basic level L=h, the echelon inventory for the first time exceeds s for the higher values of I_1: $Z_2=r+Q_1-s+L\cdot Q_1$. Since between successive basic levels I_2 increases by Q_1 units, $Z_2<Q_1$.

When h<L<f, Z_2 takes its maximum value $Z_2=Q_1$

For L=f, for some values of I_1, $I_{ech}>s+Q_2$ (non-permissible states). The number of Z_2 states will be less than Q_1: $Z_2=(n-L)\cdot Q_1+s-r$

When L>f, given the restriction that $I_{ech}\le s+Q_2$, for the all the permissible values $I_1\le r$ and $Z_2=0$.

Summing up:

$$Z_2 = \begin{cases} 0, & L < h \quad or \quad L > f \\ r + Q_1 - s + L \cdot Q_1, & L = h \\ Q_1, & h < L < f \\ (n - L) \cdot Q_1 + s - r, & L = f \end{cases}$$

O_1 states

O_1 states correspond to $p_2 > 0$ and $I_1 \leq r$. The possible values of p_2 are 1 or 2. When $L \leq h$, the O_1 has its maximum value $O_1 = 2 \cdot (r+1)$.

When $h<L<k$, the number of O_1 states decreases with each successive basic level, as the number of states where $I_{ech} \leq s$ decreases. For basic level L:

$$O_1 = 2 \cdot (r+1) - 2 \cdot (r - s + L \cdot Q_1)$$

$$O_1 = 2 \cdot (s - L \cdot Q_1 + 1)$$

With every successive level the number of permissible states decreases by $2 \cdot Q_1$.

When $L \geq k$, there can be no inventory in transit towards the vendor ($p_2=0$), and $O_1=0$.

Summing up:

$$O_1 = \begin{cases} 2 \cdot (r+1), & L \leq h \\ 2 \cdot (s - L \cdot Q_1 + 1), & h < L < k \\ 0, & L \geq k \end{cases}$$

O_2 states

O_2 states correspond to $p_2 > 0$ and $I_1 > r$. The number of possible states:

$$O_2 = \begin{cases} Q_1, & L < h \\ s - r - L \cdot Q_1, & L = h \\ 0, & L > h \end{cases}$$

6.5.1.2 Block D_0

D_0 is Z_1+Z_2 dimension square block corresponding to transitions within the same basic level and the same sub-level when there is no batch in transit to the wholesaler ($I_{ech}>s$, $p_2=0$). To facilitate the analysis, D_0 can further be reduced to constituent blocks:

$$D_0 = \begin{bmatrix} D_{01} & N_0 \\ S_0 & D_{02} \end{bmatrix}$$

D_{01} is a $Z_1 \times Z_1$ block that corresponds to transitions where $I_1 \leq r$. If we assume $dm(j) = 0, \forall j > md$, for L<k:

$$D_{01} = \begin{bmatrix}
-\mu_{11}-\lambda & d_{12}\cdot\mu_{11} & & & & & & & & & \\
 & -\mu_{12}-\lambda & & & & & & & & & \\
dm(1)\cdot\lambda & & -\mu_{11}-\lambda & d_{12}\cdot\mu_{11} & & & & & & & \\
 & dm(1)\cdot\lambda & & -\mu_{12}-\lambda & & & & & & & \\
dm(2)\cdot\lambda & & dm(1)\cdot\lambda & & -\mu_{11}-\lambda & d_{12}\cdot\mu_{11} & & & & & \\
 & dm(2)\cdot\lambda & & & & -\mu_{12}-\lambda & & & & & \\
\vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \cdots & & & & \\
dm(\frac{Z_1}{2}-2)\cdot\lambda & & dm(\frac{Z_1}{2}-3)\cdot\lambda & & dm(\frac{Z_1}{2}-4)\cdot\lambda & & \cdots & -\mu_{11}-\lambda & d_{12}\cdot\mu_{11} & & \\
 & dm(\frac{Z_1}{2}-2)\cdot\lambda & & dm(\frac{Z_1}{2}-3)\cdot\lambda & & dm(\frac{Z_1}{2}-4)\cdot\lambda & \cdots & & -\mu_{12}-\lambda & & \\
dm(\frac{Z_1}{2}-1)\cdot\lambda & & dm(\frac{Z_1}{2}-2)\cdot\lambda & & dm(\frac{Z_1}{2}-3)\cdot\lambda & & \cdots & dm(1)\cdot\lambda & & -\mu_{11}-\lambda & d_{12}\cdot\mu_{11} \\
 & dm(\frac{Z_1}{2}-1)\cdot\lambda & & dm(\frac{Z_1}{2}-2)\cdot\lambda & & dm(\frac{Z_1}{2}-3)\cdot\lambda & \cdots & & dm(1)\cdot\lambda & & -\mu_{12}-\lambda
\end{bmatrix}.$$

When L≥k, the first two columns of D_{01} correspond to $I_1=0$. If we assume $rd(j) = 0, \forall j > md$, the general structure of D_{01}:

$$D_{01} = \begin{bmatrix}
-\mu_{11} & d_{12}\cdot\mu_{11} & & & & & & & & & \\
 & -\mu_{12} & & & & & & & & & \\
rd(1)\cdot\lambda & & -\mu_{11}-\lambda & d_{12}\cdot\mu_{11} & & & & & & & \\
 & rd(1)\cdot\lambda & & -\mu_{12}-\lambda & & & & & & & \\
rd(2)\cdot\lambda & & dm(1)\cdot\lambda & & -\mu_{11}-\lambda & d_{12}\cdot\mu_{11} & & & & & \\
 & rd(2)\cdot\lambda & & dm(1)\cdot\lambda & & -\mu_{12}-\lambda & & & & & \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \cdots & & & & \\
rd(\frac{Z_1}{2}-2)\cdot\lambda & & dm(\frac{Z_1}{2}-3)\cdot\lambda & & dm(\frac{Z_1}{2}-4)\cdot\lambda & & \cdots & -\mu_{11}-\lambda & d_{12}\cdot\mu_{11} & & \\
 & rd(\frac{Z_1}{2}-2)\cdot\lambda & & dm(\frac{Z_1}{2}-3)\cdot\lambda & & dm(\frac{Z_1}{2}-4)\cdot\lambda & \cdots & & -\mu_{12}-\lambda & & \\
rd(\frac{Z_1}{2}-1)\cdot\lambda & & dm(\frac{Z_1}{2}-2)\cdot\lambda & & dm(\frac{Z_1}{2}-3)\cdot\lambda & & \cdots & dm(1)\cdot\lambda & & -\mu_{11}-\lambda & d_{12}\cdot\mu_{11} \\
 & rd(\frac{Z_1}{2}-1)\cdot\lambda & & dm(\frac{Z_1}{2}-2)\cdot\lambda & & dm(\frac{Z_1}{2}-3)\cdot\lambda & \cdots & & dm(1)\cdot\lambda & & -\mu_{12}-\lambda
\end{bmatrix}.$$

D_{02} is a $Z_2 \times Z_2$ block that corresponds to transitions where $I_1>r$. Its general structure:

$$D_{02} = \begin{bmatrix}
-\lambda & & & & \\
dm(1)\cdot\lambda & -\lambda & & & \\
dm(2)\cdot\lambda & dm(1)\cdot\lambda & \cdots & & \\
\vdots & \vdots & \cdots & -\lambda & \\
dm(Z_2-1)\cdot\lambda & dm(Z_2-2)\cdot\lambda & \cdots & dm(1)\cdot\lambda & -\lambda
\end{bmatrix}$$

S_0 is a $Z_2 \times Z_1$ block that corresponds to the triggering of a replenishment order towards the retailer. Here we have transitions from Z_2 states ($I_1>r$) to Z_1 states ($I_1 \leq r$). When L<k and assuming $dm(j) = 0, \forall j > md$:

$$S_0 = \begin{bmatrix} dm(Z_1/2)\cdot\lambda & 0 & \cdots & dm(2)\cdot\lambda & 0 & dm(1)\cdot\lambda & 0 \\ dm(Z_1/2+1)\cdot\lambda & 0 & \cdots & dm(3)\cdot\lambda & 0 & dm(2)\cdot\lambda & 0 \\ \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \vdots \\ dm(Z_1/2+Z_2-1)\cdot\lambda & 0 & \cdots & dm(Z_2+1)\cdot\lambda & 0 & dm(Z_2)\cdot\lambda & 0 \end{bmatrix}$$

When L≥k, the first two columns of S_0 correspond to $I_1=0$. If we assume $rd(j)=0, \forall j > md$, the general structure of S_0:

$$S_0 = \begin{bmatrix} rd(Z_1/2) & 0 & dm(Z_1/2-1)\cdot\lambda & 0 & \cdots & dm(2)\cdot\lambda & 0 & dm(1)\cdot\lambda & 0 \\ rd(Z_1/2+1) & 0 & dm(Z_1/2)\cdot\lambda & 0 & \cdots & dm(3)\cdot\lambda & 0 & dm(2)\cdot\lambda & 0 \\ \vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \vdots \\ rd(Z_1/2+Z_2-1) & 0 & dm(Z_1/2+Z_2-2)\cdot\lambda & 0 & \cdots & dm(Z_2+1)\cdot\lambda & 0 & dm(Z_2)\cdot\lambda & 0 \end{bmatrix}$$

N_0 is a null matrix of $Z_1 \times Z_2$ dimensions.

6.5.1.3 Block D_1

D_1 is square block of $O_1 + O_2$ dimension. It describes transitions within the same basic level and in the same sub-level when there is a batch in transit towards the wholesaler in the first Coxian phase. To facilitate the analysis, D_1 can be further reduced into constituent blocks:

$$D_1 = \begin{bmatrix} D_{11} & N_1 \\ S_1 & D_{12} \end{bmatrix}$$

D_{11} is a $O_1 \times O_1$ block describing transitions between states where I_1≤r. The first two columns correspond to $I_1=0$. The general structure of D_{11}:

$$D_{11} = \begin{bmatrix} -\mu_{21}-\mu_{11} & d_{12}\cdot\mu_{11} & & & & & & & & & \\ & -\mu_{21}-\mu_{12} & & & & & & & & & \\ rd(1)\cdot\lambda & & -\mu_{21}-\mu_{11}-\lambda & d_{12}\cdot\mu_{11} & & & & & & & \\ & rd(1)\cdot\lambda & & -\mu_{21}-\mu_{12}-\lambda & & & & & & & \\ rd(2)\cdot\lambda & & dm(1)\cdot\lambda & & -\mu_{21}-\mu_{11}-\lambda & d_{12}\cdot\mu_{11} & & & & & \\ & rd(2)\cdot\lambda & & dm(1)\cdot\lambda & & -\mu_{21}-\mu_{12}-\lambda & & & & & \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \cdots & & & & \\ rd(\frac{O_1}{2}-2)\cdot\lambda & & dm(\frac{O_1}{2}-3)\cdot\lambda & & dm(\frac{O_1}{2}-4)\cdot\lambda & & \cdots & -\mu_{21}-\mu_{11}-\lambda & d_{12}\cdot\mu_{11} & & \\ & rd(\frac{O_1}{2}-2)\cdot\lambda & & dm(\frac{O_1}{2}-3)\cdot\lambda & & dm(\frac{O_1}{2}-4)\cdot\lambda & \cdots & & -\mu_{21}-\mu_{12}-\lambda & & \\ rd(\frac{O_1}{2}-1)\cdot\lambda & & dm(\frac{O_1}{2}-2)\cdot\lambda & & dm(\frac{O_1}{2}-3)\cdot\lambda & & \cdots & dm(1)\cdot\lambda & & -\mu_{21}-\mu_{11}-\lambda & d_{12}\cdot\mu_{11} \\ & rd(\frac{O_1}{2}-1)\cdot\lambda & & dm(\frac{O_1}{2}-2)\cdot\lambda & & dm(\frac{O_1}{2}-3)\cdot\lambda & \cdots & & dm(1)\cdot\lambda & & -\mu_{21}-\mu_{12}-\lambda \end{bmatrix}.$$

D_{12} is a $O_2 \times O_2$ block corresponding to transitions between states where I_1>r. Its general structure:

$$D_{12} = \begin{bmatrix} -\mu_{21} - \lambda & & & & \\ dm(1)\cdot\lambda & -\mu_{21} - \lambda & & & \\ dm(2)\cdot\lambda & dm(1)\cdot\lambda & \cdots & & \\ \vdots & \vdots & \cdots & -\mu_{21} - \lambda & \\ dm(O_2 - 1)\cdot\lambda & dm(O_2 - 2)\cdot\lambda & \cdots & dm(1)\cdot\lambda & -\mu_{21} - \lambda \end{bmatrix}$$

S_1 describes triggering of a replenishment order towards the retailer. Here we have transitions from states where I_1>r (O_2) to states where $I_1 \le$r (O_1). S_1 is a $O_2 \times O_1$ block with general structure:

$$S_1 = \begin{bmatrix} rd(O_1/2)\cdot\lambda & 0 & dm(O_1/2-1)\cdot\lambda & 0 & dm(O_1/2-2)\cdot\lambda & 0 & \cdots & dm(2)\cdot\lambda & 0 & dm(1)\cdot\lambda & 0 \\ rd(O_1/2+1)\cdot\lambda & 0 & dm(O_1/2)\cdot\lambda & 0 & dm(O_1/2-3)\cdot\lambda & 0 & \cdots & dm(3)\cdot\lambda & 0 & dm(2)\cdot\lambda & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \vdots \\ rd(O_1/2+O_2-1)\cdot\lambda & 0 & dm(O_1/2+O_2-2)\cdot\lambda & 0 & dm(O_1/2+O_2-3)\cdot\lambda & 0 & \cdots & dm(O_2+1)\cdot\lambda & 0 & dm(O_2)\cdot\lambda & 0 \end{bmatrix}$$

Finally, N_1 is a zero block of $O_1 \times O_2$ dimensions.

6.5.1.4 Block D_2

D_2 describes transitions within the same basic level and within the same sub-level where there is an order in transit towards the wholesaler in the second Coxian phase (p_2=2). It is a $O_1 + O_2$ dimensional square block and can be further analyzed into constituent sub-blocks:

$$D_2 = \begin{bmatrix} D_{21} & N_2 \\ S_2 & D_{22} \end{bmatrix}$$

D_{21} is a $O_1 \times O_1$ block describing transitions between states where $I_1 \le$r, with the first two columns corresponding to I_1=0. The general structure of D_{21}:

$$D_{21} = \begin{bmatrix}
-\mu_{22}-\mu_{11} & d_{12}\cdot\mu_{11} & & & & & & & & & \\
 & -\mu_{22}-\mu_{12} & & & & & & & & & \\
rd(1)\cdot\lambda & & -\mu_{22}-\mu_{11}-\lambda & d_{12}\cdot\mu_{11} & & & & & & & \\
 & rd(1)\cdot\lambda & & -\mu_{22}-\mu_{12}-\lambda & & & & & & & \\
rd(2)\cdot\lambda & & dm(1)\cdot\lambda & & -\mu_{22}-\mu_{11}-\lambda & d_{12}\cdot\mu_{11} & & & & & \\
 & rd(2)\cdot\lambda & & dm(1)\cdot\lambda & & -\mu_{22}-\mu_{12}-\lambda & & & & & \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \cdots & & & & \\
rd(\frac{O_1}{2}-2)\cdot\lambda & & dm(\frac{O_1}{2}-3)\cdot\lambda & & dm(\frac{O_1}{2}-4)\cdot\lambda & & \cdots & -\mu_{22}-\mu_{11}-\lambda & d_{12}\cdot\mu_{11} & & \\
 & rd(\frac{O_1}{2}-2)\cdot\lambda & & dm(\frac{O_1}{2}-3)\cdot\lambda & & dm(\frac{O_1}{2}-4)\cdot\lambda & \cdots & & -\mu_{22}-\mu_{12}-\lambda & & \\
rd(\frac{O_1}{2}-1)\cdot\lambda & & dm(\frac{O_1}{2}-2)\cdot\lambda & & dm(\frac{O_1}{2}-3)\cdot\lambda & & \cdots & dm(1)\cdot\lambda & & -\mu_{22}-\mu_{11}-\lambda & d_{12}\cdot\mu_{11} \\
 & rd(\frac{O_1}{2}-1)\cdot\lambda & & dm(\frac{O_1}{2}-2)\cdot\lambda & & dm(\frac{O_1}{2}-3)\cdot\lambda & \cdots & & dm(1)\cdot\lambda & & -\mu_{22}-\mu_{12}-\lambda
\end{bmatrix}.$$

D_{22} is a $O_2 \times O_2$ block corresponding to transitions between states where I_1>r. Its general structure:

$$D_{22} = \begin{bmatrix} -\mu_{22}-\lambda & & & & \\ dm(1)\cdot\lambda & -\mu_{22}-\lambda & & & \\ dm(2)\cdot\lambda & dm(1)\cdot\lambda & \cdots & & \\ \vdots & \vdots & \cdots & -\mu_{22}-\lambda & \\ dm(O_2-1)\cdot\lambda & dm(O_2-2)\cdot\lambda & \cdots & dm(1)\cdot\lambda & -\mu_{22}-\lambda \end{bmatrix}$$

S_2 describes transitions from states where $I_1>r$ (O_2) to states where $I_1\leq r$ (O_1). It corresponds to triggering of a replenishment order towards the retailer. S_2 is a $O_2 \times O_1$ block identical to block S_1 described above:

$$S_2 = \begin{bmatrix} rd(O_1/2)\cdot\lambda & 0 & dm(O_1/2-1)\cdot\lambda & 0 & dm(O_1/2-2)\cdot\lambda & 0 & \cdots & dm(2)\cdot\lambda & 0 & dm(1)\cdot\lambda & 0 \\ rd(O_1/2+1)\cdot\lambda & 0 & dm(O_1/2)\cdot\lambda & 0 & dm(O_1/2-3)\cdot\lambda & 0 & \cdots & dm(3)\cdot\lambda & 0 & dm(2)\cdot\lambda & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \vdots \\ rd(O_1/2+O_2-1)\cdot\lambda & 0 & dm(O_1/2+O_2-2)\cdot\lambda & 0 & dm(O_1/2+O_2-3)\cdot\lambda & 0 & \cdots & dm(O_2+1)\cdot\lambda & 0 & dm(O_2)\cdot\lambda & 0 \end{bmatrix}$$

N_2 is a zero block of $O_1 \times O_2$ dimensions.

6.5.1.5 Block U_0

U_0 describes triggering of a replenishment order towards the vendor. It corresponds to transitions from states where $p_2=0$ (there is no order in transit towards the vendor) to states where $p_2=1$. The only event that causes a decrease in echelon inventory and may cause such triggering is the occurrence of external demand. There can be three kinds of transitions:

From states Z_1 to states O_1: Here there is triggering of a replenishment order towards the wholesaler while there is an outstanding order towards the retailer ($p_1>0$).

From states Z_2 to states O_2: In this case there is triggering of a replenishment order towards the wholesaler while there is no outstanding order towards the retailer ($p_1=0$).

From states Z_2 to states O_1: Here there is triggering of replenishment orders towards both the wholesaler and the retailer.

The exact structure of U_0 depends on which kind of transition occurs. Generally, the dimensions of the block are $(Z_1+Z_2)\times(O_1+O_2)$. When there are transitions from Z_2 to O_1 and O_2:

$$\begin{array}{cc} & \begin{array}{cc} \overbrace{\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad}^{O_1} & \overbrace{\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad}^{O_2} \end{array} \\ U_0 = & \left.\left[\begin{array}{ccccc|cccc} rd(O_2+O_1/2)\cdot\lambda & 0 & \cdots & dm(O_2+1)\cdot\lambda & 0 & dm(O_2)\cdot\lambda & \cdots & dm(2)\cdot\lambda & dm(1)\cdot\lambda \\ rd(O_2+O_1/2+1)\cdot\lambda & 0 & \cdots & dm(O_2+2)\cdot\lambda & 0 & dm(O_2+1)\cdot\lambda & \cdots & dm(3)\cdot\lambda & dm(2)\cdot\lambda \\ \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ rd(O_2+O_1/2+Z_2-1)\cdot\lambda & 0 & \cdots & dm(O_2+Z_2)\cdot\lambda & 0 & dm(O_2+Z_2-1)\cdot\lambda & \cdots & dm(Z_2+1)\cdot\lambda & dm(Z_2)\cdot\lambda \end{array}\right]\right\} Z_2 \end{array}$$

When there are transitions from Z_1 and Z_2 to O_1:

$$U_0 = \overbrace{\left[\begin{array}{ccccccc}
rd(O_1/2)\cdot\lambda & 0 & dm(O_1/2-1)\cdot\lambda & 0 & \cdots & dm(1)\cdot\lambda & 0 \\
0 & rd(O_1/2)\cdot\lambda & 0 & dm(O_1/2-1)\cdot\lambda & \cdots & 0 & dm(1)\cdot\lambda \\
rd(O_1/2+1)\cdot\lambda & 0 & dm(O_1/2)\cdot\lambda & 0 & \cdots & dm(2)\cdot\lambda & 0 \\
0 & rd(O_1/2+1)\cdot\lambda & 0 & dm(O_1/2)\cdot\lambda & \cdots & 0 & dm(2)\cdot\lambda \\
\vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\
rd(O_1/2+Z_1/2-1)\cdot\lambda & 0 & dm(O_1/2+Z_1/2-2)\cdot\lambda & 0 & \cdots & dm(Z_1/2)\cdot\lambda & 0 \\
0 & rd(O_1/2+Z_1/2-1)\cdot\lambda & 0 & dm(O_1/2+Z_1/2-2)\cdot\lambda & \cdots & 0 & dm(Z_1/2)\cdot\lambda \\
\hline
rd(O_1/2+Z_1/2)\cdot\lambda & 0 & dm(O_1/2+Z_1/2-1)\cdot\lambda & 0 & \cdots & dm(Z_1/2+1)\cdot\lambda & 0 \\
rd(O_1/2+Z_1/2+1)\cdot\lambda & 0 & dm(O_1/2+Z_1/2)\cdot\lambda & 0 & \cdots & dm(Z_1/2+2)\cdot\lambda & 0 \\
\vdots & \vdots & \vdots & \vdots & \cdots & & \\
rd(O_1/2+Z_1/2+Z_2-1)\cdot\lambda & 0 & dm(O_1/2+Z_1/2+Z_2-2)\cdot\lambda & 0 & \cdots & dm(Z_1/2+Z_2)\cdot\lambda & 0
\end{array}\right]}^{O_1}
\begin{array}{l}
\left.\vphantom{\begin{array}{c}a\\a\\a\\a\\a\\a\\a\end{array}}\right\}Z_1 \\
\left.\vphantom{\begin{array}{c}a\\a\\a\\a\end{array}}\right\}Z_2
\end{array}$$

6.5.1.6 Block U_1

Sub-matrix U_1 describes transitions from the first to the second phase of transportation towards the wholesaler. It is a $O_1 + O_2$ dimensional, diagonal block of $d_{22} \cdot \mu_{21}$.

$$U_1 = \begin{bmatrix} d_{22}\cdot\mu_{21} & & & \\ & d_{22}\cdot\mu_{21} & & \\ & & \cdots & \\ & & & d_{22}\cdot\mu_{21} \end{bmatrix}$$

6.5.1.7 Boundary states (L=0)

For the boundary states where $I_2 = 0$ (L=0) there can be no transit towards the retailer. The rules for the construction of the diagonal block are similar to but slightly different from those used for L>0. With regard to the dimensions of the constituent blocks:

$$Z_1^0 = 0$$

$$Z_2^0 = \begin{cases} 0 & ,h > 0 \\ r - s + Q_1 & ,h = 0 \end{cases}$$

$$O_1^0 = r + 1$$

$$O_2^0 = \begin{cases} Q_1 & ,h > 0 \\ s - r & ,h = 0 \end{cases}$$

<u>D_0</u>

Block D_0 is constructed according to the rules described in 6.5.1.2. Since $Z_1 = 0$, it is a $Z_2 \times Z_2$ matrix:

$$D_0 = \begin{bmatrix} -\lambda & & & \\ dm(1)\cdot\lambda & -\lambda & & \\ \vdots & \cdots & \cdots & \\ dm(Z_2-1) & \cdots & dm(1)\cdot\lambda & -\lambda \end{bmatrix}$$

<u>D_1</u>

Block D_1 is a $(O_1+O_2)\times(O_1+O_2)$ matrix that can be divided into four sub-matrices in a way similar to that described in 6.5.1.3:

$$D_1 = \begin{bmatrix} D_{11} & O_1 \\ S_1 & D_{12} \end{bmatrix}$$

D_{11} is a $O_1\times O_1$ block with general structure:

$$D_{11} = \begin{bmatrix} -\mu_{21} & & & & & & \\ rd(1)\cdot\lambda & -\mu_{21}-\lambda & & & & & \\ rd(2)\cdot\lambda & dm(1)\cdot\lambda & -\mu_{21}-\lambda & & & & \\ rd(3)\cdot\lambda & dm(2)\cdot\lambda & dm(1)\cdot\lambda & -\mu_{21}-\lambda & & & \\ \vdots & \vdots & \vdots & & \cdots & & \\ rd(O_1-2)\cdot\lambda & dm(O_1-3)\cdot\lambda & dm(O_1-4)\cdot\lambda & dm(O_1-5)\cdot\lambda & \cdots & -\mu_{21}-\lambda & \\ rd(O_1-1)\cdot\lambda & dm(O_1-2)\cdot\lambda & dm(O_1-3)\cdot\lambda & dm(O_1-4)\cdot\lambda & \cdots & dm(1)\cdot\lambda & -\mu_{21}-\lambda \end{bmatrix}$$

D_{12} is a $O_2\times O_2$ block:

$$D_{12} = \begin{bmatrix} -\mu_{21}-\lambda & & & & & & \\ dm(1)\cdot\lambda & -\mu_{21}-\lambda & & & & & \\ dm(2)\cdot\lambda & dm(1)\cdot\lambda & -\mu_{21}-\lambda & & & & \\ dm(3)\cdot\lambda & dm(2)\cdot\lambda & dm(1)\cdot\lambda & -\mu_{21}-\lambda & & & \\ \vdots & \vdots & \vdots & & \cdots & & \\ dm(O_2-2)\cdot\lambda & dm(O_2-3)\cdot\lambda & dm(O_2-4)\cdot\lambda & dm(O_2-5)\cdot\lambda & \cdots & -\mu_{21}-\lambda & \\ dm(O_2-1)\cdot\lambda & dm(O_2-2)\cdot\lambda & dm(O_2-3)\cdot\lambda & dm(O_2-4)\cdot\lambda & \cdots & dm(1)\cdot\lambda & -\mu_{21}-\lambda \end{bmatrix}$$

S_1 describes the crossing of reorder point r. Here we have transitions from states where $I_1>r$ (O_2) to states where $I_1\leq r$ (O_1). S_1 is a $O_2\times O_1$ block with general structure:

$$S_1 = \begin{bmatrix} rd(O_1)\cdot\lambda & dm(O_1-1)\cdot\lambda & dm(O_1-2)\cdot\lambda & \cdots & dm(3) & dm(2)\cdot\lambda & dm(1)\cdot\lambda \\ rd(O_1+1)\cdot\lambda & dm(O_1)\cdot\lambda & dm(O_1-1)\cdot\lambda & \cdots & dm(4)\cdot\lambda & dm(3)\cdot\lambda & dm(2)\cdot\lambda \\ rd(O_1+2)\cdot\lambda & dm(O_1+1)\cdot\lambda & dm(O_1)\cdot\lambda & \cdots & dm(5)\cdot\lambda & dm(4)\cdot\lambda & dm(3)\cdot\lambda \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots \\ rd(O_1+O_2-1)\cdot\lambda & dm(O_1+O_2-2)\cdot\lambda & dm(O_1+O_2-3)\cdot\lambda & \cdots & dm(O_2+2)\cdot\lambda & dm(O_2+1)\cdot\lambda & dm(O_2)\cdot\lambda \end{bmatrix}$$

Finally, O_1 is a zero block of $O_1 \times O_2$ dimensions

<u>D2</u>

D_2 block has dimensions $(O_1 + O_2) \times (O_1 + O_2)$ and its structure is similar to that of D_1:

$$D_2 = \begin{bmatrix} D_{21} & O_2 \\ S_2 & D_{22} \end{bmatrix}$$

D_{21} is a $O_1 \times O_1$ block with general structure:

$$D_{21} = \begin{bmatrix} -\mu_{22} & & & & & & \\ rd(1)\cdot\lambda & -\mu_{22}-\lambda & & & & & \\ rd(2)\cdot\lambda & dm(1)\cdot\lambda & -\mu_{22}-\lambda & & & & \\ rd(3)\cdot\lambda & dm(2)\cdot\lambda & dm(1)\cdot\lambda & -\mu_{22}-\lambda & & & \\ \vdots & \vdots & \vdots & & \cdots & & \\ rd(O_1-2)\cdot\lambda & dm(O_1-3)\cdot\lambda & dm(O_1-4)\cdot\lambda & dm(O_1-5)\cdot\lambda & \cdots & -\mu_{22}-\lambda & \\ rd(O_1-1)\cdot\lambda & dm(O_1-2)\cdot\lambda & dm(O_1-3)\cdot\lambda & dm(O_1-4)\cdot\lambda & \cdots & dm(1)\cdot\lambda & -\mu_{22}-\lambda \end{bmatrix}$$

D_{22} is a $O_2 \times O_2$ block:

$$D_{22} = \begin{bmatrix} -\mu_{22}-\lambda & & & & & & \\ dm(1)\cdot\lambda & -\mu_{22}-\lambda & & & & & \\ dm(2)\cdot\lambda & dm(1)\cdot\lambda & -\mu_{22}-\lambda & & & & \\ dm(3)\cdot\lambda & dm(2)\cdot\lambda & dm(1)\cdot\lambda & -\mu_{22}-\lambda & & & \\ \vdots & \vdots & \vdots & & \cdots & & \\ dm(O_2-2)\cdot\lambda & dm(O_2-3)\cdot\lambda & dm(O_2-4)\cdot\lambda & dm(O_2-5)\cdot\lambda & \cdots & -\mu_{22}-\lambda & \\ dm(O_2-1)\cdot\lambda & dm(O_2-2)\cdot\lambda & dm(O_2-3)\cdot\lambda & dm(O_2-4)\cdot\lambda & \cdots & dm(1)\cdot\lambda & -\mu_{22}-\lambda \end{bmatrix}$$

S_2 describes the crossing of reorder point r. Here we have transitions from states where $I_1 > r$ (O_2) to states where $I_1 \leq r$ (O_1). S_2 is a $O_2 \times O_1$ block with general structure:

$$S_2 = \begin{bmatrix} rd(O_1)\cdot\lambda & dm(O_1-1)\cdot\lambda & dm(O_1-2)\cdot\lambda & \cdots & dm(3) & dm(2)\cdot\lambda & dm(1)\cdot\lambda \\ rd(O_1+1)\cdot\lambda & dm(O_1)\cdot\lambda & dm(O_1-1)\cdot\lambda & \cdots & dm(4)\cdot\lambda & dm(3)\cdot\lambda & dm(2)\cdot\lambda \\ rd(O_1+2)\cdot\lambda & dm(O_1+1)\cdot\lambda & dm(O_1)\cdot\lambda & \cdots & dm(5)\cdot\lambda & dm(4)\cdot\lambda & dm(3)\cdot\lambda \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots \\ rd(O_1+O_2-1)\cdot\lambda & dm(O_1+O_2-2)\cdot\lambda & dm(O_1+O_2-3)\cdot\lambda & \cdots & dm(O_2+2)\cdot\lambda & dm(O_2+1)\cdot\lambda & dm(O_2)\cdot\lambda \end{bmatrix}$$

Finally, O_2 is a zero block of $O_1 \times O_2$ dimensions

<u>U0</u>

U_0 corresponds to triggering of a replenishment order towards the vendor. It describes transitions from states where $p_2=0$ (there is no order in transit towards the vendor) to states where $p_2=1$. U_0 is a $Z_2 \times (O_1+O_2)$ block with general structure:

$$U_0 = \begin{bmatrix} rd(O_1+O_2)\cdot\lambda & dm(O_1+O_2-1)\cdot\lambda & \cdots & dm(3)\cdot\lambda & dm(2)\cdot\lambda & dm(1)\cdot\lambda \\ rd(O_1+O_2+1)\cdot\lambda & dm(O_1+O_2)\cdot\lambda & \cdots & dm(4)\cdot\lambda & dm(3)\cdot\lambda & dm(2)\cdot\lambda \\ rd(O_1+O_2+2)\cdot\lambda & dm(O_1+O_2+1)\cdot\lambda & \cdots & dm(5)\cdot\lambda & dm(4)\cdot\lambda & dm(3)\cdot\lambda \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ rd(O_1+O_2+Z_2-1)\cdot\lambda & dm(O_1+O_2+Z_2-2)\cdot\lambda & \cdots & dm(Z_2+2)\cdot\lambda & dm(Z_2+1)\cdot\lambda & dm(Z_2)\cdot\lambda \end{bmatrix}$$

<u>U_1</u>

U_1 describes transitions from the first to the second phase of transportation towards the wholesaler. It is a O_1+O_2 dimensional, diagonal block of $d_{22}\cdot\mu_{21}$.

$$U_1 = \begin{bmatrix} d_{22}\cdot\mu_{21} & & \\ & \cdots & \\ & & d_{22}\cdot\mu_{21} \end{bmatrix}$$

6.5.2 Upper-diagonal blocks

To simplify our analysis, we define lp^L the position of the last state of basic level L in the ordering of states that was defined in paragraph 6.4.1. lp^L can be easily computed as the parameters Z_1^L, Z_2^L, O_1^L and O_2^L are known for every basic level.

$$lp^L = \sum_{i=0}^{L}\left(Z_1^i + Z_2^i + 2\cdot(O_1^i + O_2^i)\right)$$

Upper-diagonal blocks describe transitions from Level L_1 ($L_1>0$) to Level $L_2=L_1+n$. They correspond to the arrival of a replenishment order at the vendor. We denote:

$O_1^{L_1}$: The number of O_1 states for level L_1

$O_2^{L_1}$: The number of O_2 states for level L_1

$O_1^{L_2}$: The number of O_1 states for level L_2

$O_2^{L_2}$: The number of O_2 states for level L_2

$Z_1^{L_2}$: The number of Z_1 states for level L_2

$Z_2^{L_2}$: The number of Z_2 states for level L_2

In general, we can define two kinds of upper-diagonal blocks, depending on whether the arrival of the replenishment order is from the first (UD_1), or the second (UD_2) phase of the Coxian-2 transportation.

UD_1 corresponds to transitions to a higher basic level due to the arrival of a replenishment order at the wholesaler when the particular order exhibits only one transportation phase. In a sense, UD_1 is "complementary" to sub-block U_1. Its dimensions are $(O_1^{L_1}+O_2^{L_1})\times(Z_1^{L_2}+Z_2^{L_2}+O_1^{L_2}+O_2^{L_2})$.

UD_1 occurs for levels where $L_1 \cdot Q_1 \le s$, or from level 1 to level k-1 (k is the basic level where $k \cdot Q_1 > s$). Its exact structure can be defined taking into consideration the following:

- If there is a state $(L_2,1,x,y)$, then there is also a state $(L_1,1,x,y)$ and a corresponding transition $(L_1,1,x,y) \to (L_2,1,x,y)$ with probability $d_{21} \cdot \mu_{21}$.
- From a state $(L_1,1,x,y)$ there can be a transition to a state $(L_2,0,x,y)$, or $(L_2,1,x,y)$.
- If there is a state $(L_2,0,x,1)$ and there is a transition from level $L_1=0$, then there will exist a state $(0,1,x,0)$ where the retailer asks for a replenishment order but the wholesaler has no inventory on hand and the corresponding transition $(0,1,x,0) \to (L_2,0,x,1)$ with probability $d_{21} \cdot \mu_{21}$.

The general structure of sub-matrix UD_1 for $L_1>0$:

$$
\begin{array}{cc}
 & \begin{array}{cc} \overbrace{\qquad\qquad\qquad\qquad\qquad\qquad}^{Z_1^{L_2}+Z_2^{L_2}} & \overbrace{\qquad\qquad\qquad\qquad\qquad}^{O_1^{L_2}+O_2^{L_2}} \end{array} \\
UD_1 = & \left.\left[\begin{array}{cccc:ccc:cccc}
 & & & & 0 & \cdots & 0 & d_{21}\cdot\mu_{21} & & & \\
 & & & & 0 & \cdots & 0 & & d_{21}\cdot\mu_{21} & & \\
 & & & & \vdots & \cdots & \vdots & & & \cdots & \\
 & & & & 0 & \cdots & 0 & & & & d_{21}\cdot\mu_{21} \\
\hdashline
d_{21}\cdot\mu_{21} & & & & 0 & \cdots & 0 & & & & \\
 & d_{21}\cdot\mu_{21} & & & 0 & \cdots & 0 & & & & \\
 & & \cdots & & \vdots & \cdots & \vdots & & & & \\
 & & & d_{21}\cdot\mu_{21} & 0 & \cdots & 0 & & & &
\end{array}\right]\right\} O_1^{L_1}+O_2^{L_1} \\
 & \begin{array}{l} \underbrace{\qquad\qquad\qquad\qquad\qquad\qquad}_{O_1^{L_1}+O_2^{L_1}-O_1^{L_2}-O_2^{L_2}} \end{array}
\end{array}
$$

If (x,y) the position of the upper left element of UD_1:

$$x = lp^{L_1-1} + Z_1^{L_1} + Z_2^{L_1} + 1$$

$$y = lp^{L_2-1} + 1$$

In the boundary conditions where $L_1=0$, the structure of UD_1:

$$
\begin{array}{l}
\hphantom{UD_1=}\;\overbrace{\hphantom{xxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxx}}^{Z_1^{L_2}+Z_2^{L_2}}\;\overbrace{\hphantom{xxxxxxxxxxxxxxxxxxxx}}^{O_1^{L_2}}\;\overbrace{\hphantom{xxxxxxxxxxxxxx}}^{O_2^{L_2}}\\
UD_1=\left(\begin{array}{ccc|ccc|ccc|ccc|ccc}
 & & & & & & 0 & \cdots & 0 & d_{21}\cdot\mu_{21} & 0 & & & & \\
 & & & & & & 0 & \cdots & 0 & & \ddots & & & & \\
 & & & & & & 0 & \cdots & 0 & & & d_{21}\cdot\mu_{21} & & & \\
 & & & & & & 0 & \cdots & 0 & & & & d_{21}\cdot\mu_{21} & & \\
 & & & & & & 0 & \cdots & 0 & & & & & \ddots & \\
 & & & & & & 0 & \cdots & 0 & & & & & & d_{21}\cdot\mu_{21} \\
\hline
d_{21}\cdot\mu_{21} & 0 & & & & & & & & & & & & & \\
 & \ddots & & & & & & & & & & & & & \\
 & & d_{21}\cdot\mu_{21} & & & & & & & & & & & & \\
 & & & d_{21}\cdot\mu_{21} & & & & & & & & & & & \\
 & & & & \ddots & & & & & & & & & & \\
 & & & & & d_{21}\cdot\mu_{21} & & & & & & & & &
\end{array}\right)\begin{array}{l}\left.\vphantom{\begin{array}{c}1\\1\\1\\1\\1\\1\\1\\1\\1\\1\\1\\1\end{array}}\right\}O_1^{L_1}+O_2^{L_1}\end{array}
\end{array}
$$

$$\text{lower-left block: } \overbrace{\hphantom{xxxxxxxxxx}}^{Z_1^{L_2}}, \qquad \text{lower rows: } O_1^{L_1}+O_2^{L_1}-O_1^{L_2}-O_2^{L_2}$$

In this case :

$$x = Z_1^0 + Z_2^0 + 1$$

$$y = lp^{n-1} + 1$$

A similar analysis holds for UD_2. UD_2 describes the arrival of a replenishment order to the vendor from the second phase of transportation. It is a block of $(O_1^{L_1}+O_2^{L_1})\times(Z_1^{L_2}+Z_2^{L_2}+O_1^{L_2}+O_2^{L_2})$ dimensions and its structure is the same as that of UD_1 with μ_{22} substituting $d_{21}m_{21}$. For L>1:

$$
\begin{array}{l}
\hphantom{UD_2=}\;\overbrace{\hphantom{xxxxxxxxxxxxxxxxxxxxxxxxxxxx}}^{Z_1^{L_2}+Z_2^{L_2}}\;\overbrace{\hphantom{xxxxxxxxxxxxxxxxxx}}^{O_1^{L_2}+O_2^{L_2}}\\
UD_2=\left[\begin{array}{cccc|ccc|cccc}
 & & & & 0 & \cdots & 0 & \mu_{22} & & & \\
 & & & & 0 & \cdots & 0 & & \mu_{22} & & \\
 & & & & \vdots & \cdots & \vdots & & & \cdots & \\
 & & & & 0 & \cdots & 0 & & & & \mu_{22} \\
\hline
\mu_{22} & & & & 0 & \cdots & 0 & & & & \\
 & \mu_{22} & & & 0 & \cdots & 0 & & & & \\
 & & \cdots & & \vdots & \cdots & \vdots & & & & \\
 & & & \mu_{22} & 0 & \cdots & 0 & & & &
\end{array}\right]\left.\vphantom{\begin{array}{c}1\\1\\1\\1\\1\\1\\1\\1\end{array}}\right\}O_1^{L_1}+O_2^{L_1}\\
\hphantom{UD_2=}\;\underbrace{\hphantom{xxxxxxxxxxxxxxxx}}_{O_1^{L_1}+O_2^{L_1}-O_1^{L_2}-O_2^{L_2}}
\end{array}
$$

If (x,y) the position of the upper left element of UD_2:

$$x = lp^{L_1-1} + Z_1^{L_1} + Z_2^{L_1} + O_1^{L_1} + O_2^{L_1} + 1$$

$$y = lp^{L_2-1} + 1$$

When we start from the boundary states where basic level $L_1=0$, the general structure of UD_2 will be:

$$\mathrm{UD}_2=\left(\begin{array}{c|c|c|c}
 & \begin{matrix}0&\cdots&0\\0&\cdots&0\\\vdots&\vdots&\vdots\\0&\cdots&0\\0&\cdots&0\\0&\cdots&0\\0&\cdots&0\\0&\cdots&0\end{matrix} & \overbrace{\begin{matrix}\mu22&0&&&\\&&\mu22&0&\\&&&\cdots&\\&&&\mu22&0\\&&&&\\&&&&\\&&&&\\&&&&\end{matrix}}^{O_1^{L_2}} & \overbrace{\begin{matrix}&&&\\&&&\\&&&\\&&&\\\mu22&&&\\&\mu22&&\\&&\cdots&\\&&&\mu22\end{matrix}}^{O_2^{L_2}} \\ \hline
\underbrace{\begin{matrix}\mu22&0&&&\\&\mu22&0&&\\&&\cdots&&\\&&&\mu22&0\\&&&&\\&&&&\\&&&&\end{matrix}}_{Z_1^{L_2}} & \begin{matrix}&&\\&&\\&&\\&&\\\mu22&&\\&\mu22&\\&&\cdots\\&&&\mu22\end{matrix} & \multicolumn{2}{l}{\Big\}\; O_1^{L_1}+O_2^{L_1}-O_1^{L_2}-O_2^{L_2}}
\end{array}\right)\Bigg\}\,O_1^{L_1}+O_2^{L_1}$$

with the first two column blocks together spanning $Z_1^{L_2}+Z_2^{L_2}$ columns.

In this case, if (x,y) the position of the upper left element of UD_2 :

$$x=Z_1^{L_1}+Z_2^{L_1}+O_1^{L_1}+O_2^{L_1}+1$$

$$y=lp^{n-1}+1$$

6.5.3 Below-the-diagonal blocks

Blocks below the diagonal (LD_0 and LD_1) correspond to the arrival of a replenishment order at the retailer. There is a transfer of inventory from the possession of the vendor (I_2) to the retailer (I_1). Since the replenishment order is always of Q_1 units, there is always a transition to the left side adjacent basic level ($L_2=L_1-1$, skip free to the left): $I_2(t+dt)=I_2(t)-Q_1$ and $I_1(t+dt)=I_1(t)+Q_1$

In general, the possible transitions are:

$Z_1^{L_1}\rightarrow Z_1^{L_2}$ or $Z_2^{L_2}$

$O_1^{L_1}\rightarrow O_1^{L_2}$ or $O_2^{L_2}$

6.5.3.1 General case (L_1>1)

LD_0 corresponds to transitions when $p_2=0$ and has dimensions $Z_1^{L_1}\times(Z_1^{L_2}+Z_2^{L_2})$.

In the case when $Q_1\le r$ the possible transitions are $Z_1^{L_1}\rightarrow Z_1^{L_2}$ or $Z_1^{L_1}\rightarrow Z_2^{L_2}$. The first columns of LD_0 are columns of zero. Their number depends on the value of I_1 that corresponds to the first Z_1 state ($p_2=0$, $I_{ech}>s$, $I_1\le r$) of levels L_1 ($I_1^{Z_1,L_1}$) and L_2 ($I_1^{Z_1,L_2}$). For basic level $L\ge k$ the first Z_1 state corresponds to $I_1=0$, while for $L=k-1$ the first Z_1 state corresponds to $I_1=s-L\cdot Q_1+1$. In the case when $L_1<k$, the difference between $I_1^{Z_1,L_1}$ and $I_1^{Z_1,L_2}$ is Q_1. If *w* the number of the first zero columns of LD_0:

$$w = \begin{cases} 0, & L_1 < k \\ 2 \cdot Q_1, & L_1 > k \\ 2 \cdot Q_1 - 2 \cdot (s - L_2 \cdot Q_1 + 1), & L_1 = k \end{cases}$$

If $Q_1 > r$, we have only transitions $Z_1^{L_1} \to Z_2^{L_2}$ and the first $Z_1^{L_2}$ columns will have only elements of zero. Depending on the inventory at the retailer (I_1) for the first Z_2 state, zero columns will extend beyond the first $Z_1^{L_2}$ columns. If we denote with z the number of these zero columns after the first $Z_1^{L_2}$ columns of LD_0:

$$z = \begin{cases} 0, & Q_1 \le r \\ 0, & Q_1 > r, L_1 < k \\ \min\left(Q_1 - (s - (L_1 - 1) \cdot Q_1 + 1), Q_1 - r - 1\right), & Q_1 > r, L_1 = k \\ Q_1 - r - 1, & Q_1 > r, L_1 > k \end{cases}$$

The general structure of LD_0 when $Q_1 > r$:

$$LD_0 = \begin{array}{c} \begin{array}{cc} \overbrace{\qquad\qquad\qquad}^{Z_1^{L_2}} & \overbrace{\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad}^{Z_2^{L_2}} \end{array} \\ \left.\left[\begin{array}{cccccc} & & d_{11} \cdot \mu_{11} & & & \\ & & \mu_{12} & & & \\ & & & d_{11} \cdot \mu_{11} & & \\ & & & \mu_{12} & & \\ & & & & \cdots & \\ & & & & & d_{11} \cdot \mu_{11} \\ & \underbrace{\qquad}_{z} & & & & \mu_{12} \end{array}\right]\right\} Z_1^{L_1} \end{array}$$

The general structure of LD_0 when $Q_1 \le r$:

$$LD_0 = \begin{array}{c}
\begin{array}{cc} \overbrace{\hphantom{\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad}}^{Z_1^{L_2}} & \overbrace{\hphantom{\qquad\qquad\qquad\qquad\qquad}}^{Z_2^{L_2}} \end{array} \\
\left[\begin{array}{c:ccccccc:cccc}
 & d_{11}\cdot\mu_{11} & 0 & & & & & & & & & \\
 & \mu_{12} & 0 & & & & & & & & & \\
 & & & d_{11}\cdot\mu_{11} & 0 & & & & & & & \\
 & & & \mu_{12} & 0 & & & & & & & \\
 & & & & & \cdots & & & & & & \\
 & & & & & & d_{11}\cdot\mu_{11} & 0 & & & & \\
 & & & & & & \mu_{12} & 0 & & & & \\
 & & & & & & & & d_{11}\cdot\mu_{11} & & & \\
 & & & & & & & & \mu_{12} & & & \\
 & & & & & & & & & d_{11}\cdot\mu_{11} & & \\
 & & & & & & & & & \mu_{12} & & \\
 & & & & & & & & & & \cdots & \\
 & & & & & & & & & & & d_{11}\cdot\mu_{11} \\
 & & & & & & & & & & & \mu_{12}
\end{array}\right] \\
\begin{array}{l} \underbrace{\hphantom{\qquad}}_{w} \hphantom{\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad} \end{array}
\end{array} \Bigg\} Z_1^{L_1}$$

With regard to the position of LD_0 blocks in the infinitesimal generator matrix, if (x,y) is the position of the upper left element of block LD_0 corresponding to transition $L_1 \rightarrow L_2$:

$$x = lp^{L_1-1} + 1$$

$$y = lp^{L_2-1} + 1$$

LD_1 corresponds to transitions when $p_2=1$ or $p_2=2$. Its dimensions are $O_1^{L_1} \times (O_1^{L_2} + O_2^{L_2})$. When $Q_1 \leq r$, since the first O_1 state corresponds to $I_1=0$ and the replenishment order is always Q_1 units, the first $2 \cdot Q_1$ columns of LD_1 will have only elements of zero. When $Q_1>r$, we have only transitions $O_1^{L_1} \rightarrow O_2^{L_2}$ and the first $O_1^{L_2}$ columns contain only "0" elements. In such cases there will also be $z = Q_1 - r - 1$ zero columns beyond the first $O_1^{L_2}$ columns.

The general structure of LD_1 when $Q_1 > r$:

$$LD_1 = \begin{array}{c}
\begin{array}{cc} \overbrace{\hphantom{\qquad\qquad\qquad}}^{O_1^{L_2}} & \overbrace{\hphantom{\qquad\qquad\qquad\qquad\qquad\qquad\qquad}}^{O_2^{L_2}} \end{array} \\
\left[\begin{array}{c:c:cccc}
 & & d_{11}\cdot\mu_{11} & & & \\
 & & \mu_{12} & & & \\
 & & & d_{11}\cdot\mu_{11} & & \\
 & & & \mu_{12} & & \\
 & & & & \cdots & \\
 & & & & & d_{11}\cdot\mu_{11} \\
 & & & & & \mu_{12}
\end{array}\right] \\
\begin{array}{l} \hphantom{\qquad\qquad\qquad} \underbrace{\hphantom{\qquad}}_{z} \hphantom{\qquad\qquad\qquad\qquad\qquad\qquad} \end{array}
\end{array} \Bigg\} O_1^{L_1}$$

The general structure of LD_1 when $Q_1 \leq r$:

$$
LD_1 = \left.\left[\begin{array}{c:cccccccc:ccccc}
 & \multicolumn{8}{c}{\overbrace{\hspace{12em}}^{O_1^{L_2}}} & \multicolumn{5}{c}{\overbrace{\hspace{8em}}^{O_2^{L_2}}} \\
 & d_{11}\cdot\mu_{11} & 0 & & & & & & & & & & & \\
 & \mu_{12} & 0 & & & & & & & & & & & \\
 & & & d_{11}\cdot\mu_{11} & 0 & & & & & & & & & \\
 & & & \mu_{12} & 0 & & & & & & & & & \\
 & & & & & \cdots & & & & & & & & \\
 & & & & & & & d_{11}\cdot\mu_{11} & 0 & & & & & \\
 & & & & & & & \mu_{12} & 0 & & & & & \\
 & & & & & & & & & d_{11}\cdot\mu_{11} & & & & \\
 & & & & & & & & & \mu_{12} & & & & \\
 & & & & & & & & & & d_{11}\cdot\mu_{11} & & & \\
 & & & & & & & & & & \mu_{12} & & & \\
 & & & & & & & & & & & \cdots & & \\
 & & & & & & & & & & & & d_{11}\cdot\mu_{11} & \\
 & & & & & & & & & & & & \mu_{12} & \\
 \underbrace{\hspace{3em}}_{2\cdot Q_1} & & & & & & & & & & & & &
\end{array}\right]\right\} O_1^{L_1}
$$

If (x_1,y_1) the position of the upper left element of block LD_1 that corresponds to p_2=1 and transition $L_1 \rightarrow L_2$:

$$x_1 = lp^{L_1-1} + Z_1^{L_1} + Z_2^{L_1} + 1$$

$$y_1 = lp^{L_2-1} + Z_1^{L_2} + Z_2^{L_2} + 1$$

If (x_2,y_2) the position of the upper left element of block LD_1 that corresponds to p_2=2 and transition $L_1 \rightarrow L_2$:

$$x_2 = lp^{L_1-1} + Z_1^{L_1} + Z_2^{L_1} + O_1^{L_1} + O_2^{L_1} + 1$$

$$y_2 = lp^{L_2-1} + Z_1^{L_2} + Z_2^{L_2} + O_1^{L_2} + O_2^{L_2} + 1$$

6.5.3.2 Boundary states (L_1=1, L_2=0)

We investigate the boundary transitions from level L_1=1 to level L_2=0. For L_2=0 there can be no inventory in transit towards the retailer, so the dimensions of the sub-blocks are different.

With regard to LD_0:

$$w = \begin{cases} 0, & L_1 < k \\ Q_1 - (s - L_2 \cdot Q_1 + 1), & L_1 = k \end{cases}$$

The general structure of LD_0:

$$
LD_0 = \begin{array}{l}
\overbrace{\left.\begin{bmatrix}
0 & \cdots & 0 & d_{11}\cdot\mu_{11} & & & & \\
0 & \cdots & 0 & \mu_{12} & & & & \\
0 & \cdots & 0 & & d_{11}\cdot\mu_{11} & & & \\
0 & \cdots & 0 & & \mu_{12} & & & \\
0 & \cdots & 0 & & & d_{11}\cdot\mu_{11} & & \\
0 & \cdots & 0 & & & \mu_{12} & & \\
\vdots & \vdots & \vdots & & & & \cdots & \\
0 & \cdots & 0 & & & & & d_{11}\cdot\mu_{11} \\
0 & \cdots & 0 & & & & & \mu_{12}
\end{bmatrix}\right\} Z_1^{L_1}}^{Z_1^{L_2}+Z_2^{L_2}} \\
\underbrace{\hphantom{0 \;\cdots\; 0}}_{w}
\end{array}
$$

If the position of the upper left element of block $LD_0^{L_2=0}$ in the infinitesimal generator matrix is (x,y):

$x = lp^0 + 1$

$y = 1$

With regard to LD_1, the general structure of the block will be:

$$
LD_1 = \begin{array}{l}
\overbrace{\left.\begin{bmatrix}
0 & \cdots & 0 & d_{11}\cdot\mu_{11} & & & & \\
0 & \cdots & 0 & \mu_{12} & & & & \\
0 & \cdots & 0 & & d_{11}\cdot\mu_{11} & & & \\
0 & \cdots & 0 & & \mu_{12} & & & \\
0 & \cdots & 0 & & & d_{11}\cdot\mu_{11} & & \\
0 & \cdots & 0 & & & \mu_{12} & & \\
\vdots & \vdots & \vdots & & & & \cdots & \\
0 & \cdots & 0 & & & & & d_{11}\cdot\mu_{11} \\
0 & \cdots & 0 & & & & & \mu_{12}
\end{bmatrix}\right\} O_1^{L_1}}^{O_1^{L_2}+O_2^{L_2}} \\
\underbrace{\hphantom{0 \;\cdots\; 0}}_{Q_1}
\end{array}
$$

If (x_1,y_1) the position of $LD_1^{L_2=0}$ block that corresponds to $p_2=1$:

$x_1 = lp^0 + Z_1^1 + Z_2^1 + 1$

$y = Z_1^0 + Z_2^0 + 1$

If (x_2,y_2) the position of $LD_1^{L_2=0}$ block that corresponds to $p_2=2$:

$x_2 = lp^0 + Z_1^1 + Z_2^1 + O_1^1 + O_2^1 + 1$

$y_2 = Z_1^0 + Z_2^0 + O_1^0 + O_2^0 + 1$

6.5.4 General structure of the infinitesimal generator matrix

Having defined the exact structure and position of each substituent block, we can construct the infinitesimal generator matrix Q. Summing-up our previous analysis, there are in total $Nl+1$ diagonal blocks, the first one corresponding to the boundary conditions where I_2=0.

Below the diagonal blocks occur for transitions from every basic level, except from basic level 0. In general, for every diagonal block of level L>0 there are three below the diagonal blocks, each one describing transitions for different p_2 sublevels. LD_0 corresponds to p_2=0, the first LD_1 block corresponds to p_2=1 and the second LD_1 block to p_2=2. Below the diagonal blocks always describe transitions to the adjacent left side basic level.

Upper diagonal blocks occur from basic level *L=0* to basic level *L=k-1*. For every basic level there are two upper-diagonal blocks, the first corresponding to p_2=1 and the second to p_2=2. Between a diagonal block and the respective upper diagonal block there is an interval of *n-1* basic levels.

Without taking into consideration the fact that the dimensions of the various blocks are different for different basic levels, the outlay of the infinitesimal generator matrix can be given schematically:

$$\begin{pmatrix}
D^{L=0} \xleftrightarrow{\text{n-1 Basic Levels}} \begin{matrix} UD_1^{L_1=0} \\ UD_2^{L_1=0} \end{matrix} & & & & \\
LD_0^{L=1}\; LD_1^{L=1}\; LD_1^{L=1} & D^{L=1} \xleftrightarrow{\text{n-1 Basic Levels}} \begin{matrix} UD_1^{L_1=1} \\ UD_2^{L_1=1} \end{matrix} & & & \\
& LD_0^{L=2}\; LD_1^{L=2}\; LD_1^{L=2} & D^{L=2} \xleftrightarrow{\text{n-1 Basic Levels}} \begin{matrix} UD_1^{L_1=2} \\ UD_2^{L_1=2} \end{matrix} & & \\
& \cdots & \cdots & \cdots & \\
& & LD_0^{L=k-1}\; LD_1^{L=k-1}\; LD_1^{L=k-1} & D^{L=k-1} \xleftrightarrow{\text{n-1 Basic Levels}} \begin{matrix} UD_1^{L_1=k-1} \\ UD_2^{L_1=k-1} \end{matrix} & \\
& & & \cdots \quad \cdots & \\
& & & LD_0^{L=Nl}\; LD_1^{L=Nl}\; LD_1^{L=Nl} & D^{L=Nl}
\end{pmatrix}$$

6.6 Performance Measures

Having constructed the infinitesimal generator matrix, we can compute the stationary probabilities for each possible state of the system. We denote as **X** the vector of the stationary probabilities and X(i) the i^{th} element of the vector which corresponds to the i^{th} state in the hierarchy of states defined according to the rules of paragraph 6.4.1. If **Q** the infinitesimal generator matrix, then in the steady state:

$$X \cdot Q = 0$$

$$\sum_i X(i) = 1$$

From the above, a system of linear equations can be extracted and the vector X can be computed numerically. Performance measures about the system are computed algorithmically using the stationary probabilities and taking advantage of the hierarchy of states.

As already mentioned, we define lp^L the position of the last state of basic level L in the ordering of states that was defined in paragraph 6.4.1:

$$lp^L = \sum_{i=0}^{L} \left(Z_1^i + Z_2^i + 2 \cdot (O_1^i + O_2^i) \right)$$

6.6.1 Average Inventory at the Wholesaler (WIP$_2$)

WIP$_2$ is the average inventory at the possession of the vendor. It includes both inventory on hand at the vendor and inventory in transit towards the Retailer. It can be computed iteratively:

$$WIP_2 = \sum_{L=1}^{Nl} \sum_{j=lp^{L-1}+1}^{lp^L} L \cdot Q_1 \cdot X(j)$$

6.6.2 Stock-out probability at the retailer (SO)

Stock-out probability at the retailer is the probability that external demand will occur, but the retailer will have no inventory on hand to meet it even partially. As the external demand is assumed to be independent and identically distributed, SO is the probability that I_1=0. In general for L>0, zero inventory at the retailer corresponds to the first two O_1 states of D_1 sub-matrices, to the first two O_1 states of D_2 blocks, and when L≥k to the first two Z_1 states of D_0 sub-matrices. In the case of *L=0*, zero inventory at the retailer corresponds to the first O_1 state of D_1 and the first O_1 state of D_2 blocks.

$$SO = SO^0 + SO^Z + SO^{O1} + SO^{O2}$$

$$SO^0 = X(Z_1^0 + Z_2^0 + 1) + X(Z_1^0 + Z_2^0 + O_1^0 + O_2^0 + 1)$$

$$SO^Z = \sum_{L=1}^{Nl} n_z^L \cdot \left(X(lp^{L-1} + 1) + X(lp^{L-1} + 2) \right)$$

$$SO^{O1} = \sum_{L=1}^{Nl} n_o^L \cdot \left(X(lp^{L-1} + Z_1^L + Z_2^L + 1) + X(lp^{L-1} + Z_1^L + Z_2^L + 2) \right)$$

$$SO^{O2} = \sum_{L=1}^{Nl} n_o^L \cdot \left(X(lp^{L-1} + Z_1^L + Z_2^L + O_1^L + O_2^L + 1) + X(lp^{L-1} + Z_1^L + Z_2^L + O_1^L + O_2^L + 2) \right)$$

$$n_z^L = \begin{cases} 1\ if\ Z_1^L > 0\ and\ L \geq k \\ 0\ in\ any\ other\ case \end{cases}$$

$$n_o^L = \begin{cases} 1\ if\ O_1^L > 0 \\ 0\ if\ O_1^L = 0 \end{cases}$$

6.6.3 Average inventory at the retailer (WIP$_1$)

We define a $r+Q_1$ dimensional vector I_R such that its ith element is equal to the probability of the inventory at the retailer being i units $\left(I_R(i) = prob(I_1 = i)\right)$. The values of $I_R(i)$ can be calculated iteratively. The iterative process is outlined below.

For basic level L=0:

For the states corresponding to p$_2$=0, for $i=1$ to $i = Z_1^0 + Z_2^0$:

$$I_R(s+i) = X(i)$$

For the states corresponding to p$_2$=1, for $i=2$ to $i = O_1^0 + O_2^0$

$$I_R(i-1) = I_R(i-1) + X(Z_1^0 + Z_2^0 + i)$$

For the states corresponding to p$_2$=2, for i=2 to $i = O_1^0 + O_2^0$

$$I_R(i-1) = I_R(i-1) + X(Z_1^0 + Z_2^0 + O_1^0 + O_2^0 + i)$$

For basic level L, 1≤ L≤Nl

For the states corresponding to p$_2$=0:

If L<k:

For $i = 1,3,5,.., Z_1^L - 1$

$$I_R\left(s - L\cdot Q_1 + \frac{i+1}{2}\right) = I_R\left(s - L\cdot Q_1 + \frac{i+1}{2}\right) + X(lp^{L-1} + i) + X(lp^{L-1} + i + 1)$$

If L≥k:

For $i = 3,5,7,.., Z_1^L - 1$

$$I_R\left(\frac{i-1}{2}\right) = I_R\left(\frac{i-1}{2}\right) + X(lp^{L-1} + i) + X(lp^{L-1} + i + 1)$$

If L=h

For $i=1$ to $i = Z_2^L$

$$I_R(s - L\cdot Q_1 + i) = I_R(s - L\cdot Q_1 + i) + X(lp^{L-1} + Z_1^L + i)$$

If L≠ h

For $i=1$ to $i = Z_2^L$

$$I_R(r+i) = I_R(r+i) + X(lp^{L-1} + Z_1^L + i)$$

For the states corresponding to p_2=1:
For $i = 3,5,7,..,O_1^L$

$$I_R\left(\frac{i-1}{2}\right) = I_R\left(\frac{i-1}{2}\right) + X(lp^{L-1} + Z_1^L + Z_2^L + i) + X(lp^{L-1} + Z_1^L + Z_2^L + i + 1)$$

For $i = 1$ to $i = O_2^L$

$$I_R(r+i) = I_R(r+i) + X(lp^{L-1} + Z_1^L + Z_2^L + O_1^L + i)$$

For the states corresponding to p_2=2:
For $i = 3,5,7,..,O_1^L$

$$I_R\left(\frac{i-1}{2}\right) = I_R\left(\frac{i-1}{2}\right) + X(lp^{L-1} + Z_1^L + Z_2^L + O_1^L + O_2^L + i) + X(lp^{L-1} + Z_1^L + Z_2^L + O_1^L + O_2^L + i + 1)$$

For $i = 1$ to $i = O_2^L$

$$I_R(r+i) = I_R(r+i) + X(lp^{L-1} + Z_1^L + Z_2^L + O_1^L + O_2^L + O_1^L + i)$$

Having constructed vector I_R, we can easily calculate the average inventory on hand at the retailer as a simple sum:

$$WIP_1 = \sum_{i=1}^{r+Q_1} i \cdot I_R(i)$$

6.6.4 Throughput

Throughput is the number of product units sold by the retailer per unit of time. Essentially, it expresses the flow of products through the system.

We define a $r+Q_1$ dimensional vector I_{RO} such that its i^th^ element is equal to the probability that the inventory at the retailer is equal to or greater than i units $\left(I_{RO}(i) = prob(I_1 \geq i)\right)$. Vector I_{RO} can be calculated using vector I_R (paragraph 6.6.3).

$$I_{RO}(i) = \sum_{j=i}^{r+Q_1} I_R(j)$$

We define an *md*-dimensional vector met such that the i^th^ element $met(i)$ is equal to the probability that external demand of i units occurs while the inventory on hand at the retailer is equal to or greater than i:

$$met(i) = prob(d = i \mid I_1 \geq i)$$

Since the external demand is independent of the inventory at the retailer:

$$met(i) = dm(i) \cdot I_{RO}(i)$$

Similarly, we define an $r+Q_1$ dimensional vector *unmet* such that the ith element $unmet(i)$ is equal to the probability that the inventory on hand at the retailer is exactly *i* and external demand in excess of *i* occurs:

$unmet(i) = prob(I_1 = i \mid d > i)$

Since the external demand is independent of the inventory at the retailer:

$unmet(i) = \left(rd(i) - dm(i)\right) \cdot I_R(i)$

The total output is the sum of the output when external demand is fully met and the output when external demand is partially met.

$$Throughput = \lambda \cdot \sum_{i=1}^{\min(r+Q_1, dmd)} i \cdot \left(met(i) + unmet(i)\right)$$

6.6.5 Order Fill Rate (OFR)

Order Fill Rate is the percentage of external customers whose demand is fully met by the inventory on hand at the retailer.

$$OFR = \frac{probability\ that\ the\ external\ demand\ is\ fully\ met}{probability\ that\ external\ demand\ occurs}$$

$$OFR = \frac{prob(d=1)\cdot prob(I_1 \geq 1) + prob(d=2)\cdot prob(I_1 \geq 2) + \ldots + prob(d=md)\cdot prob(I_1 \geq md)}{prob(d=1) + prob(d=2) + \ldots + prob(d=md)}$$

OFR can be calculated using vector *met* from paragraph 6.6.4:

$$OFR = \sum_{i=1}^{\min(r+Q_1, dmd)} met(i)$$

6.6.6 Service level (SL)

Service level is the percentage of external demand, in terms of product units, that is met from the inventory on hand at the retailer.

The average demand E_d:

$$E_d = \sum_{i=1}^{md} i \cdot dm(i)$$

And service level *SL*:

$$SL = \frac{Throughput}{\lambda \cdot E_d}$$

6.6.7 Average lost sales

Average lost sales per unit of time (ALS_{time}) can be easily computed using service level and the average demand:

$$ALS_{time} = (1 - SL) \cdot E_d$$

To calculate the average lost sales per lost order (order partially met or totally lost), order fill rate must be taken into account:

$$ALS_{order} = \frac{ALS_{time}}{(1-OFR)}$$

6.6.8 Average inventory in transit towards the retailer (Intransit$_1$)

First we calculate the utilization of the resource for transportation towards the retailer (u_1). It can be expressed as the percentage of time that there is a replenishment order in transit towards the retailer (p_1>0):

$$u_1 = \sum_{L=1}^{Nl}\left[\sum_{i=1}^{Z_1^L} X(lp^{L-1}+i) + \sum_{i=1}^{O_1^L} X(lp^{L-1}+Z_1^L+Z_2^L+i) + \sum_{i=1}^{O_1^L} X(lp^{L-1}+Z_1^L+Z_2^L+O_1^L+O_2^L+i)\right]$$

Since only full replenishment orders of Q_1 units are sent to the retailer:

$$Intransit_1 = u_1 \cdot Q_1$$

6.6.9 Average inventory in transit towards the vendor (Intransit$_2$)

First we calculate the utilization of the resource for transportation towards the vendor (u_2). It can be expressed as the percentage of time that there is a replenishment order in transit towards the vendor (p_2>0):

$$u_2 = \sum_{i=Z_1^0+Z_2^0+1}^{lp^0} X(i) + \sum_{L=1}^{Nl} \sum_{i=Z_1^L+Z_2^L+1}^{lp^L-lp^{L-1}} X(lp^{L-1}+i)$$

Since we have assumed that the Manufacturer is saturated and always full replenishment orders of Q_2 units are sent to the vendor:

$$Intransit_2 = u_2 \cdot Q_2$$

6.7 Illustrative example

To illustrate the algorithm described above, we present the analysis for a simple example with echelon reorder point s=4, vendor's replenishment order Q_2=8, retailer's reorder point r=2, and retailer's replenishment order Q_1=4. Each external customer may ask from 1 to 4 product units with a respective probability dm_i for each case $(md = 4)$.

6.7.1 States definition and state transitions

6.7.1.1 States definition

The system is a continuous time Markov Process. At any time t, the state of the system can be defined by a four dimensional vector

$\bar{S}_t = \left(I_2(t), p_2(t), I_1(t),\ p_1(t)\right)$

$I_2(t)$ is the vendor's inventory at time t. According to the assumptions made, $I_2(t)$ includes the inventory on hand at the vendor as well as any inventory in transit towards the retailer. The possible values of $I_2(t)$ are multiples of Q_1=4, while its maximum value is $s+Q_2$=12. The permissible values: $I_2(t) = \{0, 4, 8, 12\}$

$p_2(t)$ is the phase of the replenishment order towards the vendor. Its possible values are $p_2(t) = \{0, 1, 2\}$. In order for a replenishment order to be in transit towards the wholesaler ($p_2(t) > 0$), the echelon inventory must be equal to or less than the echelon reorder point $\left(I_1(t) + I_2(t) \leq s\right)$. Correspondingly, $p_2(t) = 0$ only when the echelon inventory is greater than the echelon reorder point $\left(I_1(t) + I_2(t) > s\right)$.

$I_1(t)$ is the inventory on hand at the retailer at time t. Its maximum permissible value is $r+Q_1$=6 and its possible values are $I_1(t) = \{0, 1, 2, 3, 4, 5, 6\}$

$p_1(t)$ is the phase of the replenishment order towards the retailer. Its possible values are $p_1(t) = \{0, 1, 2\}$. $p_1(t) > 0$ as long as the inventory at the retailer is equal to or less than the reorder point r $\left(I_1(t) \leq r\right)$. Correspondingly, $p_1(t) = 0$ when the retailer's inventory is greater than the reorder point r $\left(I_1(t) > r\right)$. An exception occurs for the boundary conditions when $I_2(t) = 0$, in which case $p_2(t) = 0$ for any value of $I_1(t)$.

The state space of the Markov process is comprised of all the possible vectors $\bar{S}_t$ and its dimension can be computed as a function of the design variables through an iterative process. For the system under consideration there are 34 possible states. These states are ordered linearly, using a lexicographical ordering and moving from lower to higher values. First the states are ordered according to $I_2(t)$(basic levels). Within each basic level the ordering is done based on the transportation phase towards the vendor $p_2(t)$, then according to inventory at the retailer $I_1(t)$, and finally according to the transportation phase towards the retailer $p_1(t)$. The possible states and their respective hierarchy are given in figure 6.3

Figure 6.3: States for s=4, Q_2=8, r=2, Q_1=4					
s/n	State	$I_2(t)$	Transportation towards the vendor - $p_2(t)$	$I_1(t)$	Transportation towards the retailer - $p_1(t)$
1	0050	0	no	5	no
2	0060	0	no	6	no
3	0100	0	1st phase	0	no
4	0110	0	1st phase	1	no
5	0120	0	1st phase	2	no
6	0130	0	1st phase	3	no
7	0140	0	1st phase	4	no
8	0200	0	2nd phase	0	no
9	0210	0	2nd phase	1	no
10	0220	0	2nd phase	2	no
11	0230	0	2nd phase	3	no
12	0240	0	2nd phase	4	no
13	4011	4	no	1	1st phase
14	4012	4	no	1	2nd phase
15	4021	4	no	2	1st phase
16	4022	4	no	2	2nd phase
17	4030	4	no	3	no
18	4040	4	no	4	no
19	4050	4	no	5	no
20	4060	4	no	6	no
21	4101	4	1st phase	0	1st phase
22	4102	4	1st phase	0	2nd phase
23	4201	4	2nd phase	0	1st phase
24	4202	4	2nd phase	0	2nd phase
25	8001	8	no	0	1st phase
26	8002	8	no	0	2nd phase
27	8011	8	no	1	1st phase
28	8012	8	no	1	2nd phase
29	8021	8	no	2	1st phase
30	8022	8	no	2	2nd phase
31	8030	8	no	3	no
32	8040	8	no	4	no
33	d001	12	no	0	1st phase
34	d002	12	no	0	2nd phase

6.7.1.2 State transitions

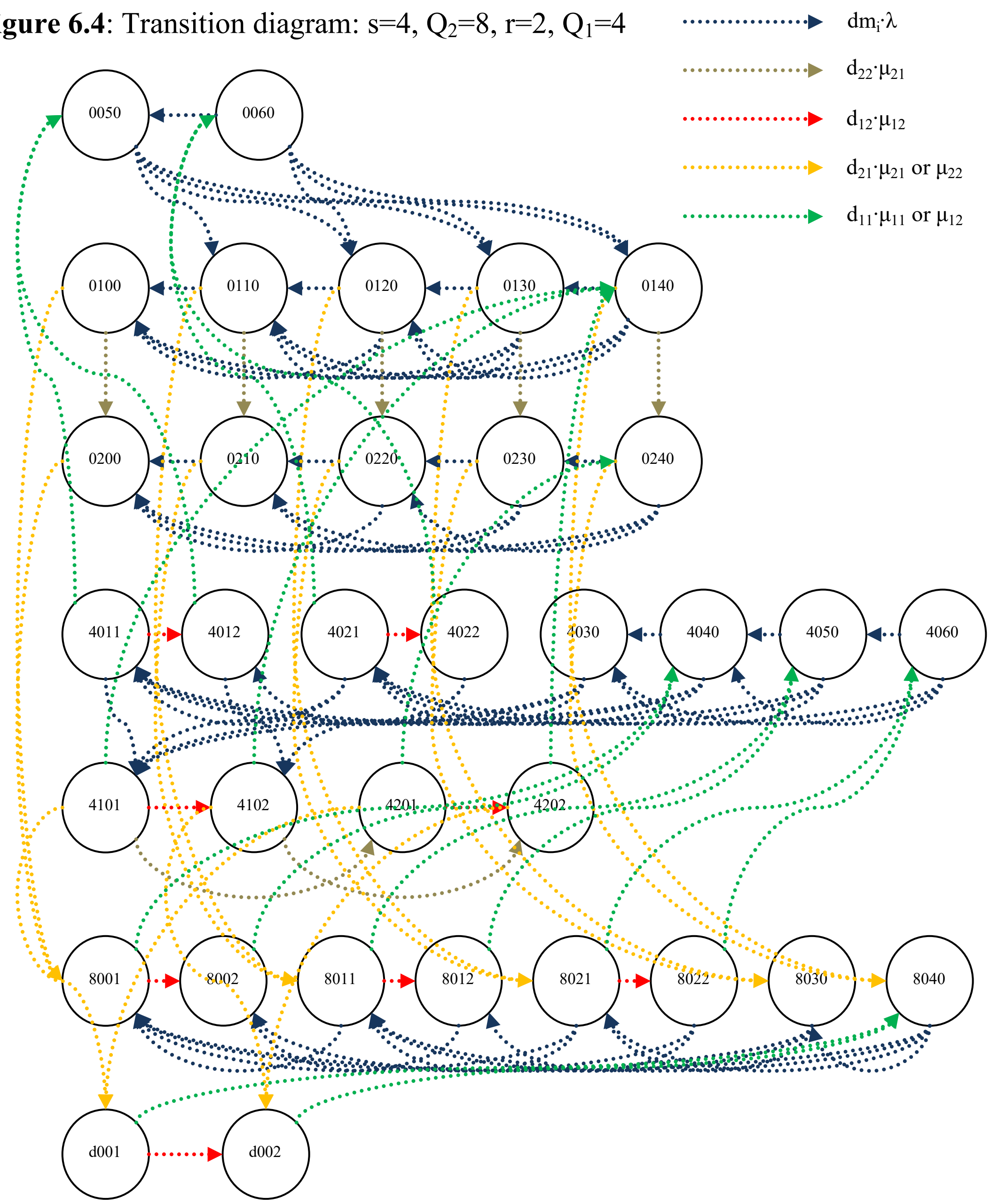
With the methodological restriction that in infinitesimal time *dt* only one event may occur, the state of the system can be altered instantaneously by five kinds of events:

1. The transition from the first to the second phase of the Coxian-2 transportation towards the vendor. $p_2(t) = 1 \rightarrow p_2(t+dt) = 2$. The instantaneous transition rate of such transitions is $d_{22} \cdot \mu_{21}$.
2. The arrival of an outstanding order at the vendor. The incoming order increases the inventory at the vendor by Q_2=8 product units $\left(I_2(t+dt) = I_2(t) + Q_2\right)$. Since in the example under consideration Q_2>s, there can be no reordering from the vendor $p_2(t+dt) = 0$. If $I_2(t) = 0$ and $I_1(t) \leq 2$, on replenishment order arrival, Q_1=4 units are immediately forwarded for transportation towards the retailer ($p_1(t+dt) = 1$). Nominally, these units are still in the possession of the vendor, so $I_2(t+dt) = Q_2$. In general, the arrival of the replenishment order can be from the first or the second Coxian phase. In the first case the instantaneous transition rate is $d_{21} \cdot \mu_{21}$, while in the second case the respective transition rate is μ_{22} .

3. The transition from the first to the second phase of the Coxian-2 transportation towards the retailer. The corresponding transition is $p_1(t) = 1 \rightarrow p_1(t + dt) = 2$ and the respective instantaneous transition rate is $d_{12} \cdot \mu_{11}$.
4. The arrival of an outstanding order at the retailer. On arrival Q_1=4 product units are transferred from the possession of the vendor to the retailer. The inventory on hand at the retailer increases by Q_1 units such that $I_1(t+dt) = I_1(t) + Q_1$, while the inventory of the vendor decreases correspondingly: $I_2(t+dt) = I_2(t) - Q_1$. In the example under consideration Q_1>r so there can be no reordering from the retailer $\left(p_1(t+dt) = 0\right)$. In general, the arrival of the replenishment order can be from the first or the second Coxian phase. In the first case, the instantaneous transition rate is $d_{11} \cdot \mu_{11}$. In the second case, the respective rate is μ_{12}.
5. The occurrence of external demand at the retailer. Each customer may ask from 1 to 4 units with respective probability *dm(i)*. The inventory on hand at the retailer decreases correspondingly, while any excessive demand is lost. If $I_1(t+dt) \leq 2$, $p_1(t) = 0$ and $I_2(t+dt) \geq 4$, a new replenishment order is sent from the vendor $(p_1(t+dt) = 1)$. If $I_1(t+dt) + I_2(t+dt) \leq s$ and $p_2(t) = 0$, a new replenishment order is asked by the vendor from the manufacturer $(p_2(t+dt) = 1)$. The instantaneous transition rate for external demand i is $dm_i \cdot \lambda$.

The transition diagram for the system under consideration is given in figure 6.4.

Figure 6.4: Transition diagram: s=4, Q_2=8, r=2, Q_1=4



6.7.2 The Infinitesimal Generator Matrix

$f = 2$: Basic Level L=2 is the lowest level at which the system may reach its maximum capacity

$k = 2$: Basic level L=2 is lowest basic level where the inventory at the vendor exceeds the echelon reorder point

$h = 0$: The echelon inventory may exceed the echelon reorder point even for basic level L=0

$Nl = 3$: Basic level L=3 is the highest basic level. The maximum value of the inventory at the vendor is $3 \cdot Q_1 = 12$. The total number of basic levels is 3+1=4 (3 levels for L>0 plus level L=0)

$$rd_1 = dm(1) + dm(2) + dm(3) + dm(4) = 1$$
$$rd_2 = dm(2) + dm(3) + dm(4)$$
$$rd_3 = dm(3) + dm(4)$$
$$rd_4 = dm(4)$$

6.7.2.1 Diagonal blocks

The diagonal blocks D are analyzed into constituent blocks:

$$D = \begin{bmatrix} D_0 & U_0 & \\ & D_1 & U_1 \\ & & D_2 \end{bmatrix}$$

<u>Basic Level L=0</u>

$$Z_1^0 = 0$$
$$Z_2^0 = 2$$
$$O_1^0 = 3$$
$$O_2^0 = 2$$

Block D_0 is a 2×2 block:

$$D_0 = \begin{bmatrix} -\lambda & 0 \\ dm_1 \cdot \lambda & -\lambda \end{bmatrix}$$

Block D_1 is a 5×5 matrix that can be divided into four sub-matrices:

$$D_1 = \left[\begin{array}{ccc:cc} -\mu_{21} & 0 & 0 & 0 & 0 \\ rd(1) \cdot \lambda & -\mu_{21} - \lambda & 0 & 0 & 0 \\ rd(2) \cdot \lambda & dm(1) \cdot \lambda & -\mu_{21} - \lambda & 0 & 0 \\ \hdashline rd(3) \cdot \lambda & dm(2) \cdot \lambda & dm(1) \cdot \lambda & -\mu_{21} - \lambda & 0 \\ rd(4) \cdot \lambda & dm(3) \cdot \lambda & dm(2) \cdot \lambda & dm(1) \cdot \lambda & -\mu_{21} - \lambda \end{array}\right]$$

D_2 is also a 5×5 block:

$$D_2 = \left[\begin{array}{ccc|cc} -\mu_{22} & 0 & 0 & 0 & 0 \\ rd(1)\cdot\lambda & -\mu_{22}-\lambda & 0 & 0 & 0 \\ rd(2)\cdot\lambda & dm(1)\cdot\lambda & -\mu_{22}-\lambda & 0 & 0 \\ \hline rd(3)\cdot\lambda & dm(2)\cdot\lambda & dm(1)\cdot\lambda & -\mu_{22}-\lambda & 0 \\ rd(4)\cdot\lambda & dm(3)\cdot\lambda & dm(2)\cdot\lambda & dm(1)\cdot\lambda & -\mu_{22}-\lambda \end{array}\right]$$

U_0 is a 2×5 block :

$$U_0 = \begin{bmatrix} 0 & dm(4)\cdot\lambda & dm(3)\cdot\lambda & dm(2)\cdot\lambda & dm(1)\cdot\lambda \\ 0 & 0 & dm(4)\cdot\lambda & dm(3)\cdot\lambda & dm(2)\cdot\lambda \end{bmatrix}$$

U_1 is a 5×5 diagonal block of $d_{22}\cdot\mu_{21}$.

$$U_1 = \begin{bmatrix} d_{22}\cdot\mu_{21} & 0 & 0 & 0 & 0 \\ 0 & d_{22}\cdot\mu_{21} & 0 & 0 & 0 \\ 0 & 0 & d_{22}\cdot\mu_{21} & 0 & 0 \\ 0 & 0 & 0 & d_{22}\cdot\mu_{21} & 0 \\ 0 & 0 & 0 & 0 & d_{22}\cdot\mu_{21} \end{bmatrix}$$

<u>Basic Level L=1</u>

$Z_1^1 = 4$

$Z_2^1 = 4$

$O_1^1 = 2$

$O_2^1 = 0$

D_0 is 8×8 block

$$D_0 = \left[\begin{array}{cccc|cccc} -\mu_{11}-\lambda & d_{12}\cdot\mu_{11} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -\mu_{12}-\lambda & 0 & 0 & 0 & 0 & 0 & 0 \\ dm(1)\cdot\lambda & 0 & -\mu_{11}-\lambda & d_{12}\cdot\mu_{11} & 0 & 0 & 0 & 0 \\ 0 & dm(1)\cdot\lambda & 0 & -\mu_{12}-\lambda & 0 & 0 & 0 & 0 \\ \hline dm(2)\cdot\lambda & 0 & dm(1)\cdot\lambda & 0 & -\lambda & 0 & 0 & 0 \\ dm(3)\cdot\lambda & 0 & dm(2)\cdot\lambda & 0 & dm(1)\cdot\lambda & -\lambda & 0 & 0 \\ dm(4)\cdot\lambda & 0 & dm(3)\cdot\lambda & 0 & dm(2)\cdot\lambda & dm(1)\cdot\lambda & -\lambda & 0 \\ 0 & 0 & dm(4)\cdot\lambda & 0 & dm(3)\cdot\lambda & dm(2)\cdot\lambda & dm(1)\cdot\lambda & -\lambda \end{array}\right]$$

D_1 and D_2 are 2×2 blocks:

$$D_1 = \begin{bmatrix} -\mu_{21}-\mu_{11} & d_{12}\cdot\mu_{11} \\ 0 & -\mu_{21}-\mu_{12} \end{bmatrix}$$

$$D_2 = \begin{bmatrix} -\mu_{22}-\mu_{11} & d_{12}\cdot\mu_{11} \\ 0 & -\mu_{22}-\mu_{12} \end{bmatrix}$$

U_0 is a 8×2 block:

$$U_0 = \left[\begin{array}{cc} rd(1)\cdot\lambda & 0 \\ 0 & rd(1)\cdot\lambda \\ rd(2)\cdot\lambda & 0 \\ 0 & rd(2)\cdot\lambda \\ \hdashline rd(3)\cdot\lambda & 0 \\ rd(4)\cdot\lambda & 0 \\ 0 & 0 \\ 0 & 0 \end{array}\right]$$

Finally, U_1 is a 2×2 diagonal block of $d_{22}\cdot\mu_{21}$

<u>Basic Level L=2</u>

$Z_1^2 = 6$

$Z_2^2 = 2$

$O_1^2 = 0$

$O_2^2 = 0$

D_0 is 8×8 block:

$$D_0 = \left[\begin{array}{cccccc:cc} -\mu_{11} & d_{12}\cdot\mu_{11} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -\mu_{12} & 0 & 0 & 0 & 0 & 0 & 0 \\ rd(1)\cdot\lambda & 0 & -\mu_{11}-\lambda & d_{12}\cdot\mu_{11} & 0 & 0 & 0 & 0 \\ 0 & rd(1)\cdot\lambda & 0 & -\mu_{12}-\lambda & 0 & 0 & 0 & 0 \\ rd(2)\cdot\lambda & 0 & dm(1)\cdot\lambda & 0 & -\mu_{11}-\lambda & d_{12}\cdot\mu_{11} & 0 & 0 \\ 0 & rd(2)\cdot\lambda & 0 & dm(1)\cdot\lambda & 0 & -\mu_{12}-\lambda & 0 & 0 \\ \hdashline rd(3)\cdot\lambda & 0 & dm(2)\cdot\lambda & 0 & dm(1)\cdot\lambda & 0 & -\lambda & 0 \\ rd(4)\cdot\lambda & 0 & dm(3)\cdot\lambda & 0 & dm(2)\cdot\lambda & 0 & dm(1)\cdot\lambda & -\lambda \end{array}\right]$$

Since $O_1^2 = 0$ and $O_2^2 = 0$, blocks D_1, D_2, U_0, and U_1 do not occur for L=2.

Basic Level L=3

$Z_1^3 = 2$

$Z_2^3 = 0$

$O_1^3 = 0$

$O_2^3 = 0$

D_0 is a 2×2 block

$$D_0 = \begin{bmatrix} -\mu_{11} & d_{12} \cdot \mu_{11} \\ 0 & -\mu_{12} \end{bmatrix}$$

Since $O_1^3 = 0$ and $O_2^3 = 0$, the rest of the blocks do not occur for L=3.

6.7.2.2 Upper-diagonal blocks

Upper diagonal blocks describe transitions from basic level L_1 to basic level L_2=L_1+2.

From basic level L_1=0

UD_1 is a 5×8 block with the upper left element at position (3, 25):

$$UD_1 = \begin{bmatrix} d_{21} \cdot \mu_{21} & 0 & & & & & & \\ & 0 & d_{21} \cdot \mu_{21} & 0 & & & & \\ & & & 0 & d_{21} \cdot \mu_{21} & 0 & & \\ & & & & & 0 & d_{21} \cdot \mu_{21} & \\ & & & & & & & d_{21} \cdot \mu_{21} \end{bmatrix}$$

Similarly, UD_2 is a 5×8 block with the upper left element at position (8, 25):

$$UD_2 = \begin{bmatrix} \mu_{22} & 0 & & & & & & \\ & 0 & \mu_{22} & 0 & & & & \\ & & & 0 & \mu_{22} & 0 & & \\ & & & & & 0 & \mu_{22} & \\ & & & & & & & \mu_{22} \end{bmatrix}$$

From basic level L_1=1

UD_1 is a 2×2 block with its upper left element at position (21, 33):

$$UD_1 = \begin{bmatrix} d_{21} \cdot \mu_{21} & 0 \\ 0 & d_{21} \cdot \mu_{21} \end{bmatrix}$$

UD_2 is also a 2×2 block with the upper left element at position (23, 33)

$$UD_2 = \begin{bmatrix} \mu_{22} & 0 \\ 0 & \mu_{22} \end{bmatrix}$$

6.7.2.3 Below the diagonal blocks

<u>From basic level L_1=1</u>

LD_0 is a 4×2 block with its upper left element at position (13, 1) of the infinitesimal generator matrix:

$w = 0$

$$LD_0 = \begin{bmatrix} d_{11} \cdot \mu_{11} & 0 \\ \mu_{12} & 0 \\ 0 & d_{11} \cdot \mu_{11} \\ 0 & \mu_{12} \end{bmatrix}$$

LD_1 is a 2×5 block:

$$LD_1 = \begin{bmatrix} 0 & 0 & 0 & 0 & d_{11} \cdot \mu_{11} \\ 0 & 0 & 0 & 0 & \mu_{12} \end{bmatrix}$$

The upper left element of LD_1 for transitions from p_2=1 is at position (21, 3) of the infinitesimal generator matrix. For transitions from p_2=2, the position of the respective element is at (23, 8)

<u>From basic level L_1=2</u>

Since $Q_1 \geq r+1$, we have only transitions $Z_1^2 \to Z_2^1$ and the first Z_1^1 columns will have only elements of zero.

$z = 1$

LD_0 is a 6×8 block:

$$LD_0 = \left[\begin{array}{ccccc:ccc} 0 & 0 & 0 & 0 & 0 & d_{11} \cdot \mu_{11} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \mu_{12} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & d_{11} \cdot \mu_{11} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \mu_{12} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & d_{11} \cdot \mu_{11} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mu_{12} \end{array}\right]$$

The position of the upper left element is at (25, 13)

Since for basic level L=2 there are no O_1 states, there are no corresponding LD_1 matrices.

From basic level L_1=3

LD_0 is a 2×8 block:

$z = 1$

$$LD_0 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & d_{11}\cdot\mu_{11} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mu_{12} \end{bmatrix}$$

The position of the upper left element is at (33, 25)

Since for basic level L=3 there are no O_1 states, there are no corresponding LD_1 matrices. The complete structure of the infinitesimal generator matrix is given in figure 6.5.

6.7.3 Performance measures

We denote X(i) the stationary probability for the i^{th} state.

Average inventory at the possession of the vendor - WIP_2

$$WIP_2 = 4\cdot\sum_{i=12}^{24} X(i) + 8\cdot\sum_{i=25}^{32} X(i) + 12\cdot\sum_{i=33}^{34} X(i)$$

Stock-out probability at the retailer – SO

$$SO^0 = X(3) + X(8)$$

$$SO^Z = X(25) + X(26) + X(33) + X(34)$$

$$SO^{O1} = X(21) + X(22)$$

$$SO^{O2} = X(23) + X(24)$$

$$SO = SO^0 + SO^Z + SO^{O1} + SO^{O2}$$

Average inventory at the retailer – WIP_1

$$I_R(1) = X(4) + X(9) + X(13) + X(14) + X(27) + X(28)$$

$$I_R(2) = X(5) + X(10) + X(15) + X(16) + X(29) + X(30)$$

$$I_R(3) = X(6) + X(11) + X(17) + X(31)$$

$$I_R(4) = X(7) + X(12) + X(18) + X(32)$$

$$I_R(5) = X(1) + X(19)$$

$$I_R(6) = X(2) + X(20)$$

$$WIP_1 = \sum_{i=1}^{6} i\cdot I_R(i)$$

Figure 6.5: The infinitesimal generator matrix for s=4, Q_2=8, r=2, Q_1=4 and maximum demand per external customer md=4

		1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30	31	32	33	34
	state	0050	0060	0100	0110	0120	0130	0140	0200	0210	0220	0230	0240	4011	4012	4021	4022	4030	4040	4050	4060	4101	4102	4201	4202	8001	8002	8011	8012	8021	8022	8030	8040	d001	d002
1	0050	-λ			dm(4)*λ	dm(3)*λ	dm(2)*λ	dm(1)*λ																											
2	0060	dm(1)*λ	-λ			dm(4)*λ	dm(3)*λ	dm(2)*λ																											
3	0100			-μ21					d22*μ21																	d21*μ21									
4	0110			λ	-μ21-λ					d22*μ21																		d21*μ21							
5	0120			rd(2)*λ	dm(1)*λ	-μ21-λ					d22*μ21																			d21*μ21					
6	0130			rd(3)*λ	dm(2)*λ	dm(1)*λ	-μ21-λ					d22*μ21																				d21*μ21			
7	0140			rd(4)*λ	dm(3)*λ	dm(2)*λ	dm(1)*λ	-μ21-λ					d22*μ21																				d21*μ21		
8	0200								-μ22																	μ22									
9	0210								λ	-μ22-λ																		μ22							
10	0220								rd(2)*λ	dm(1)*λ	-μ22-λ																			μ22					
11	0230								rd(3)*λ	dm(2)*λ	dm(1)*λ	-μ22-λ																				μ22			
12	0240								rd(4)*λ	dm(3)*λ	dm(2)*λ	dm(1)*λ	-μ22-λ																				μ22		
13	4011	d11*μ11												-μ11-λ	d12*μ11							λ													
14	4012	μ12													-μ12-λ								λ												
15	4021		d11*μ11											dm(1)*λ		-μ11-λ	d12*μ11					rd(2)*λ													
16	4022		μ12												dm(1)*λ		-μ12-λ						rd(2)*λ												
17	4030													dm(2)*λ		dm(1)*λ		-λ				rd(3)*λ													
18	4040													dm(3)*λ		dm(2)*λ		dm(1)*λ	-λ			rd(4)*λ													
19	4050													dm(4)*λ		dm(3)*λ		dm(2)*λ	dm(1)*λ	-λ															
20	4060															dm(4)*λ		dm(3)*λ	dm(2)*λ	dm(1)*λ	-λ														
21	4101							d11*μ11														-μ21-μ11	d12*μ11	d22*μ21										d21*μ21	
22	4102							μ12															-μ21-μ12		d22*μ21										d21*μ21
23	4201												d11*μ11											-μ22-μ11	d12*μ11									μ22	
24	4202												μ12												-μ22-μ12										μ22
25	8001																		d11*μ11							-μ11	d12*μ11								
26	8002																		μ12								-μ12								
27	8011																			d11*μ11						λ		-μ11-λ	d12*μ11						
28	8012																			μ12							λ		-μ12-λ						
29	8021																				d11*μ11					rd(2)*λ		dm(1)*λ		-μ11-λ	d12*μ11				
30	8022																				μ12						rd(2)*λ		dm(1)*λ		-μ12-λ				
31	8030																									rd(3)*λ		dm(2)*λ		dm(1)*λ		-λ			
32	8040																									rd(4)*λ		dm(3)*λ		dm(2)*λ		dm(1)*λ	-λ		
33	d001																																d11*μ11	-μ11	d12*μ11
34	d002																																μ12		-μ12

Throughput

$$I_{RO}(i) = \sum_{j=i}^{r+Q_1} I_R(j)$$

$$I_{RO}(1) = I_R(1) + I_R(2) + I_R(3) + I_R(4) + I_R(5) + I_R(6)$$
$$I_{RO}(2) = I_R(2) + I_R(3) + I_R(4) + I_R(5) + I_R(6)$$
$$I_{RO}(3) = I_R(3) + I_R(4) + I_R(5) + I_R(6)$$
$$I_{RO}(4) = I_R(4) + I_R(5) + I_R(6)$$
$$I_{RO}(5) = I_R(5) + I_R(6)$$
$$I_{RO}(6) = I_R(6)$$

$$met(i) = dm(i) \cdot I_{RO}(i)$$
$$unmet(i) = \left(rd(i) - dm(i)\right) \cdot I_R(i)$$
$$Throughput = \lambda \cdot \sum_{i=1}^{6} i \cdot \left(met(i) + unmet(i)\right)$$

Utilization of transportation resource towards the Retailer

$$u_1 = \sum_{i=13}^{16} X(i) + \sum_{i=21}^{24} X(i) + \sum_{i=25}^{30} X(i) + \sum_{i=33}^{34} X(i)$$

Utilization of transportation resource towards the Vendor

$$u_2 = \sum_{i=3}^{12} X(i) + \sum_{i=21}^{24} X(i)$$

The rest of the performance measures can be calculated from the above metrics through simple relations.

6.7.4 Validation of the algorithmic results

The system of linear equations that describes the system was constructed manually and solved in Mathematica to get the vector of stationary probabilities. Then the performance measures of the system were calculated as described in section 6.7.3. The results were practically identical to those produced algorithmically. The algorithmic results were also contrasted to simulation results (see section 6.8). Five replications of 2000000 time units each were used. The algorithmic results were within the confidence interval provided by simulation. In the table below are given the results for μ_{11}=2.0, μ_{12}=0.4, d_{12}=0.2, μ_{21}=1.25, μ_{22}=0.5, d_{22}=0.1, λ=0.5, and $dm = (0.1, 0.2, 0.3, 0.4)$.

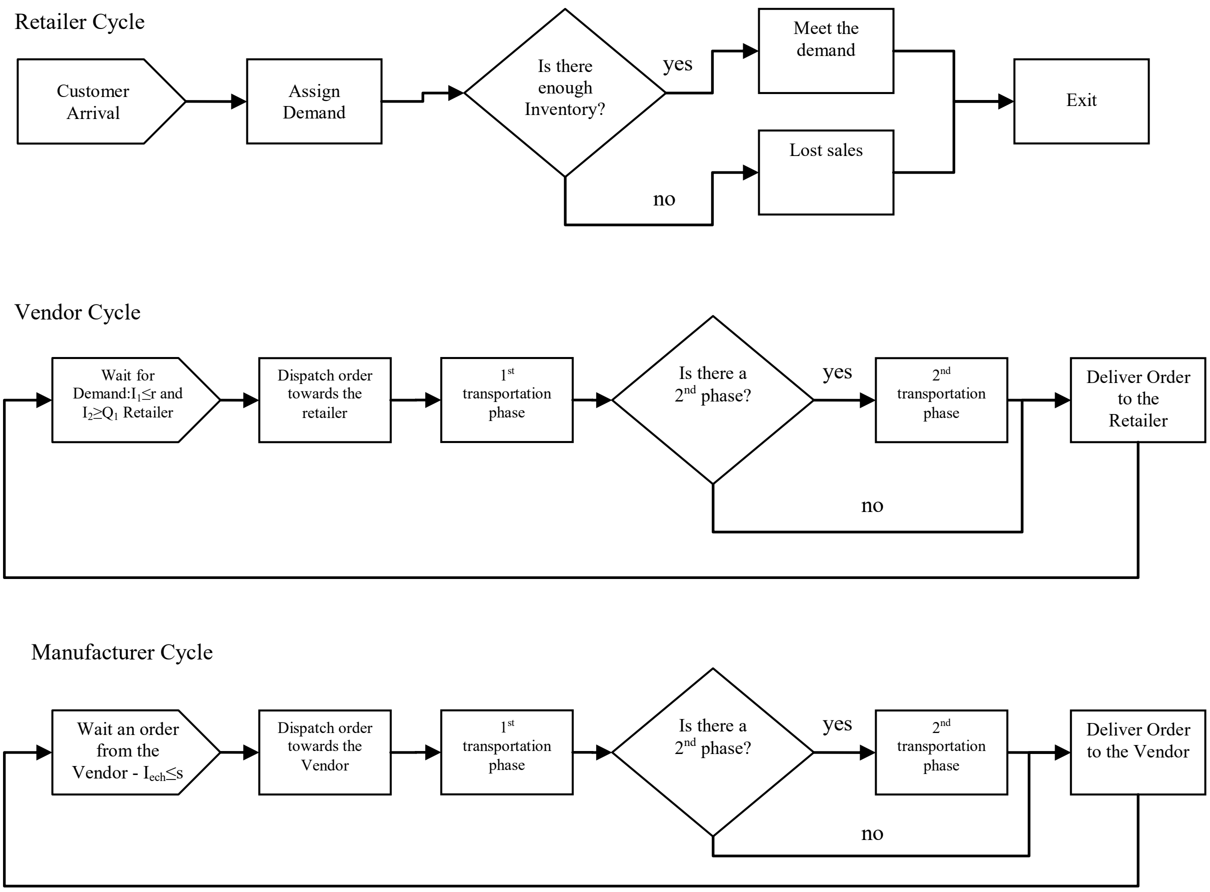
Performance measure	Algorithm - Matlab	Manually- Mathematica	Simulation – Arena (95% C.I.)
OFR	0.659517	0.659517	0.660 ± 0.001
SL	0.717521	0.717521	0.718 ± 0.001
WIP_1	3.106957	3.106959	3.107 ± 0.001
WIP_2	4.748908	4.748911	4.751 ± 0.001
Throuput	1.076281	1.076281	1.077 ± 0.001
$Intransit_1$	1.076281	1.076281	1.076 ± 0.001

6.8 Validation of the model

6.8.1 Simulation model

Several simple scenarios we solved manually using Mathematica and in every case the results were identical to those produced algorithmically. However, such an approach is not practical for a more thorough validation of the developed algorithm since for bigger systems several thousand states may be involved. For a more vigorous testing of the algorithm, a simulation model of the system under consideration was developed. The system was modeled as a series of cycles, each cycle describing the interface between successive members of the network. The basic logic of the simulation model is given in figure 6.6.

Figure 6.6: Simulation model logic



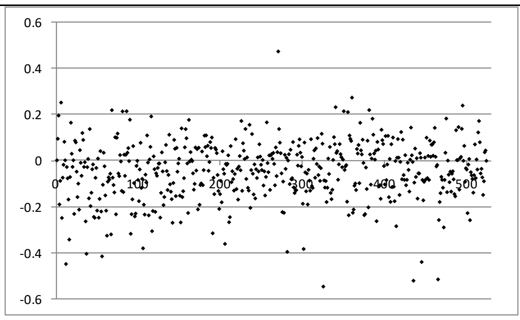
The simulation model was constructed in Arena simulation package, Version 12.00.00 – CPR 9. Test runs were executed to determine the specific parameters of the simulation runs that would give statistically rigorous results within a reasonable computation time. A simulation time of 1600000 time units was selected as it was deemed long enough to provide statistically acceptable results. To eliminate any effects of the initial conditions, a warm-up period of 100000 time units was also selected.

Eight different performance measures were included in the analysis: Order Fill Rate (OFR), Service Level (SL), Average inventory at the retailer (WIP_1), Average inventory at the possession of the vendor (WIP_2), Average inventory in transit towards the retailer ($Intransit_1$), Average inventory in transit towards the Vendor ($Intransit_2$), the average lost sales per lost customer (ALS_{order}), and the Throughput.

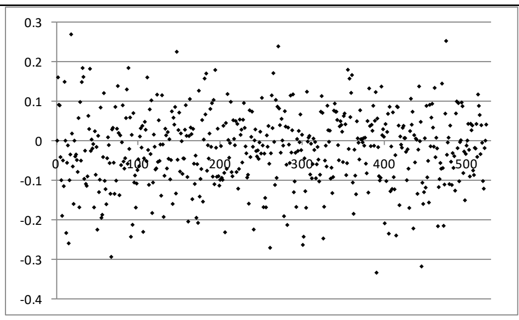
6.8.2 Simulation results

Figure 6.7: % Difference of Simulation and Analytic Results. λ=0.5, μ_{11}=2, μ_{12}=0.4, d_{12}=0.2, μ_{21}=1.25, μ_{22}=0.5, d_{22}=0.1, $dm = [0.4,0.3,0.2,0.1]$, $0 \le s \le 5$, $1 \le n \le 5$, $0 \le r \le s$, $1 \le Q_1 \le 5$

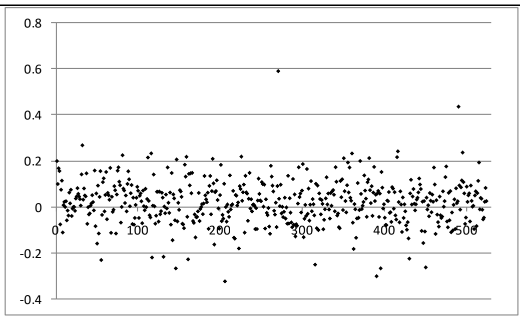
OFR



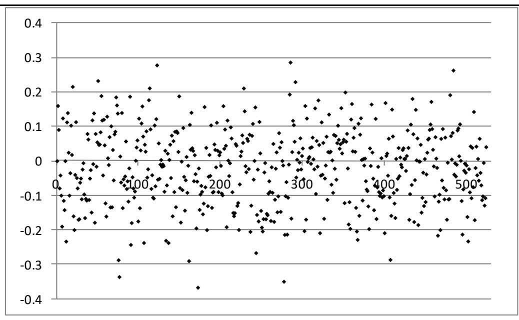
SL



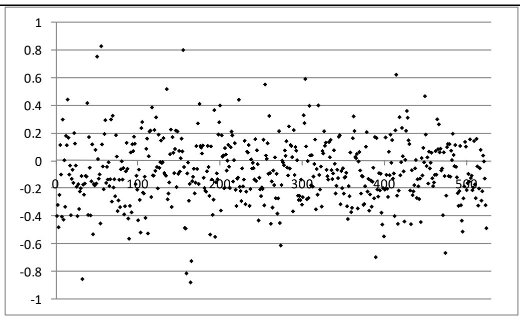
WIP_1



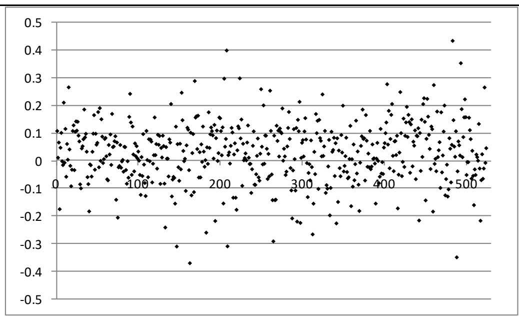
Throughput



$Intransit_1$



ALS_{order}



More than 1400 scenarios were tested for different combinations of system parameters. Balanced, supply constrained, and demand constrained systems were included in the analysis. Simulation results were found to be consistent with the results of the analytic algorithm. The observed deviation was well within the limits of the expected variability due to the statistical nature of simulation.

Some examples of the comparison between the analytic solution and the simulation results are given in Fig. 6.7. In the graphs the deviation is given as the percentage difference between the analytical method and simulation:

$$\%\,deviation = 100 \times \frac{analytic - simulation}{analytic}$$

Some of the corresponding data are given in the Appendix.

6.9 Model Performance and limitations

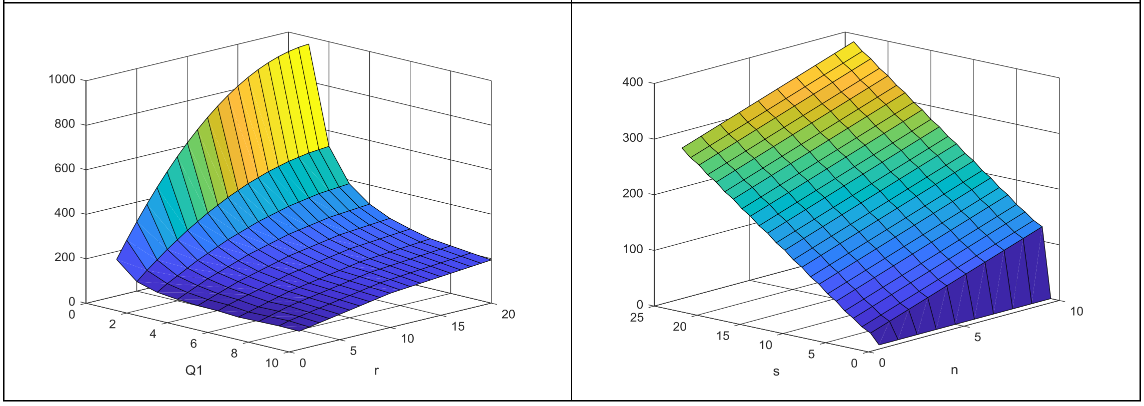
The algorithm was programmed in Matlab, version 2018a, 9.4.0.813654. For the runs commented here a computer with Core i-3-4005U CPU at 1.70 GHz processor and 4GB installed RAM was used. Its operating system was Windows 7 – Ultimate, 64-bit.

The proposed algorithm is valid for any combination of the decision variables and for any given system parameters. However, as the system under consideration becomes bigger, the dimension of the infinitesimal generator matrix increases and the algorithm may become computationally demanding. The number of the possible states depends on the relation of Q_1 with r and s, as well as the absolute values of r, s, and n (Q_2=n·Q_1). In general, higher numbers of states are observed for low Q_1 values and high s, r and n values. It must be noted that due to the assumptions that have been made about the system, the number of states remains relatively low over a wide range of the decision variable values, allowing us flexibility to model a wide range of problems. Some examples of the number of states as a function of the decision variables are given in figure 6.8.

The multiplication of problem's dimensions is a common drawback with models based on Markov analysis (Mehmood and Lu, 2011) and restricts the application of exact Markov models in real scale systems. Rising computational power tends to alleviate the problem, but it still remains one of the main drawbacks of such approaches.

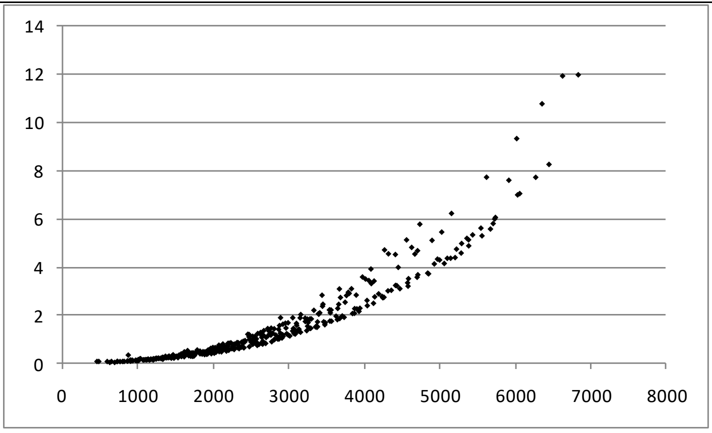
Figure 6.8: Number of states as a function of the decision variables

s=20, n=2 | r=4, Q_1=2



Computational time depends mainly on the number of possible states. The exact relation between the decision variables affects the number of steps that are required for the solution of the system, so it also has an effect. An example of computational time as a function of the infinitesimal generator's dimensions is given in figure 6.9.

Figure 6.9: Computational time (sec) as a function of the infinitesimal generator matrix dimension



The main practical limitation of the model is the required RAM memory, which in its turn depends on the dimensions of the infinitesimal generator matrix. Even in a computer of moderate performance, the developed model offers a satisfactory degree of flexibility. The biggest problem solved in the testing computer comprised of 22902 states (s=90, n=40, r=80, Q_1=1). At this stage of our research priority was given to the tractability of the algorithm in relation to the theory, so neither algorithmic efficiency (number of steps to the solution), nor memory consumption were taken into consideration during the development of the computer program. Both could be improved by rephrasing the program code and by exploiting embedded features of Matlab such as sparse matrices.

The proposed algorithm offers certain advantages compared to alternative approaches such as simulation. Firstly, the analytic approach gives better results in terms of speed. Even for relatively big systems, the exact algorithm is significantly faster than simulation and in most cases the difference in computation time is several orders of magnitude. Secondly, the exact algorithm poses no limits on precision. In contrast, simulation provides results in the form of confidence intervals and this can be problematic especially when low values of the performance measures are concerned where the specific value is comparable to the margin of error. Finally, the exact algorithm is easier to integrate in a more comprehensive scheme, as for example, in the context of an optimization model.

6.10 Numerical Results

6.10.1 Balanced systems

We examine balanced systems where the average time between successive arrivals of external customers is equal to both the average transportation time from the manufacturer to the vendor and the average transportation time from the vendor to the retailer: $\frac{1}{\lambda} = \frac{1}{\mu_{21}} + d_{22} \cdot \frac{1}{\mu_{22}} = \frac{1}{\mu_{11}} + d_{12} \cdot \frac{1}{\mu_{12}}$

For the examples presented below we use μ_{11}=1, μ_{12}=0.2, d_{12}=0.2, μ_{21}=1, μ_{22}=0.2, d_{22}=0.2, λ=0.5 and $dm = (0.6, 0.4)$ (there is 0.6 probability that an external customer will ask 1 product unit and 0.4 probability that he will ask 2 units).

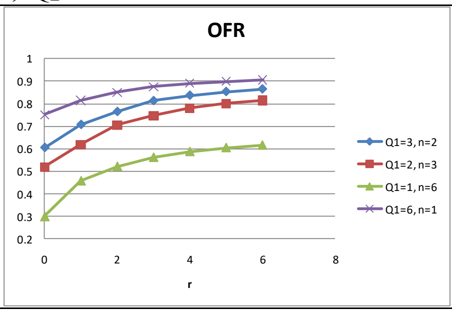
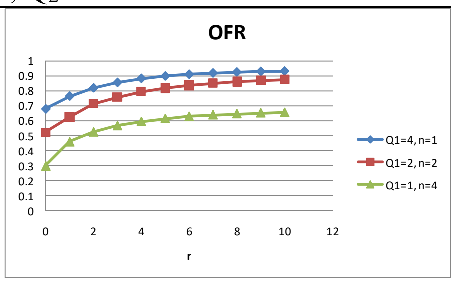
6.10.1.1 The effect of the decision variables on the performance measures

We investigate the effect of each decision variable on the performance measures.

<u>Retailer's reorder point – r</u>

We investigate the effect of the retailer's reorder point (r) on the performance measures.

<u>Order Fill rate and Service Level</u>

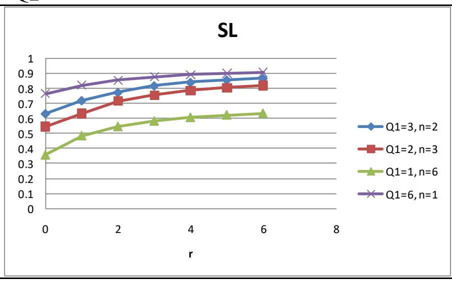
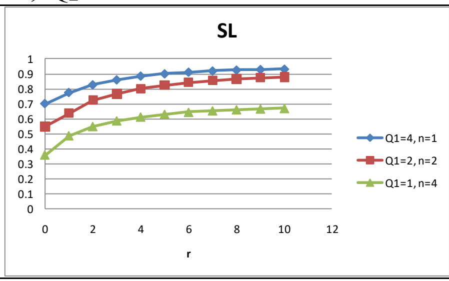
<table>
<tr><td colspan="2">Order Fill Rate as a function of reorder point r</td></tr>
<tr><td>s=6, Q_2=6</td><td>s=10, Q_2=4</td></tr>
<tr><td>
</td><td>
</td></tr>
</table>

By increasing r we can increase OFR but with decreasing elasticity. The higher the value of r, the more difficult it is to affect OFR through it. Beyond a point, the effect of the retailer's reorder point becomes negligible as OFR reaches a plateau. In general, the value of r is more important for lower Q_1 values, but demand characteristics may also have an effect.

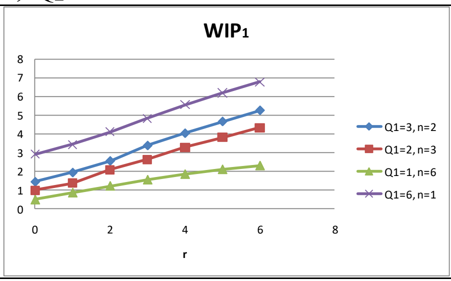
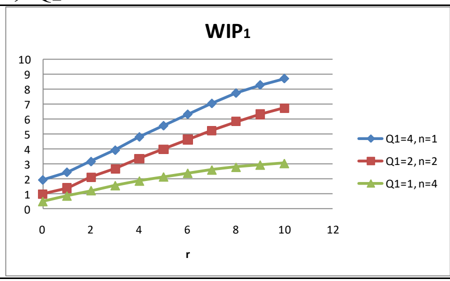
Service Level (SL) is related to Order Fill Rate (OFR):

$$\frac{1-SL}{1-FR}=\frac{ALS_{order}}{E_x}$$

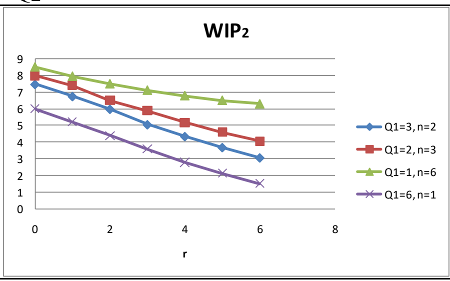
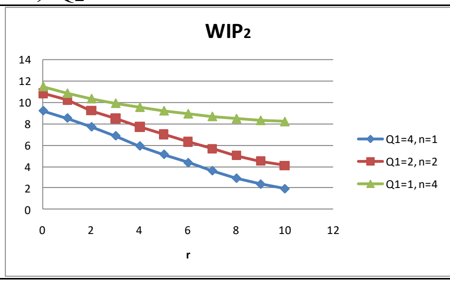
For the given demand characteristics, both performance measures exhibit a similar behavior.

Service Level as a function of reorder point *r*	
s=6, Q_2=6	s=10, Q_2=4
	

Average inventories

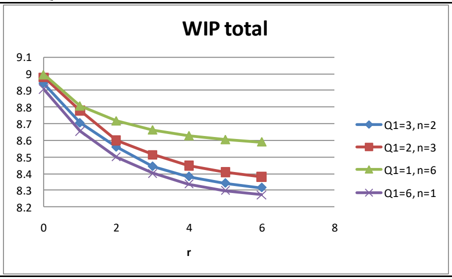
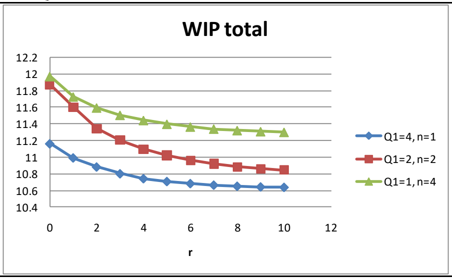
Retailer's WIP as a function of reorder point *r*	
s=6, Q_2=6	s=10 , Q_2=4
	

Increasing *r* causes an almost linear increase in retailer's inventory. The effect of *r* on WIP_1 is less pronounced in the case of base stock policy (Q_1=1).

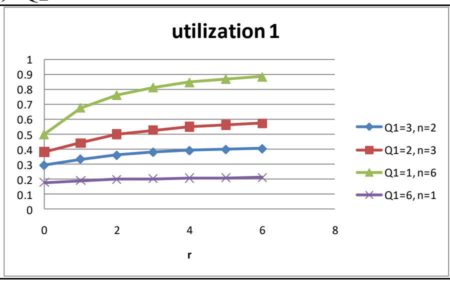
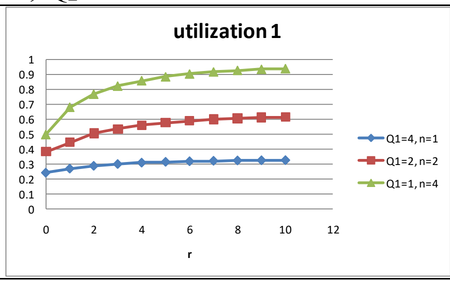
Wholesaler's WIP as a function of reorder point *r*	
s=6, Q_2=6	s=10 , Q_2=4
	

The increase of *r* for a given echelon inventory policy (s,Q_2) causes an almost linear decrease in wholesaler's inventory. The effect is less pronounced in the case of base stock policy (Q_1=1).

For low *r* values total inventory decreases with increasing *r*, but beyond a point the effect becomes negligible. In effect, for a given echelon inventory policy (s, Q_2) the increase of *r* causes a transfer of system inventory downstream from the wholesaler to the retailer.

Total average inventory as a function of reorder point *r*	
s=6, Q_2=6	s=10 , Q_2=4
	

Utilization of resources

Utilization of transportation resource towards the retailer as a function of reorder point *r*	
s=6, Q_2=6	s=10 , Q_2=4
	

By the definition of balanced systems, utilization for transportation towards the retailer is in a linear relation with the utilization for transportation towards the wholesaler:

$$Q_1 \cdot utilization_1 = Q_2 \cdot utilization_2$$

For low *r* values, increasing *r* tends to increase utilization and their relation can be described with good precision as logarithmic. For high *r* values or high Q_1 values the effect becomes negligible. The effect of the retailer's reorder point is most important in the case of base stock policies.

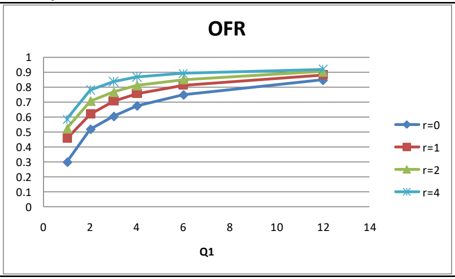
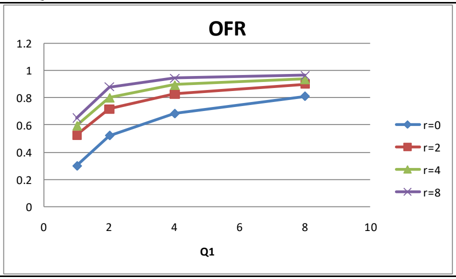
Retailer's replenishment order – Q_1

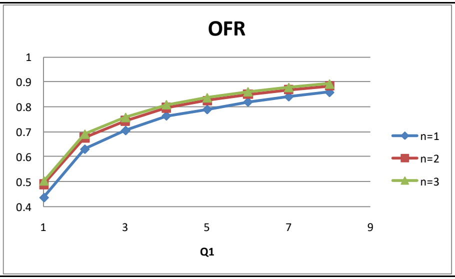
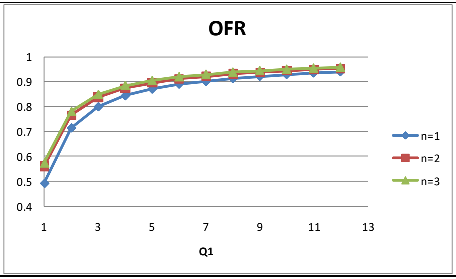
We investigate the effect of the retailer's replenishment order (Q_1) on the performance measures.

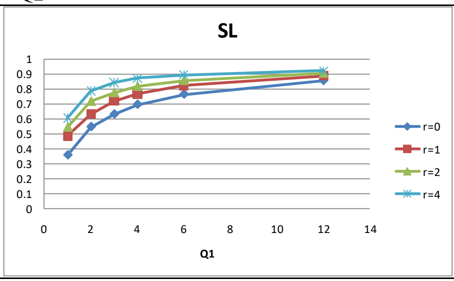
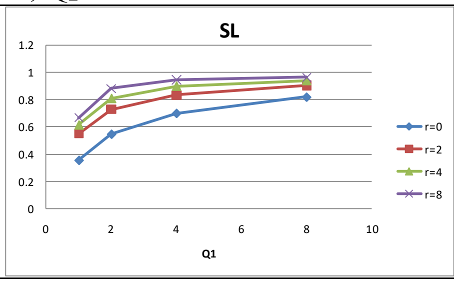
Order Fill rate and Service Level

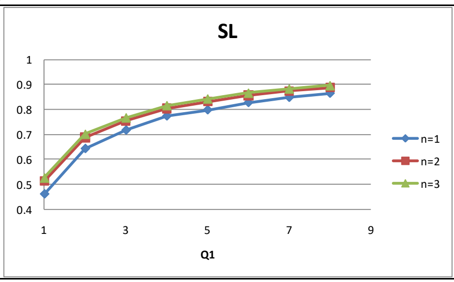
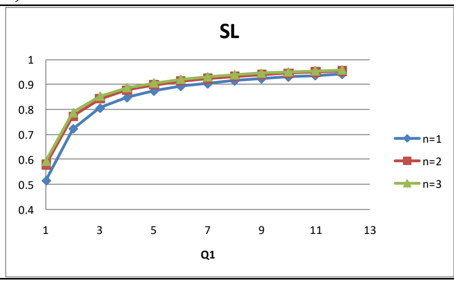
Increasing Q_1 causes OFR to increase, but with decreasing elasticity. The behavior of OFR with changing Q_1 is similar for all tested reorder point *r* values, while when both Q_1 and Q_2 are allowed to change simultaneously (n=constant), predictably the effect of Q_1 is more important.

As has already been explained, the behavior of SL with changing Q_1 is similar to that of OFR.

<table>
<tr><td colspan="2">Order Fill Rate as a function of replenishment order Q_1 (Q_2=const)</td></tr>
<tr><td>s=4, Q_2=12</td><td>s=10, Q_2=8</td></tr>
<tr><td>
</td><td>
</td></tr>
</table>

<table>
<tr><td colspan="2">Order Fill Rate as a function of replenishment order Q_1 (n=const)</td></tr>
<tr><td>s=4, r=2</td><td>s=6, r=4</td></tr>
<tr><td>
</td><td>
</td></tr>
</table>

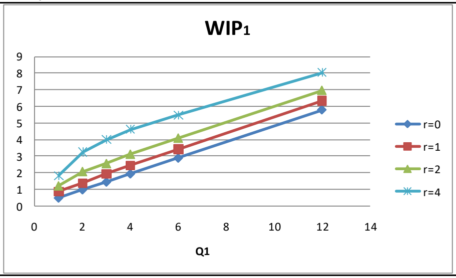
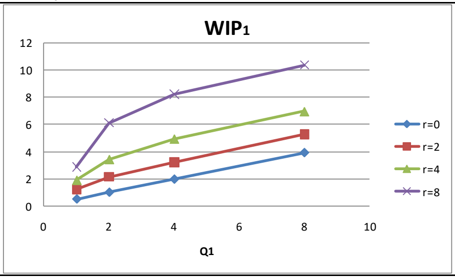
<table>
<tr><td colspan="2">Service level as a function of replenishment order Q_1 (Q_2=const)</td></tr>
<tr><td>s=4, Q_2=12</td><td>s=10, Q_2=8</td></tr>
<tr><td>
</td><td>
</td></tr>
</table>

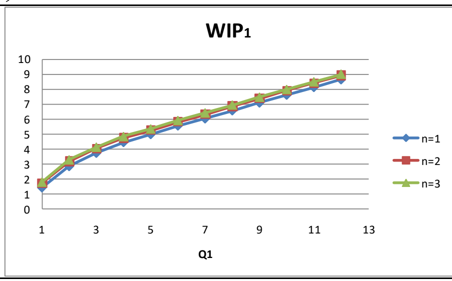
<table>
<tr><td colspan="2">Service level as a function of replenishment order Q_1 (n=const)</td></tr>
<tr><td>s=4, r=2</td><td>s=6, r=4</td></tr>
<tr><td>
</td><td>
</td></tr>
</table>

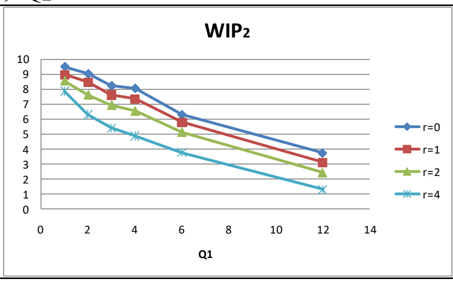
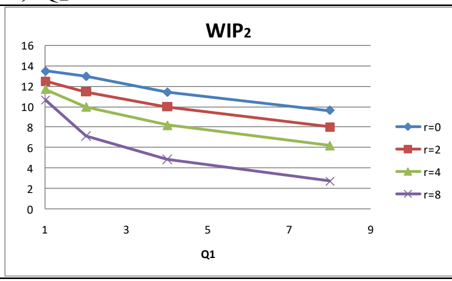
Average inventories

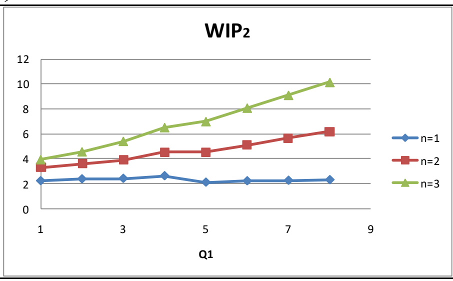
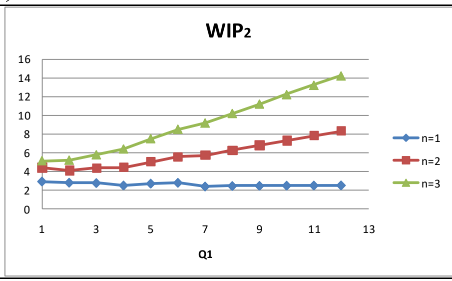
For a given vendor's policy, an increase in Q_1 causes an increase in WIP_1 and a corresponding decrease in WIP_2. The effect in total WIP is the sum of these changes. The effect of Q_1 in WIP total depends on the specific values of the decision variables, and in certain scenarios local minima were observed. In any case, the change in total average inventory was relatively small.

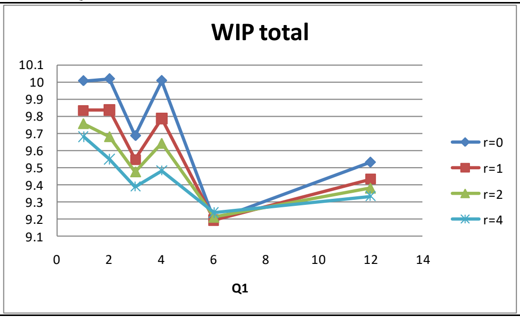
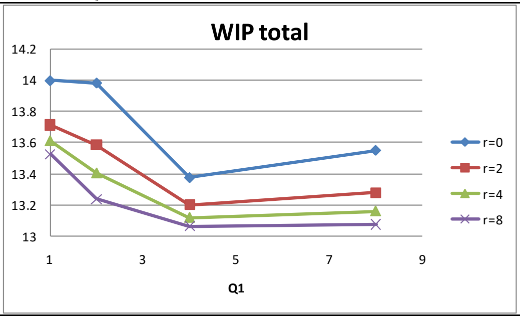
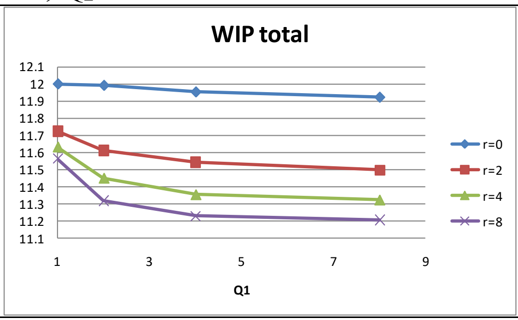
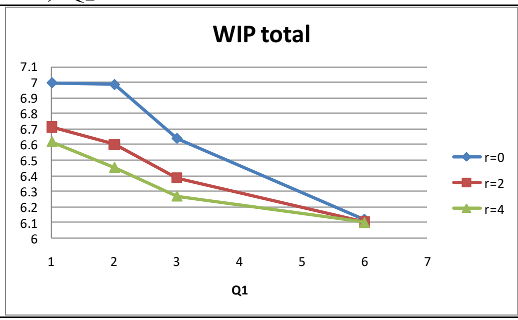
When Q_2 is allowed to change along with Q_1 (n=constant), increasing Q_1 causes a significant increase in all average inventories.

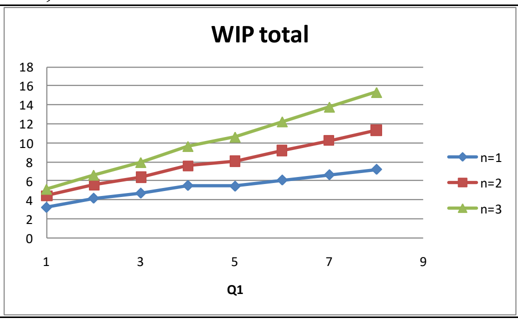
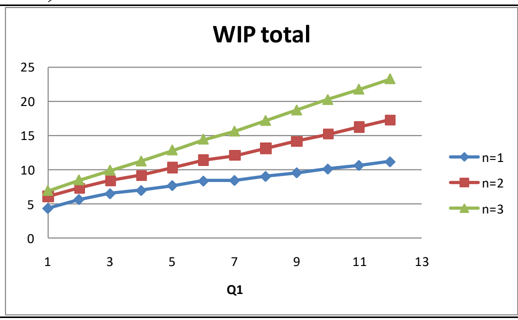
Average inventory at the retailer as a function of the replenishment order Q_1 (Q_2=const)	
s=4, Q_2=12	s=10, Q_2=8
	

Average inventory at the retailer as a function of the replenishment order Q_1 (n=const)	
s=4, r=2	s=6, r=4
	

Average inventory at the vendor as a function of replenishment order Q_1 (Q_2=const)	
s=4, Q_2=12	s=10, Q_2=8
	

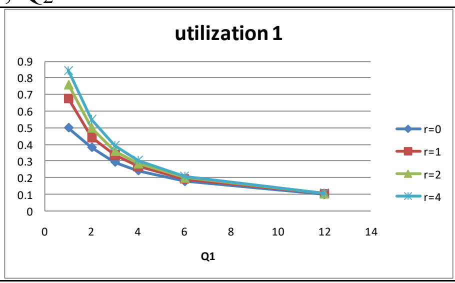
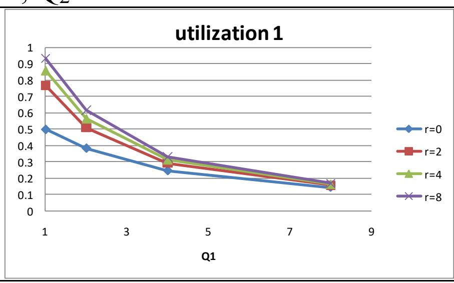
Average inventory at the vendor as a function of replenishment order Q_1 (n=const)	
s=4, r=2	s=6, r=4
	

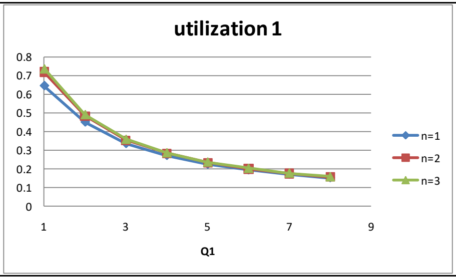
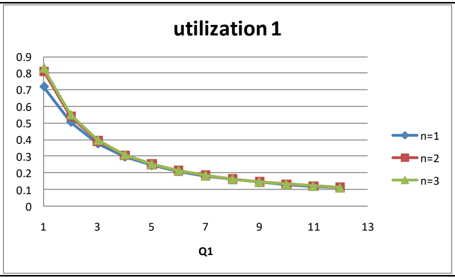
<table>
<tr><td colspan="2">Average total inventory as a function of replenishment order Q_1 (Q_2=const)</td></tr>
<tr><td>s=4, Q_2=12</td><td>s=10, Q_2=8</td></tr>
<tr><td>
</td><td>
</td></tr>
<tr><td>s=8, Q_2=8</td><td>s=4, Q_2=6</td></tr>
<tr><td>
</td><td>
</td></tr>
</table>

<table>
<tr><td colspan="2">Average total inventory as a function of replenishment order Q_1 (n=const)</td></tr>
<tr><td>s=4, r=2</td><td>s=6, r=4</td></tr>
<tr><td>
</td><td>
</td></tr>
</table>

Utilization of resources

Predictably, the utilization of the transportation resource towards the retailer decreases as the amount of the replenishment orders increases (less replenishment orders are required to transfer a given amount of inventory). In comparison with the effect of Q_1, the effect of Q_2 is negligible.

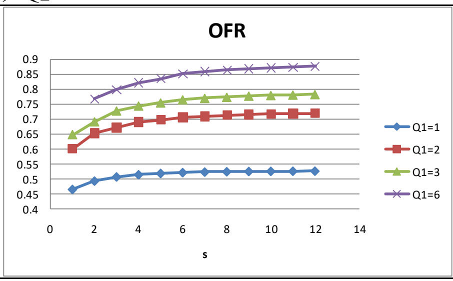
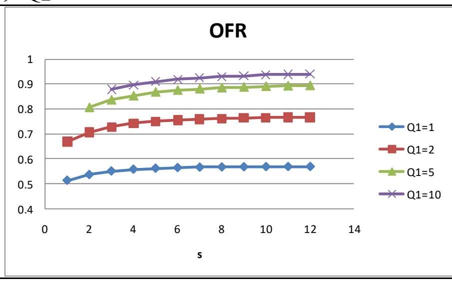
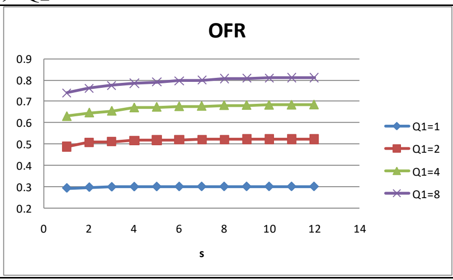
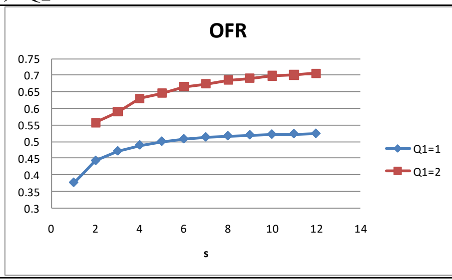
<table>
<tr><td colspan="2">Utilization of transportation resource towards the retailer as a function of replenishment order Q_1 (Q_2=const)</td></tr>
<tr><td>s=4, Q_2=12</td><td>s=10, Q_2=8</td></tr>
<tr><td>
</td><td>
</td></tr>
</table>

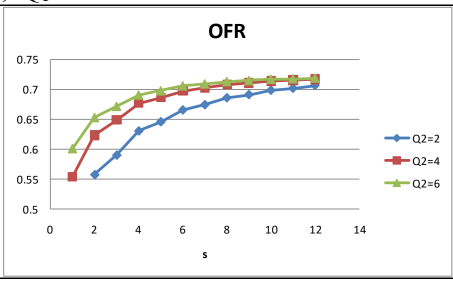
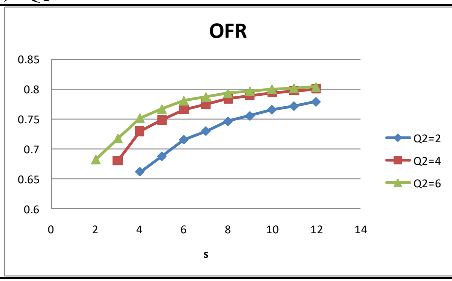
<table>
<tr><td colspan="2">Utilization of transportation resource towards the retailer as a function of replenishment order Q_1 (n=const)</td></tr>
<tr><td>s=4, r=2</td><td>s=6, r=4</td></tr>
<tr><td>
</td><td>
</td></tr>
</table>

Echelon reorder point – s

We investigate the effect of the echelon reorder point (*s*) on the performance measures.

Order Fill rate and Service Level

<table>
<tr><td colspan="2">Order Fill Rate as a function of echelon reorder point s for different Q_1 values</td></tr>
<tr><td>r=2, Q_2=6</td><td>r=3, Q_2=10</td></tr>
<tr><td>
</td><td>
</td></tr>
<tr><td>r=0, Q_2=8</td><td>r=2, Q_2=2</td></tr>
<tr><td>
</td><td>
</td></tr>
</table>

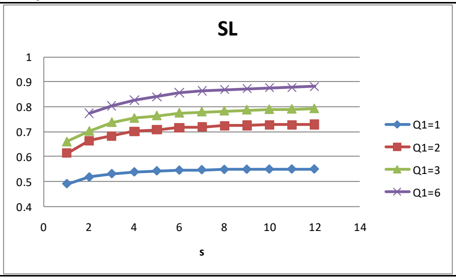
<table>
<tr><td colspan="2">Order Fill Rate as a function of echelon reorder point s for different Q_2 values</td></tr>
<tr><td>r=2, Q_1=2</td><td>r=4, Q_1=2</td></tr>
<tr><td>
</td><td>
</td></tr>
</table>

OFR can be increased by increasing echelon reorder point but the enhancement of OFR that can be achieved is limited. The importance of *s* depends on the Q_2 value, the effect of s being more pronounced for low Q_2.

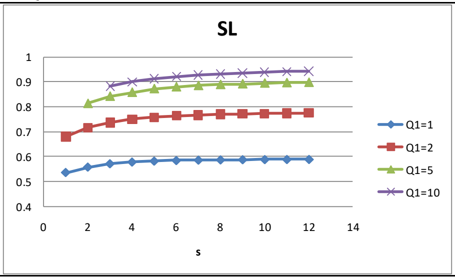
For the given demand characteristics, the behavior of SL with changing *s* is similar to that of OFR.

Service Level as a function of echelon reorder point s

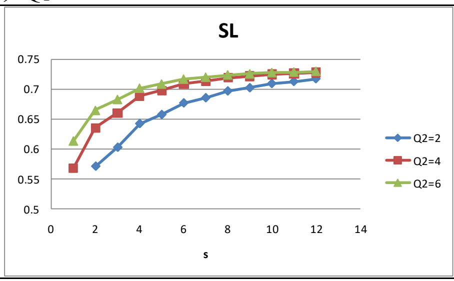
r=2, Q_2=6



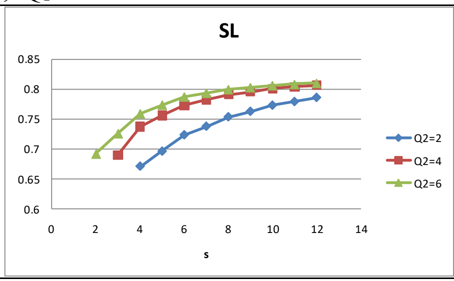
r=3, Q_2=10



r=2, Q_1=2



r=4, Q_1=2

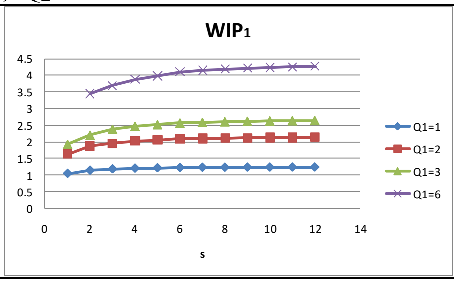


Average inventories

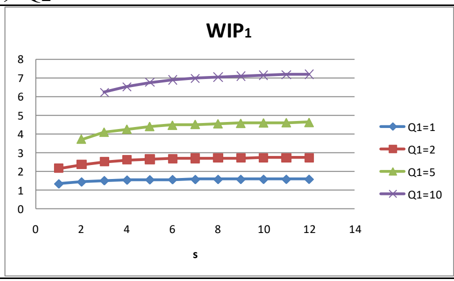
The increase of *s* initially causes a relatively mild increase in the average retailer's inventory and for higher *s* values the effect become negligible. In contrast, increasing *s* causes an almost linear increase in the average inventory of the wholesaler. As a result, the average total inventory increases significantly with *s*.

Average inventory at the retailer as a function of echelon reorder point *s*

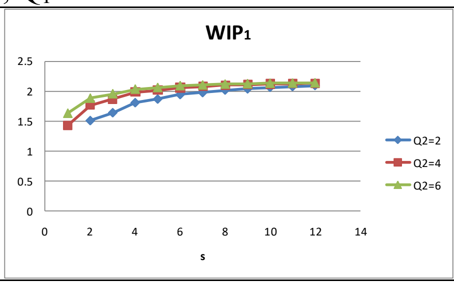
r=2, Q_2=6



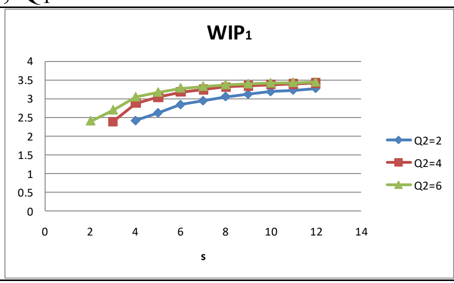
r=3, Q_2=10



r=2, Q_1=2

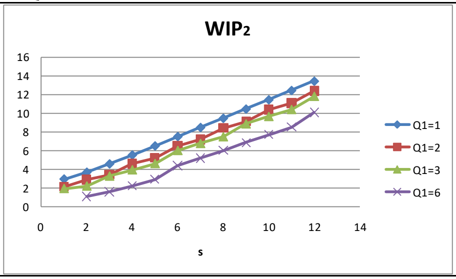


r=4, Q_1=2

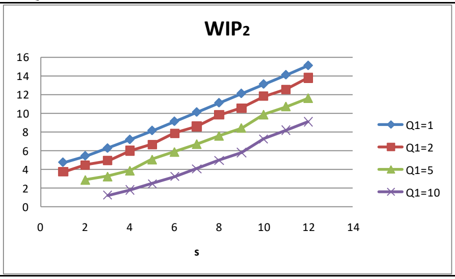


Average inventory at the wholesaler as a function of echelon reorder point *s*

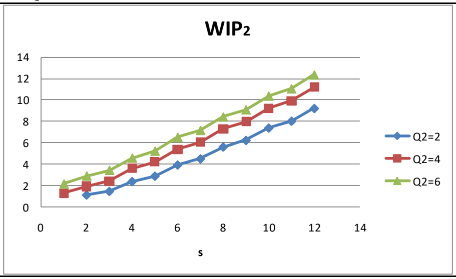
r=2, Q_2=6



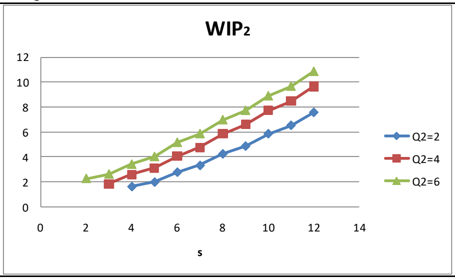
r=3, Q_2=10



r=2, Q_1=2

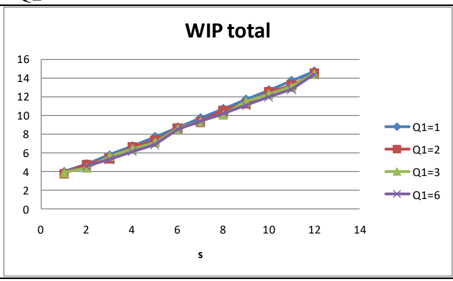


r=4, Q_1=2

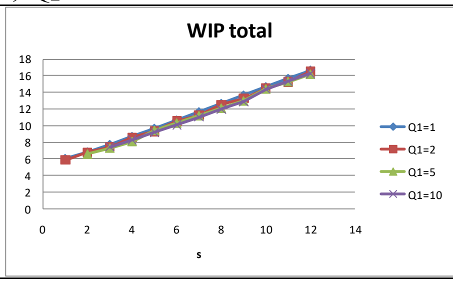


Average total inventory as a function of echelon reorder point *s*

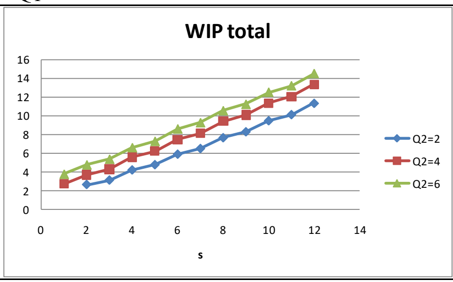
r=2, Q_2=6



r=3, Q_2=10



r=2, Q_1=2

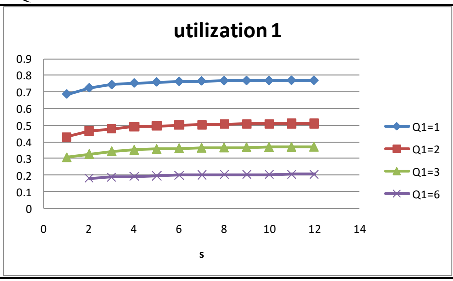


r=4, Q_1=2

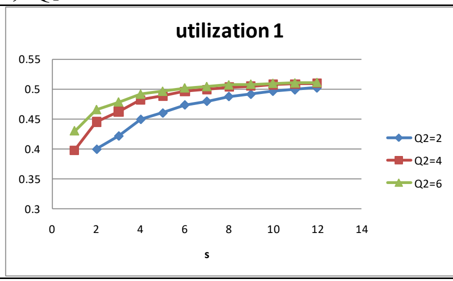
Utilization of resources

Utilization of transportation resource towards the retailer as a function of echelon reorder point s

r=2, Q_2=6



r=2, Q_1=2

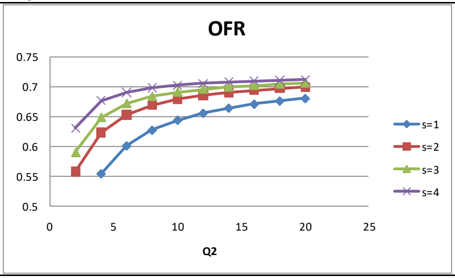
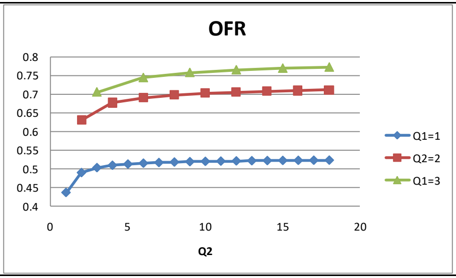


Utilization of transportation resource towards the retailer tends to increase mildly with increasing *s*. The effect becomes less important for high Q_2 values.

Vendor's replenishment order – Q_2

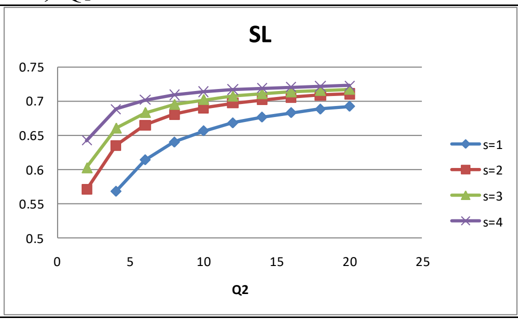
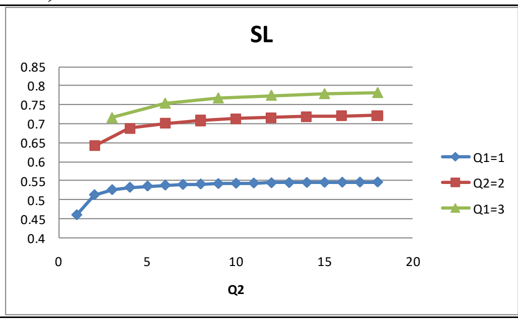
We investigate the effect of the vendor's replenishment order (Q_2) on the performance measures.

Order Fill rate and Service Level

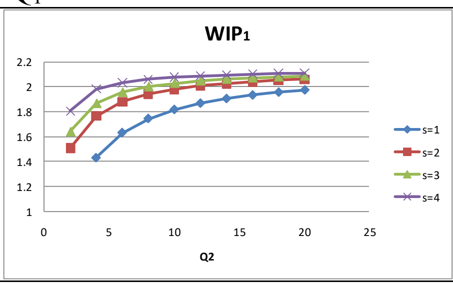
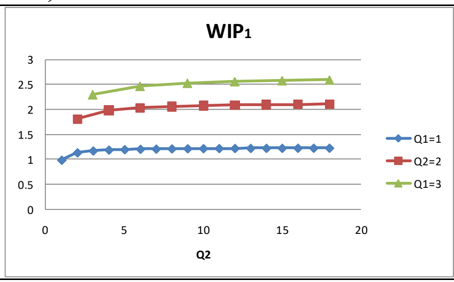
Order Fill Rate as a function of replenishment order Q_2	
r=2, Q_1=2	s=4, r=2
	

For low *s* values OFR can be increased by increasing the echelon replenishment order Q_2. However, beyond a point, the effect becomes negligible. The behavior of OFR with changing Q_2 is not affected by *s* or Q_1 values (The Q_2-OFR curves have similar shape for different s values and different Q_1 values).

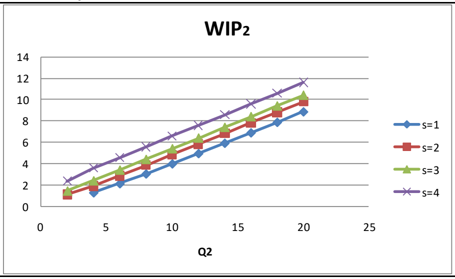
As already explained, for the given demand characteristics, SL exhibits a similar behavior.

Service level as a function of replenishment order Q_2	
r=2, Q_1=2	s=4, r=2
	

Average inventories

Average inventory at the retailer as a function of replenishment order Q_2	
r=2, Q_1=2	s=4, r=2
	

The average inventory at the retailer is affected mildly by Q_2, and only where the lowest possible Q_2 values are concerned. In contrast, the average inventory at the wholesaler increases with Q_2 according to a relation that can be described with good precision as linear. Total system inventory, as the sum of WIP_1 and WIP_2, increases almost linearly with increasing Q_2.

<table>
<tr><td colspan="2">Average inventory at the wholesaler as a function of replenishment order Q_2</td></tr>
<tr><td>r=2, Q_1=2</td><td>s=4, r=2</td></tr>
<tr><td>


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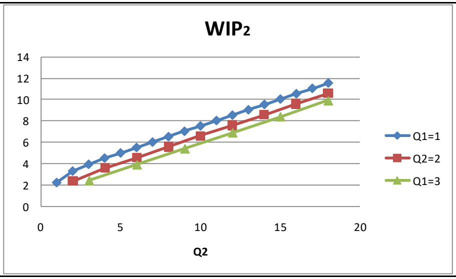
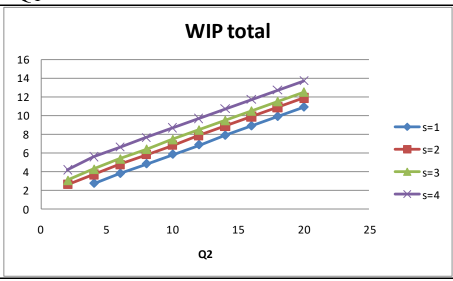


</td></tr>
</table>

<table>
<tr><td colspan="2">Total average inventory as a function of replenishment order Q_2</td></tr>
<tr><td>r=2, Q_1=2</td><td>r=3, Q_1=1</td></tr>
<tr><td>


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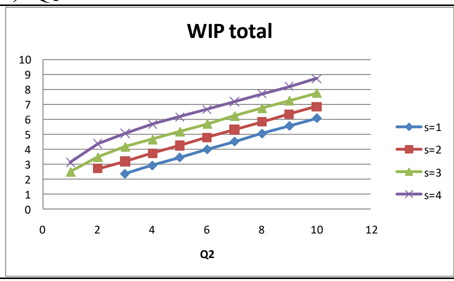
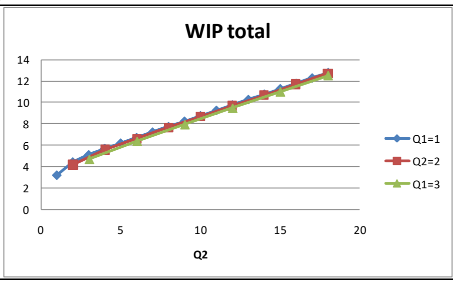


</td></tr>
<tr><td>s=4, r=2</td><td>s=6, r=5</td></tr>
<tr><td>


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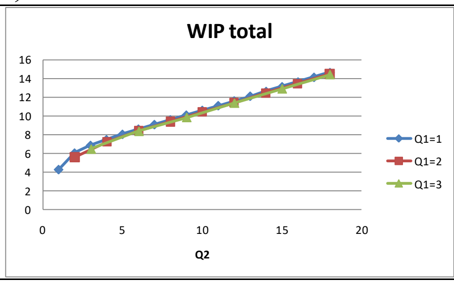
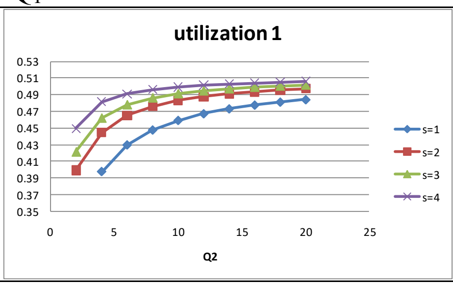


</td></tr>
</table>

Utilization of resources

<table>
<tr><td colspan="2">Order Fill Rate as a function of replenishment order Q_2</td></tr>
<tr><td>r=2, Q_1=2</td><td>s=4, r=2</td></tr>
<tr><td>


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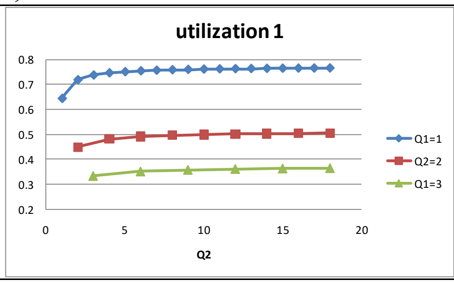


</td></tr>
</table>

Increasing the echelon replenishment order Q_2, causes an increase in transportation resource utilization. The effect is mild and reflects the increase in Service Level and the corresponding increase of output.

6.10.1.2 Combined Effect of the decision variables on the performance measures

Expanding the analysis of the previous section, we investigate the behavior of the performance measures when two of the decision variables are changed simultaneously.

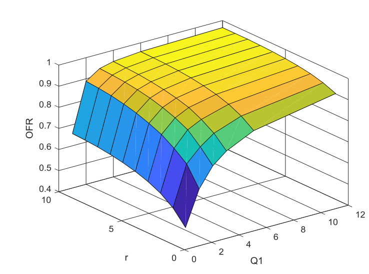
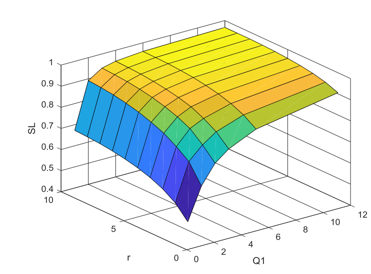
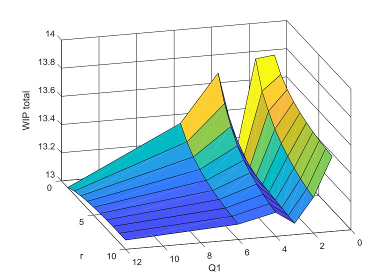
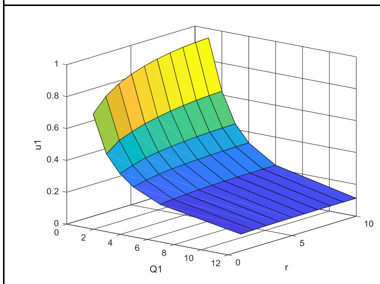
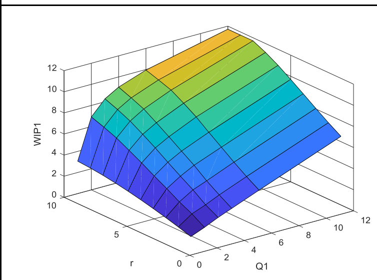
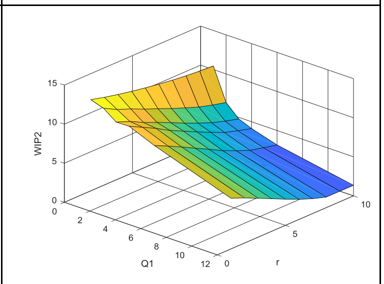
As explained in 6.10.1.1. SL and OFR are related to each other. For the demand characteristics that were used for the analysis (low variance), Order Fill Rate and Service Level exhibit a similar behavior with OFR taking slightly lower values in comparison with SL. Moreover, utilization of resources is in linear relation with SL through system Output. If T_1 the average replenishment time for the retailer and T_2 the average replenishment time for the wholesaler:

$$Throughput = \lambda \cdot E_x \cdot SL = \frac{utilization_1 \cdot Q_1}{T_1} = \frac{utilization_2 \cdot Q_2}{T_2}$$

For the above reasons, and so as to economize space, OFR and utilizations are not included in the following analysis.

Retailer's inventory policy

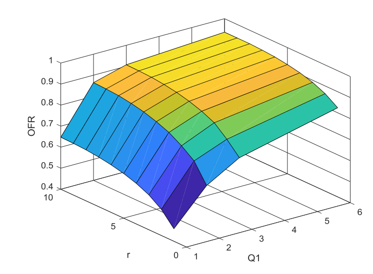
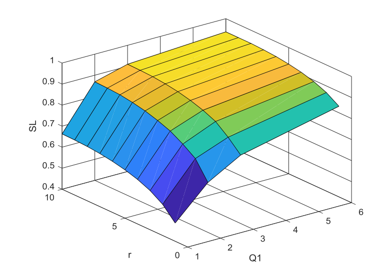
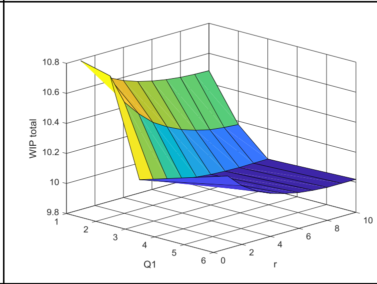
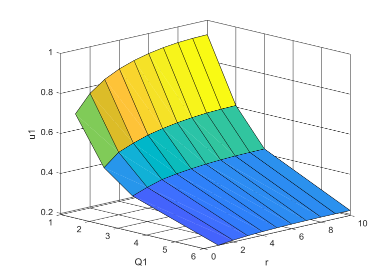
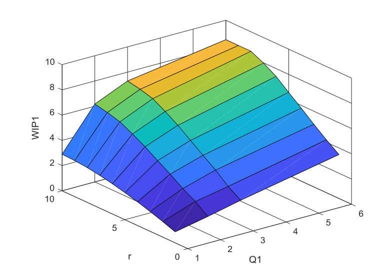
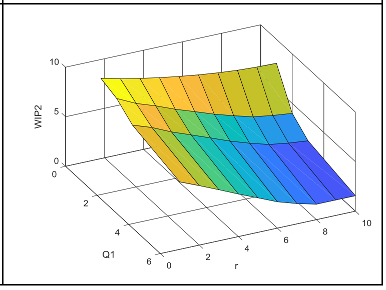
We investigate the simultaneous change of both parameters of the retailer's inventory policy (r,Q_1).

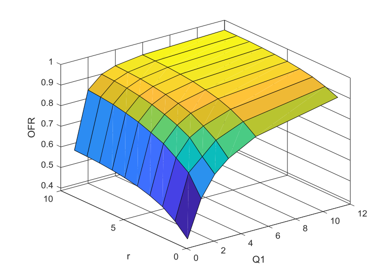
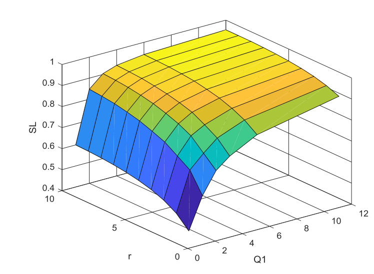
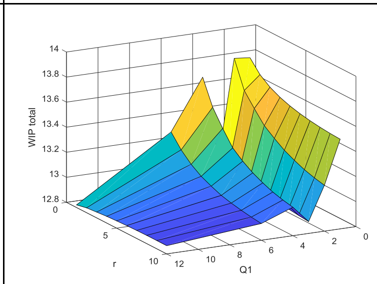
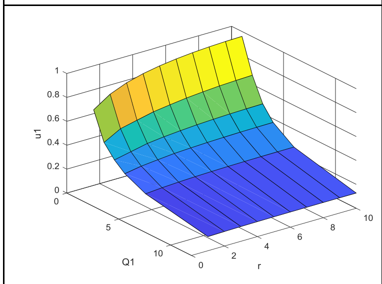
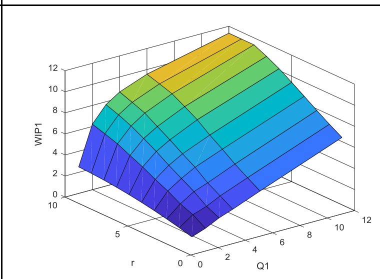
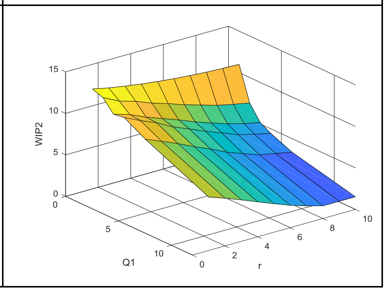
s=8, Q_2=12, dm=(0.6,0.4)		
OFR	SL	WIP total
		
Utilization1	WIP_1	WIP_2
		

Both r and Q_1 are important for customer satisfaction. Service level and Order Fill Rate can be increased by increasing r or Q_1. For both parameters the effect is more important when low parameter values are concerned. In most of the tested cases the effect of Q_1 is slightly more important than that of *r*.

Average inventory in the system tends to decrease with increasing r and Q_1, but in either case the effect is mild. In most scenarios Q_1 was more effective in decreasing

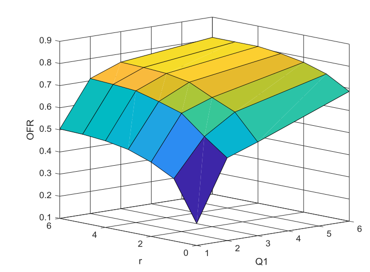
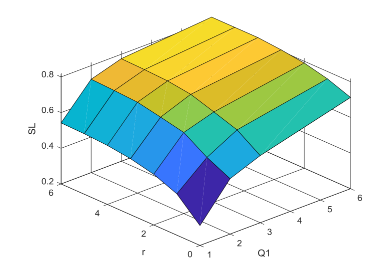
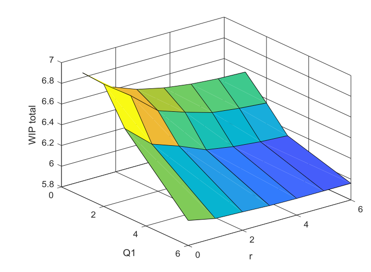
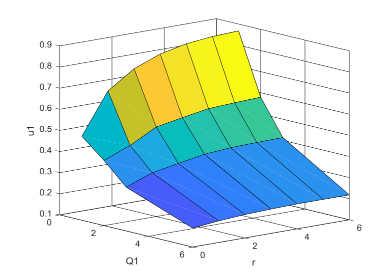
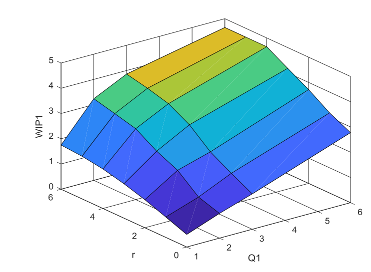
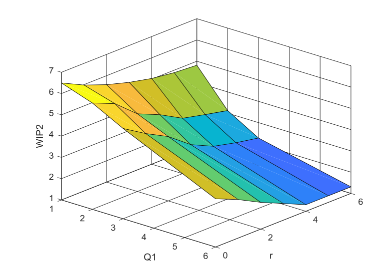
WIP total, and in some cases local minima where observed for intermediate Q_1 values.

s=8, Q_2=6, dm=(0.6, 0.4)		
OFR	SL	WIP total
		
Utilization1	WIP_1	WIP_2
		

s=8, Q_2=12, dm=(0.4,0.6)		
OFR	SL	WIP total
		
Utilization1	WIP_1	WIP_2
		

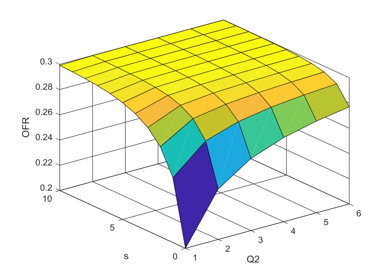
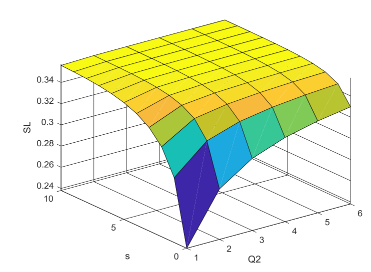
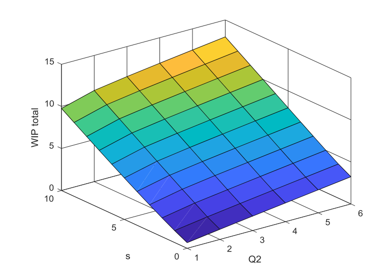
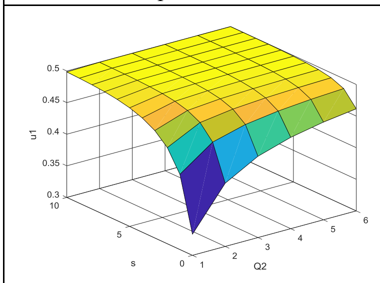
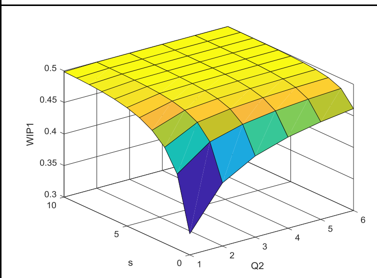
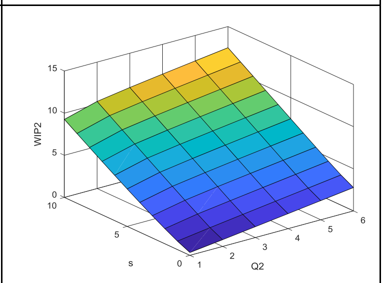
In general, a policy of high *r* and high Q_1 is desirable as it increases the performance of the system while actually decreasing the total inventory in the system. On the downside, increasing *r* or Q_1 will cause a significant increase in the average inventory of the retailer, Q_1 having a greater effect on WIP_1 than r.

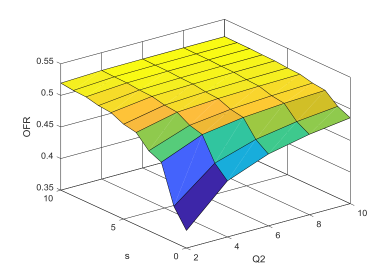
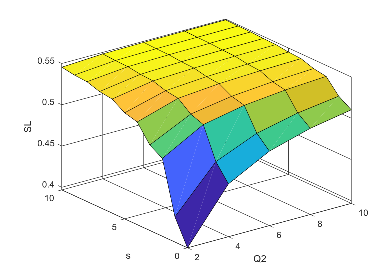
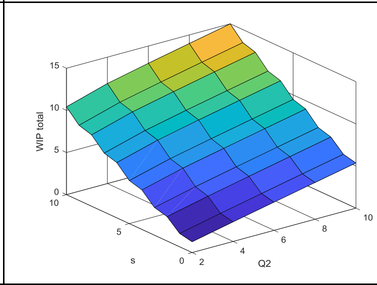
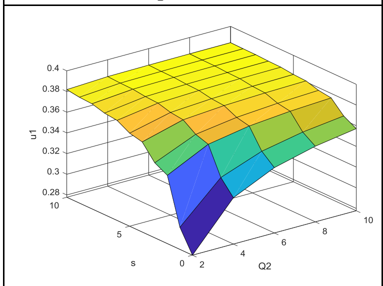
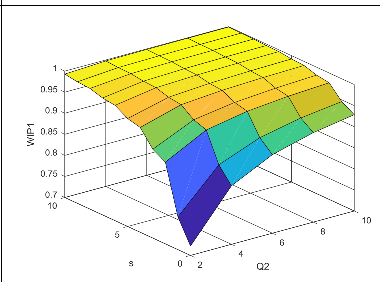
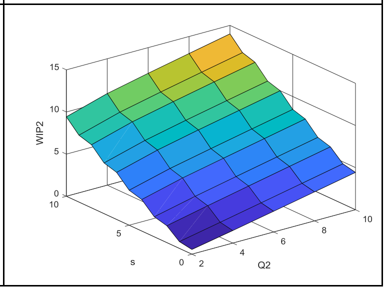
Slight changes in the external demand characteristics do not affect the behavior of the system.

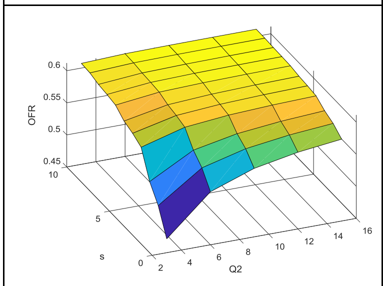
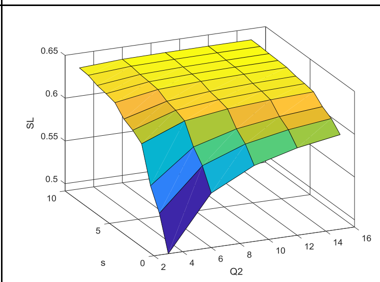
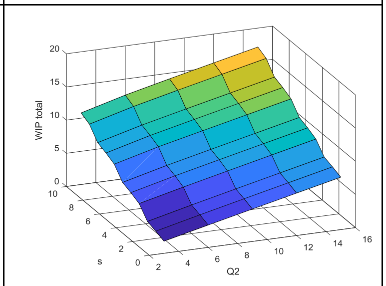
s=4, Q_2=6, dm=(0.4,0.6)		
OFR	SL	WIP total
		
Utilization1	WIP_1	WIP_2
		

Echelon inventory policy

We investigate the simultaneous change of vendor's inventory policy parameters s and Q_2.

r=4, Q_1=1, dm=(0.6,0.4)		
OFR	SL	WIP total
		
$Utilization_1$	WIP_1	WIP_2
		

<table>
<tr><td colspan="3">r=4, Q_1=2, dm=(0.6,0.4)</td></tr>
<tr><td>OFR</td><td>SL</td><td>WIP total</td></tr>
<tr><td>
</td><td>
</td><td>
</td></tr>
<tr><td>Utilization_1</td><td>WIP_1</td><td>WIP_2</td></tr>
<tr><td>
</td><td>
</td><td>
</td></tr>
</table>

<table>
<tr><td colspan="3">r=6, Q_1=3, dm=(0.6,0.4)</td></tr>
<tr><td>OFR</td><td>SL</td><td>WIP total</td></tr>
<tr><td>
</td><td>
</td><td>
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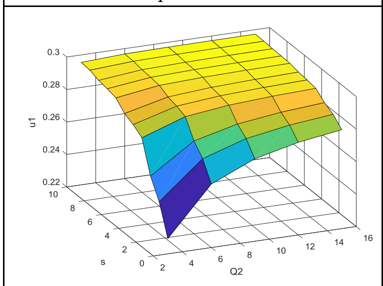
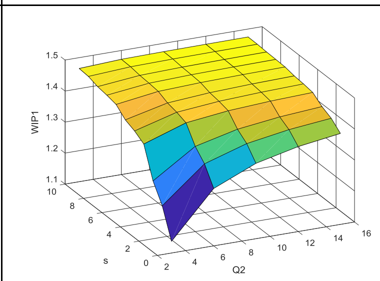
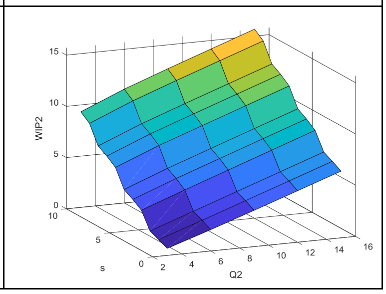
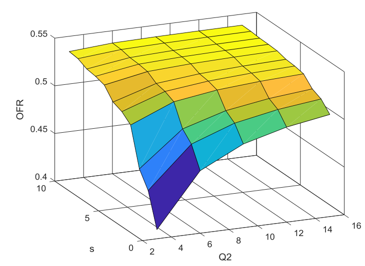
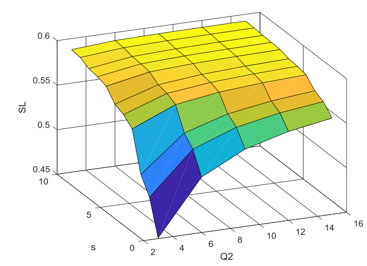
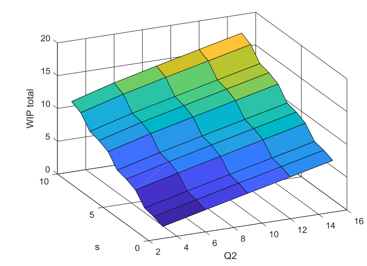
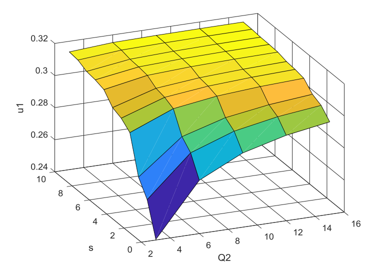
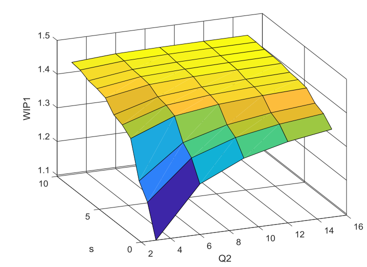
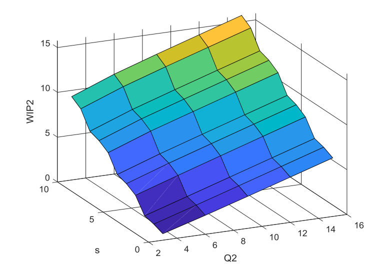
<tr><td>Utilization_1</td><td>WIP_1</td><td>WIP_2</td></tr>
<tr><td>
</td><td>
</td><td>
</td></tr>
</table>

Both echelon reorder point *s* and echelon replenishment order Q_2 are positively correlated with OFR and SL, but in general the effect of s is more important than that of Q_2. With respect to average inventories, the contribution of *s* is greater for both WIP_1 and WIP_2. As a result, *s* is also more important for the total average inventory in the system. As a conclusion, echelon inventory policy can be used to enhance customer satisfaction, but any improvement of service level must be compensated with higher inventories in the system. The effect of each parameter is affected by the value of the other. In general, the lower the value of the one parameter, the stronger the effect of the other.

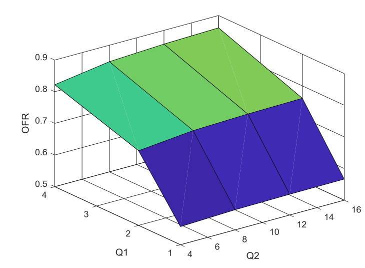
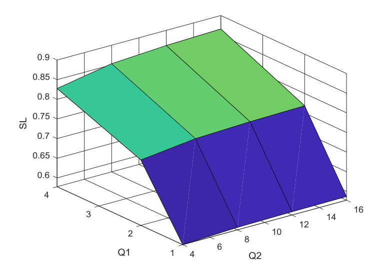
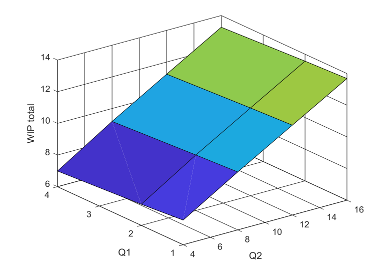
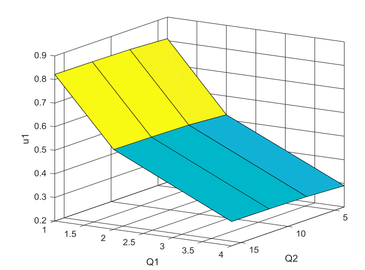
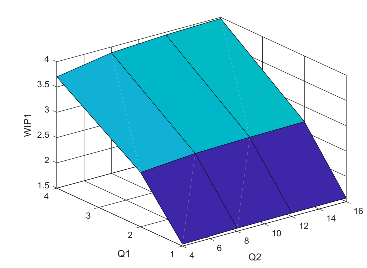
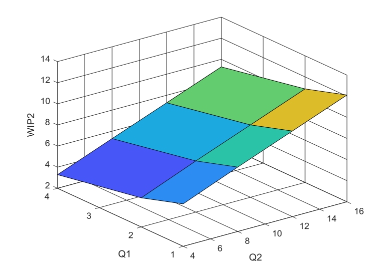
r=4, Q_1=3, dm=(0.4,0.6)		
OFR	SL	WIP total
		
$Utilization_1$	WIP_1	WIP_2
		

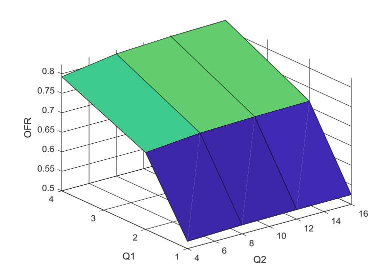
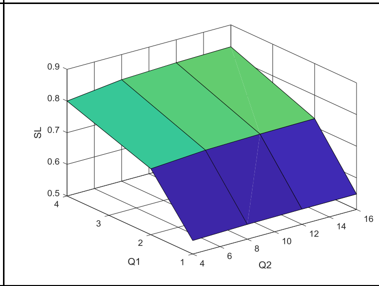
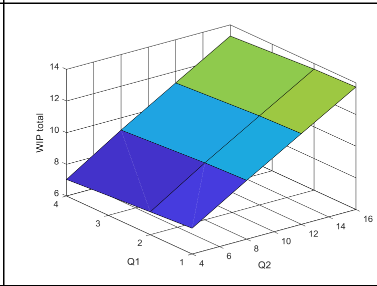
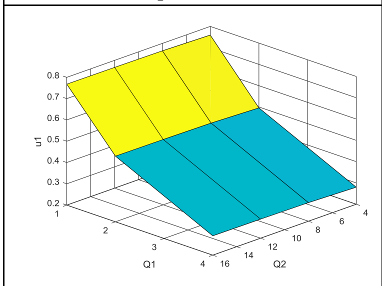
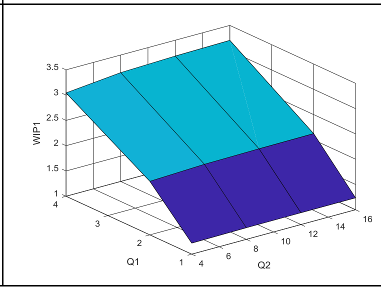
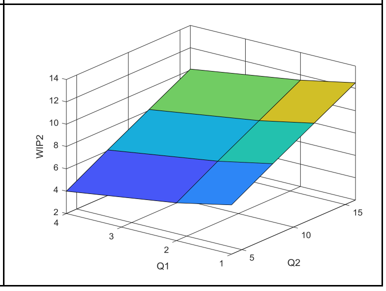
Slight changes in the characteristics of the external demand do not alter the behavior of the system.

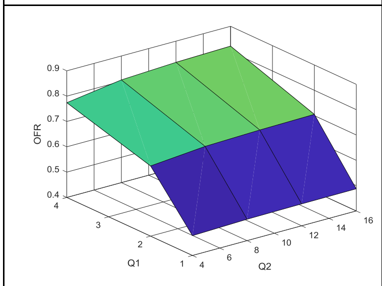
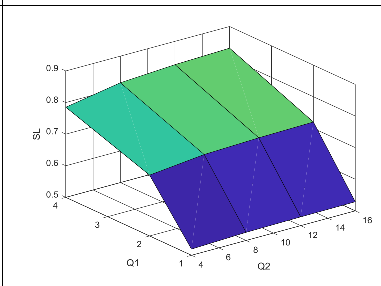
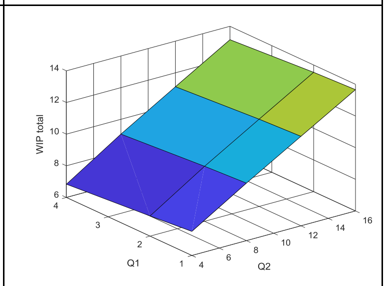
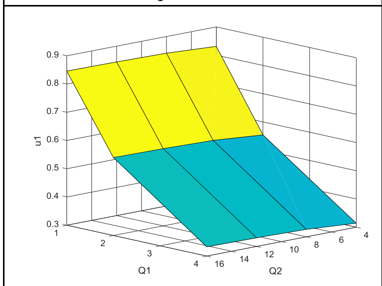
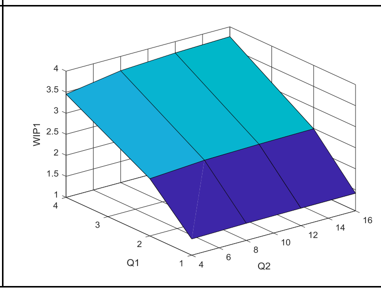
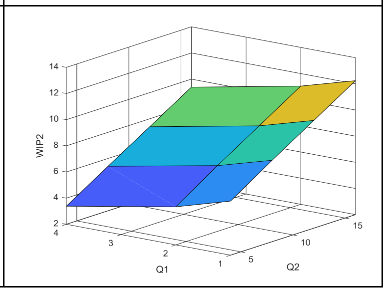
Replenishment orders

We investigate the effect of the simultaneous change of the replenishment orders towards the vendor (Q_2) and the retailer (Q_1) for given reorder points.

Q_2 is of some significance for OFR and SL only when the lowest possible values of Q_2 are concerned. In any case its effect is negligible in relation to that of Q_1.

r=3, s=6, dm=(0.6,0.4)		
OFR	SL	WIP total
		
$Utilization_1$	WIP_1	WIP_2
		

r=2, s=6, dm=(0.6,0.4)		
OFR	SL	WIP total
		
Utilization$_1$	WIP$_1$	WIP$_2$
		

r=3, s=6, dm=(0.4,0.6)		
OFR	SL	WIP total
		
Utilization$_1$	WIP$_1$	WIP$_2$
		

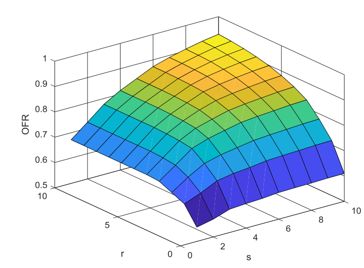
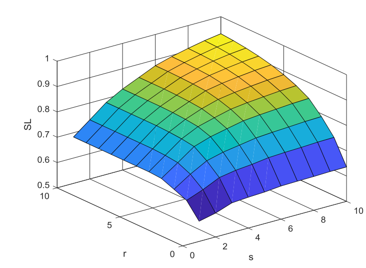
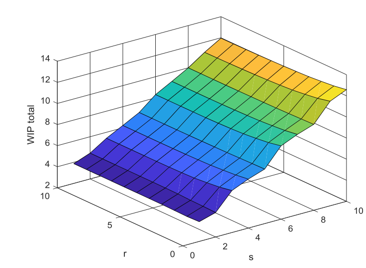
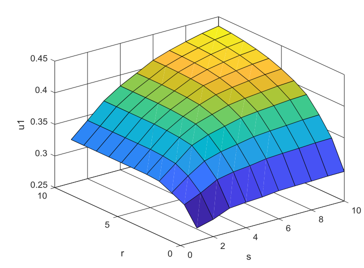
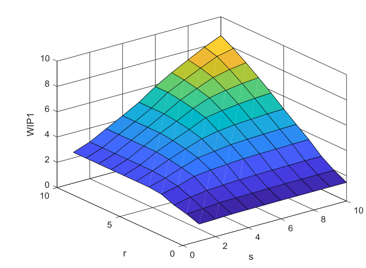
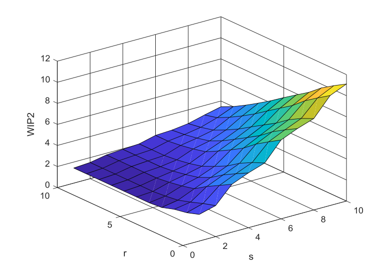
With respect to average inventories, average inventory at the retailer depends strongly, but in a concave fashion, on Q_1. In contrast, increasing Q_2 causes a slight increase of WIP_1, a change that reflects the higher availability of inventory at the retailer. Average inventory at the wholesaler rises significantly with Q_2, while on the other hand, raising Q_1 causes a decrease in WIP_2.

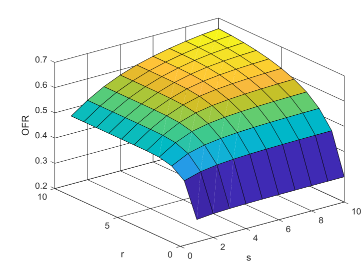
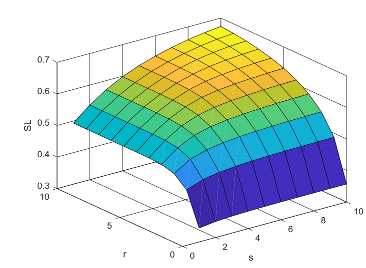
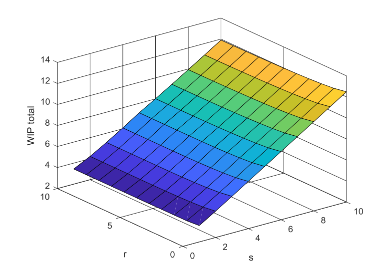
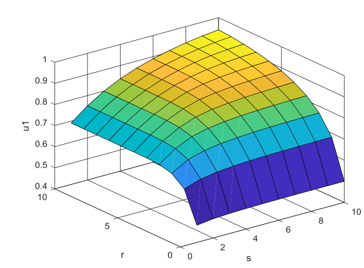
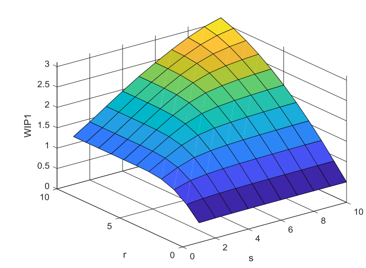
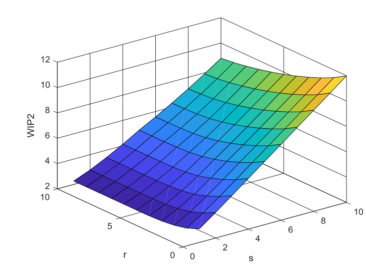
With regard to average total inventory, the increase of Q_1 may actually decrease WIP total, while Q_2 has an almost linear contribution. In conclusion, for an efficient inventory policy for the system, high Q_1 and low Q_2 values must be employed. It should be reminded that by the assumptions of the system Q_2 is a multiple of Q_1.

Small changes in the external demand characteristics do not alter the general behavior of the system.

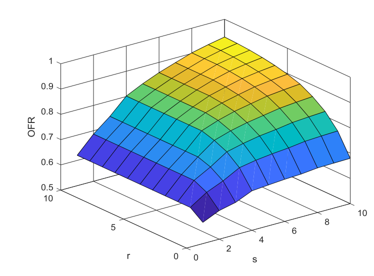
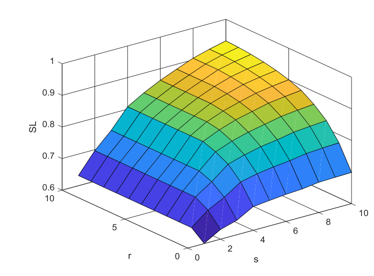
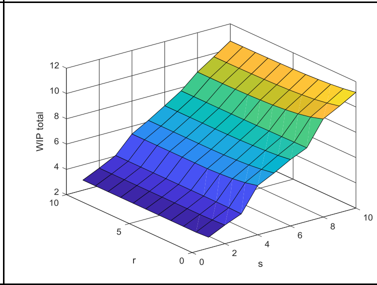
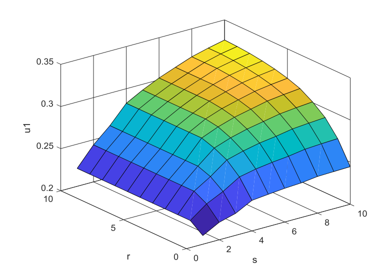
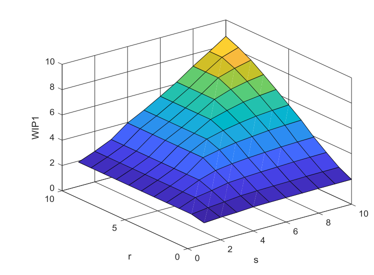
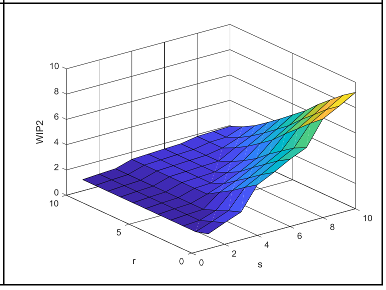
Reorder points

We investigate the simultaneous change of the echelon reorder point *s* and the retailer's reorder point *r* for given replenishment orders (Q_1, Q_2)

Q_1=3,Q_2=6, dm=(0.6,0.4)		
OFR	SL	WIP total
		
Utilization$_1$	WIP$_1$	WIP$_2$
		

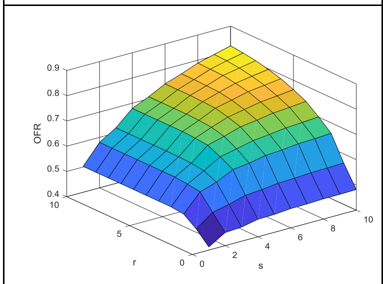
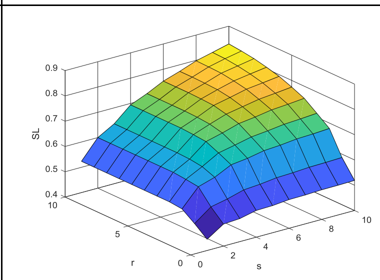
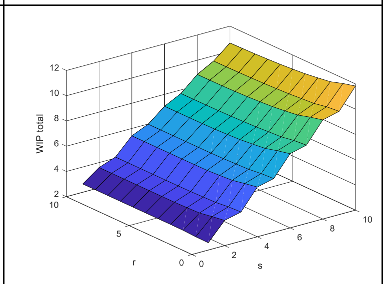
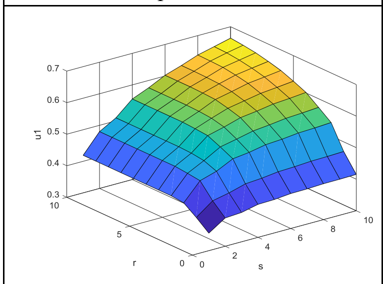
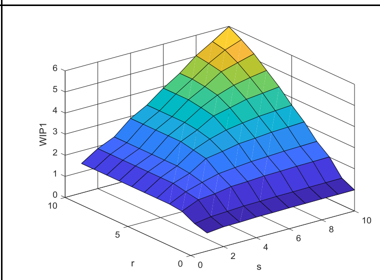
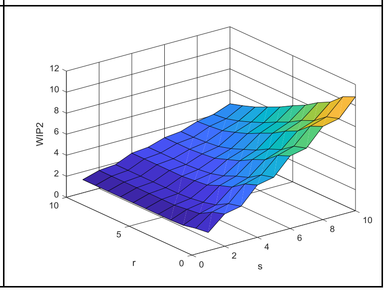
Q_1=1, Q_2=5, dm=(0.6,0.4)		
OFR	SL	WIP total
		
Utilization$_1$	WIP$_1$	WIP$_2$
		

The effect on Order Fill Rate and Service Level is similar for both *r* and *s*. The performance measures are increasing with increasing values of the reorder points, but the elasticities are decreasing.

Q_1=4, Q_2=4, dm=(0.6,0.4)		
OFR	SL	WIP total
		
Utilization$_1$	WIP$_1$	WIP$_2$
		

With respect to retailer's inventory, both *r* and *s* are positively correlated with WIP$_1$. Predictably, *r* is important only for higher *s* values. Conversely, *s* is more important for higher *r* values.

As far as WIP$_2$ is concerned, *s* is positively correlated with the average inventory at the wholesaler and its effect is more significant for lower *r* values. Increasing *r* causes WIP$_2$ to fall, with the decrease being more pronounced for high *s* values.

Q_1=2, Q_2=4, dm=(0.4,0.6)		
OFR	SL	WIP total
		
Utilization$_1$	WIP$_1$	WIP$_2$
		

For given values of the other decision variables, average total inventory is only slightly affected by the retailers reorder point and only for low *r* values. For higher *r* values the effect is almost negligible. On the other hand, echelon reorder point *s* is strongly positively correlated with WIP total.

From our analysis we conclude that in general a high value of *r* in relation to *s* is desirable for a system which achieves high customer satisfaction with minimal inventories.

Small changes in the external demand characteristics do not alter the general behavior of the system.

6.10.1.3 Effect of the demand characteristics on the performance measures

We investigate the effect of external demand variability on the performance measures. Different scenarios are explored where the average external demand remains constant, but its variance is changed.

$$Var = \sum_{i=1}^{md} dm(i) \cdot (i - E_x)^2$$

Some results are given below both in the form of a table and graphically.

Performance measures as a function of external demand variance. s=8, Q_2=8, r=4, Q_1=4, E_x=1.8, μ_{11}=1, μ_{12}=0.2, d_{12}=0.2, μ_{21}=1, μ_{22}=0.2, d_{22}=0.2, λ=0.5

dm(1)	dm(2)	dm(3)	dm(4)	dm(5)	dm(6)	Var	OFR	SL	WIPtotal	WIP_1	WIP_2	$util_1$
0.2	0.8	0	0	0	0	0.16	0.819	0.832	11.189	4.493	6.696	0.374
0.45	0.3	0.25	0	0	0	0.66	0.811	0.820	11.184	4.474	6.710	0.369
0.39	0.45	0.15	0	0	0.01	0.66	0.811	0.820	11.187	4.479	6.708	0.369
0.48	0.28	0.2	0.04	0	0	0.8	0.809	0.818	11.197	4.485	6.712	0.368
0.6	0	0.4	0	0	0	0.96	0.808	0.815	11.208	4.498	6.710	0.367
0.65	0.1	0.1	0.1	0.05	0	1.56	0.802	0.803	11.251	4.524	6.727	0.361
0.7	0.1	0	0.1	0.1	0	1.96	0.796	0.794	11.273	4.534	6.739	0.357
0.7	0.05	0.15	0.02	0.01	0.07	2.1	0.793	0.789	11.286	4.556	6.730	0.355
0.82	0	0	0	0.1	0.08	2.96	0.780	0.771	11.341	4.577	6.764	0.347
0.84	0	0	0	0	0.16	3.36	0.772	0.761	11.379	4.610	6.769	0.343

Performance measures as a function of external demand variance. s=4, Q_2=4, r=4, Q_1=2, E_x=1.8, μ_{11}=1, μ_{12}=0.2, d_{12}=0.2, μ_{21}=1, μ_{22}=0.2, d_{22}=0.2, λ=0.5

dm(1)	dm(2)	dm(3)	dm(4)	dm(5)	dm(6)	Var	OFR	SL	WIPtotal	WIP_1	WIP_2	$util_1$
0.2	0.8	0	0	0	0	0.16	0.624	0.644	5.422	2.477	2.944	0.579
0.45	0.3	0.25	0	0	0	0.66	0.605	0.628	5.385	2.486	2.898	0.566
0.39	0.45	0.15	0	0	0.01	0.66	0.611	0.630	5.390	2.493	2.897	0.567
0.48	0.28	0.2	0.04	0	0	0.8	0.607	0.626	5.390	2.500	2.890	0.563
0.6	0	0.4	0	0	0	0.96	0.599	0.621	5.393	2.509	2.884	0.559
0.65	0.1	0.1	0.1	0.05	0	1.56	0.611	0.611	5.410	2.570	2.839	0.550
0.7	0.1	0	0.1	0.1	0	1.96	0.615	0.603	5.418	2.609	2.809	0.543
0.7	0.05	0.15	0.02	0.01	0.07	2.1	0.614	0.601	5.421	2.629	2.792	0.541
0.82	0	0	0	0.1	0.08	2.96	0.614	0.584	5.440	2.715	2.724	0.526
0.84	0	0	0	0	0.16	3.36	0.615	0.576	5.454	2.768	2.686	0.519

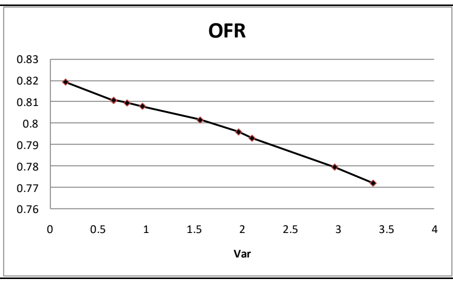
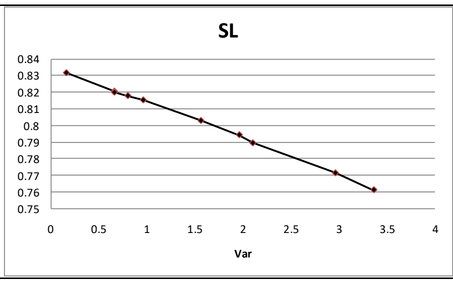
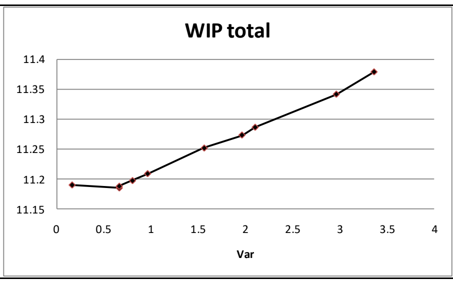
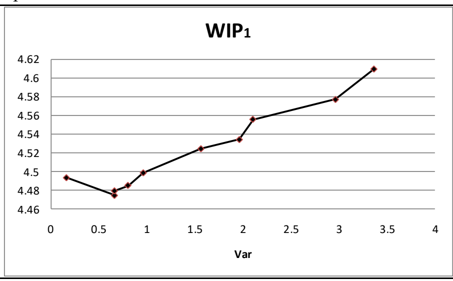
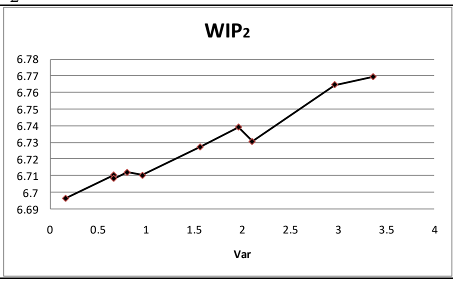
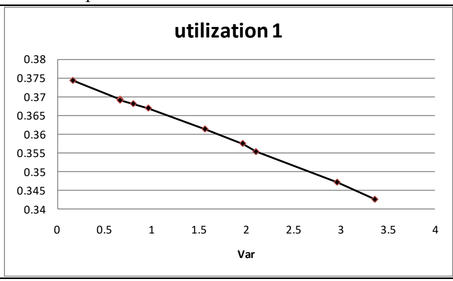
With regard to Order Fill Rate (the percentage of external orders that are fully met), the exact values of the external demand parameters are more important than demand variance, and no safe prediction can be made. On the other hand, service level (the percentage of external demand in terms of product units that is met from inventory on hand) consistently decreases with increasing variance.

Average inventory at the retailer also tends to increase with increased demand variance. This is somewhat paradoxical, as service level decreases. Demand variability disrupts the system in such a way that inventory at the retailer increases at the same time when customer satisfaction decreases.

Average inventory at the wholesaler is less affected by external demand variance, and the effect depends on the specific values of the system parameters. For average total inventory, changes in WIP_1 dominate, so WIP total also tends to increase with increased variance.

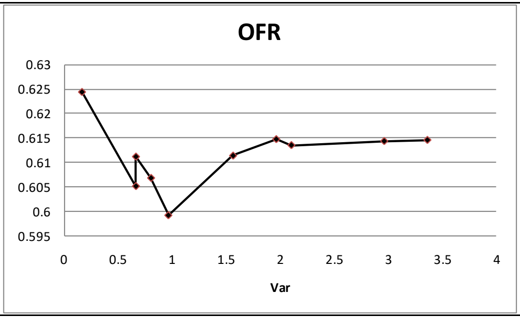
The utilization of transportation resources is directly related to service level and predictably exhibits the same behavior.

In conclusion, increasing the variance of external demand is detrimental to the performance of the system. The effects are more pronounced to the retailer and include both accumulation of inventory and lower service levels and ultimately lower system output.

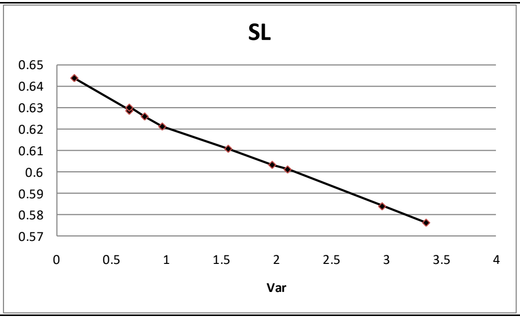
The effect of external demand variance on the performance measures. s=8, Q_2=8 ,r=4, Q_1=4, Average demand E_x=1.8, μ_{11}=1, μ_{12}=0.2, d_{12}=0.2, μ_{21}=1, μ_{22}=0.2, d_{22}=0.2, λ=0.5	
OFR	
	
WIP total	WIP_1
	
WIP_2	$Utilization_1$
	

The effect of external demand variance on the performance measures. s=4, Q_2=4 ,r=4, Q_1=2, Average demand Ex=1.8, μ_{11}=1, μ_{12}=0.2, d_{12}=0.2, μ_{21}=1, μ_{22}=0.2, d_{22}=0.2, λ=0.5

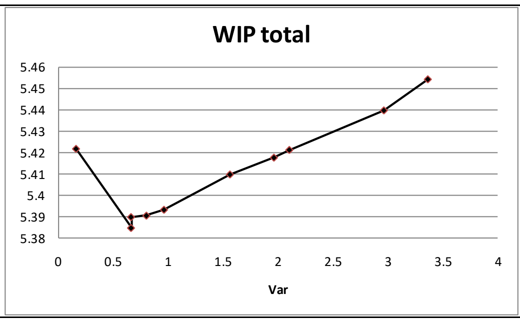
OFR



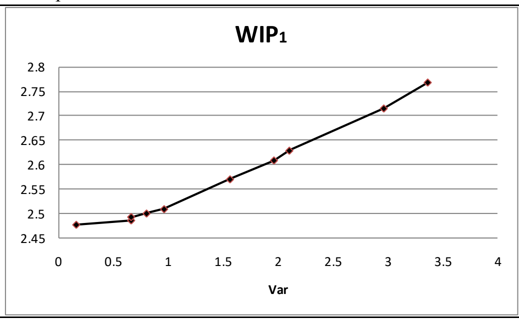
SL



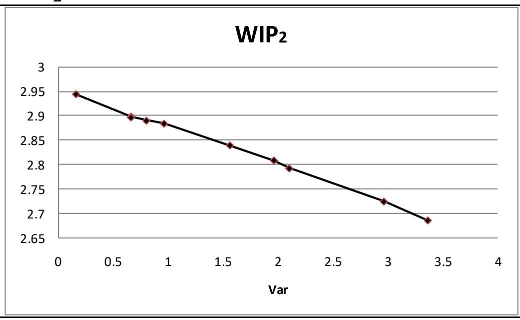
WIP total



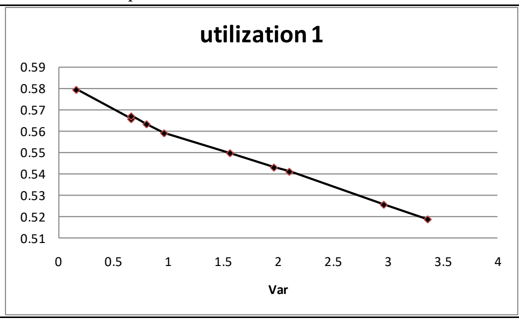
WIP_1



WIP_2



$Utilization_1$

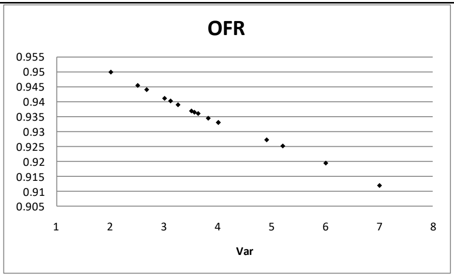


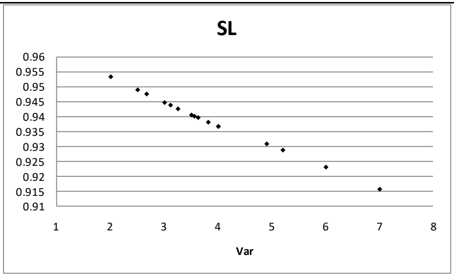
6.10.1.4 Effect of the lead time variance on the performance measures

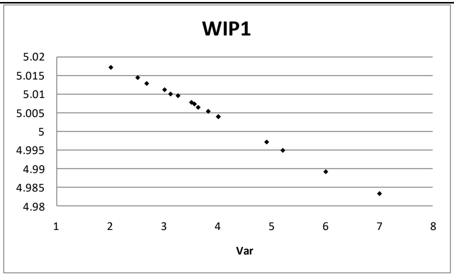
We use the parameters of the Coxian distribution to investigate the effect of the retailer's lead time variance on the performance of the system. For our analysis we kept constant the transportation time parameters for transportation from the manufacturer to the vendor, and changed the time parameters for transportation from the vendor to the retailer. The average transportation time from the vendor to the retailer (T_1) was kept constant, but its variance changed according to the Coxian-2 parameters:

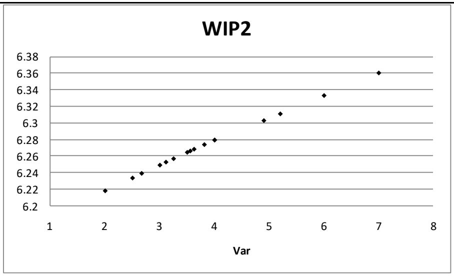
$$Var[T_1] = \frac{\mu_{12}^2 + d_{12}\mu_{11}^2(2-d_{12})}{\mu_{11}^2\mu_{12}^2}$$

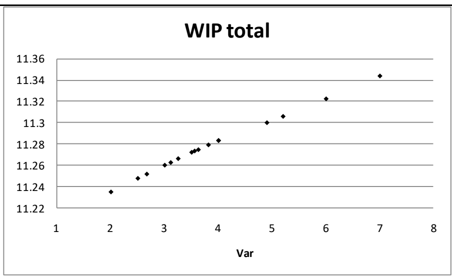
The effect of retailer's Lead time Variance on performance measures. s=8, Q_2=8, r=4, Q_1=4, dm=(0.6, 0.4), μ_{21}=2/3, μ_{22}=2/3, d_{22}=1/3, λ=0.5, T_1=2

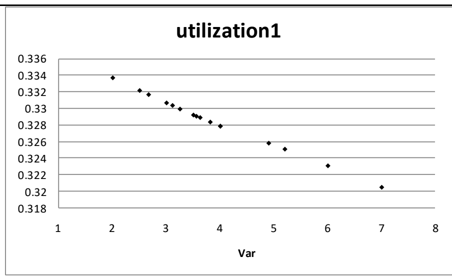








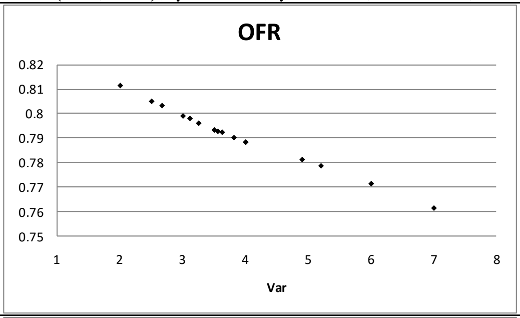


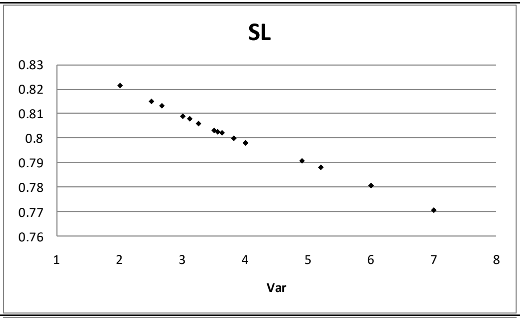


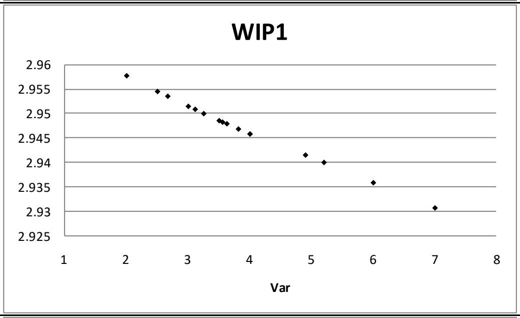
The increase in the Retailer's lead time variance is detrimental to the performance of the system. OFR and SL decrease. The average inventory at the retailer also tends to decrease, but in general WIP_1 is the least affected performance measure and for many scenarios, in absolute values the changes are negligible. The effect is stronger for WIP_2 which increases as the variance increases. As a result, total average inventory also increases with increasing variance. In most cases the changes in all examined performance measures can be captured quite accurately with simple linear relations (in most cases an R^2 value above 0.99 is observed for linear fit). As was the case for external demand variance, the increase of lead time variance causes customer satisfaction to fall and at the same time more inventory is accumulated in the system.

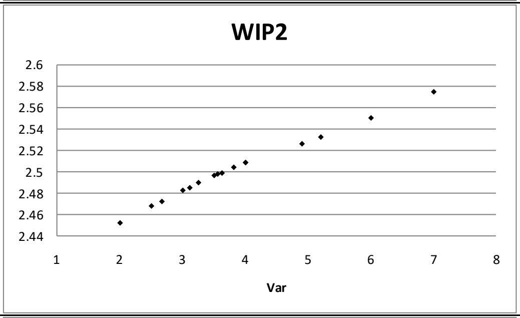
Some of the corresponding numerical data are given in appendix 6.13

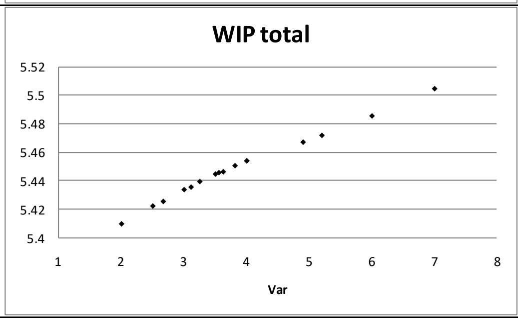
The effect of retailer's Lead time Variance on performance measures. s=4, Q_2=4, r=4, Q_1=2, dm=(0.6, 0.4), μ_{21}=2/3, μ_{22}=2/3, d_{22}=1/3, λ=0.5, T_1=2

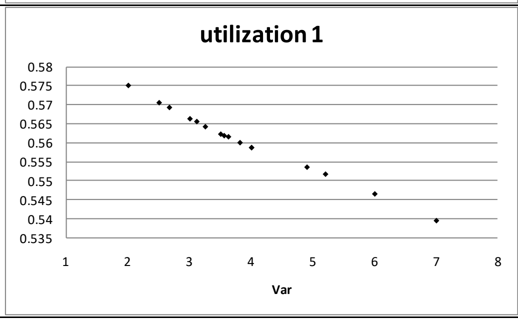












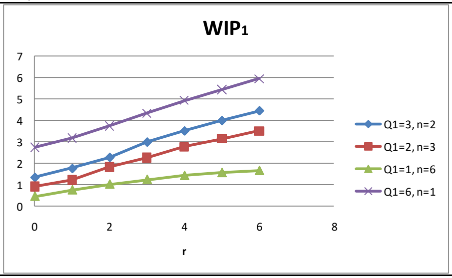
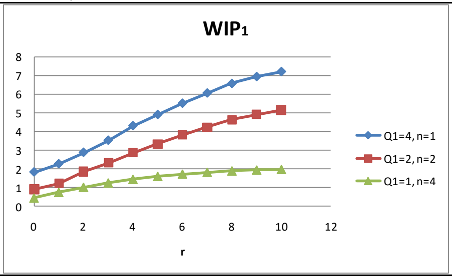
6.10.2 Supply Constrained systems

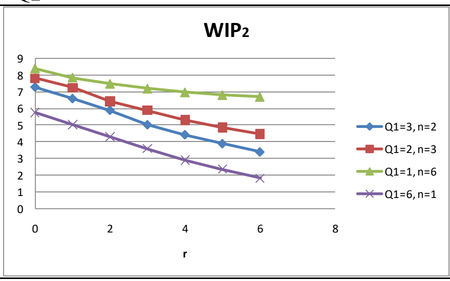
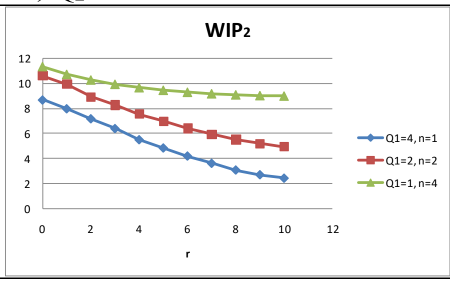
In supply constrained systems the average transportation time from the manufacturer to the vendor is longer than the average transportation time from the vendor to the retailer, which in its turn is longer than the average time between successive arrivals of external customers: $\frac{1}{\mu_{21}} + d_{22} \cdot \frac{1}{\mu_{22}} > \frac{1}{\mu_{11}} + d_{12} \cdot \frac{1}{\mu_{12}} > \frac{1}{\lambda}$

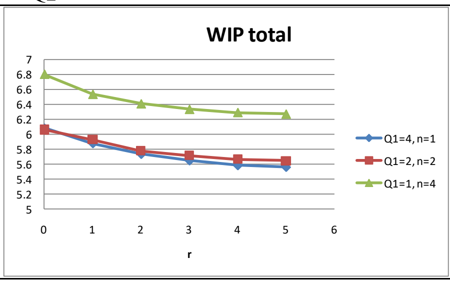
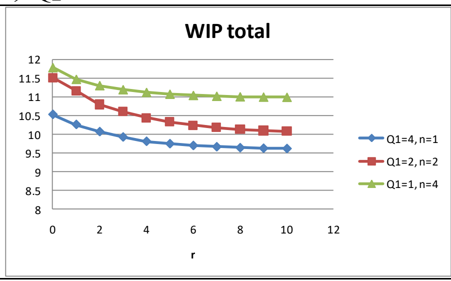
For the examples presented below we use μ_{11}=1, μ_{12}=0.2, d_{12}=0.3, μ_{21}=1, μ_{22}=0.2, d_{22}= 0.4, λ=0.5 and $dm = (0.6, 0.4)$ (there is 0.6 probability that an external customer will ask 1 product unit and 0.4 probability that he will ask 2 units).

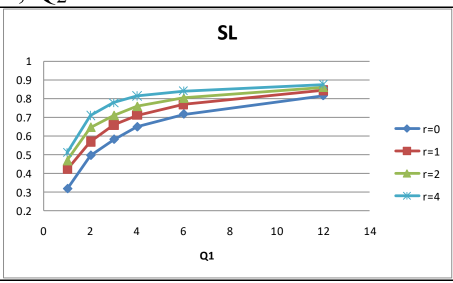
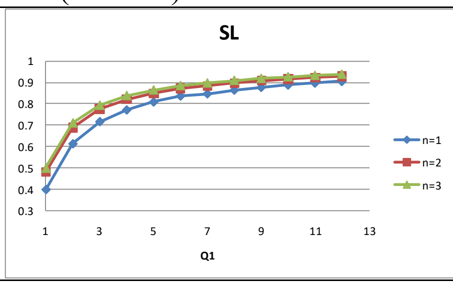
6.10.2.1 The effect of the decision variables on the performance measures

In comparison with balanced systems, for given external demand characteristics, supply constrained systems exhibit slightly lower Order Fill rates and Service Levels, as well as lower average inventories at the retailer. For average inventory at the wholesaler, as well as for average total inventory, both higher and lower values were observed, depending on the specific values of the decision variables for the scenario under consideration. In any case the behavior of the system (the shape of the respective curves) for changing decision variables is the same for both balanced and supply constrained systems.

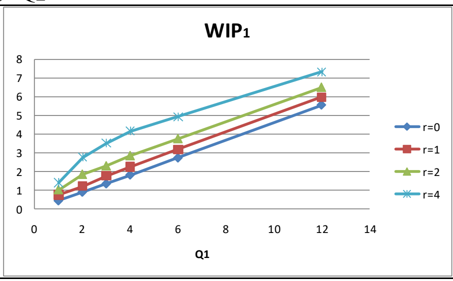
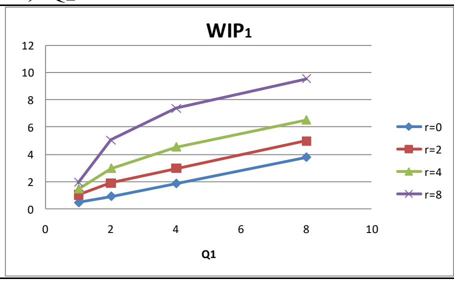
<table>
<tr><td colspan="2">Retailer’s WIP as a function of reorder point r</td></tr>
<tr><td>s=6, Q_2=6</td><td>s=10 , Q_2=4</td></tr>
<tr><td>
</td><td>
</td></tr>
</table>

<table>
<tr><td colspan="2">Wholesaler’s WIP as a function of reorder point r</td></tr>
<tr><td>s=6, Q_2=6</td><td>s=10 , Q_2=4</td></tr>
<tr><td>
</td><td>
</td></tr>
</table>

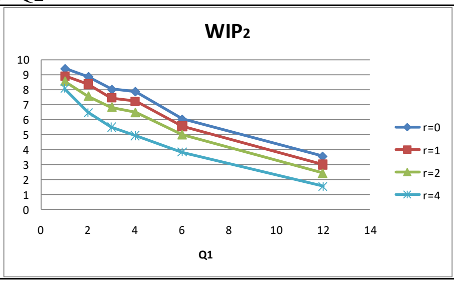
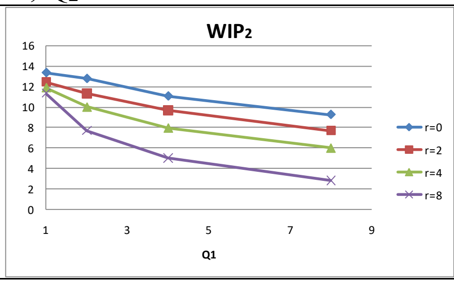
<table>
<tr><td colspan="2">Total average inventory as function of reorder point r</td></tr>
<tr><td>s=5 Q_2=4</td><td>s=10 , Q_2=4</td></tr>
<tr><td>
</td><td>
</td></tr>
</table>

<table>
<tr><td colspan="2">Service level as a function of replenishment order Q_1</td></tr>
<tr><td>s=4, Q_2=12</td><td>s=6, r=4 (n=const)</td></tr>
<tr><td>
</td><td>
</td></tr>
</table>

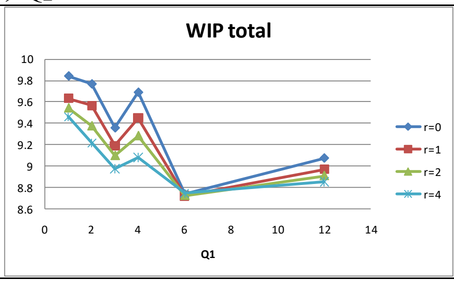
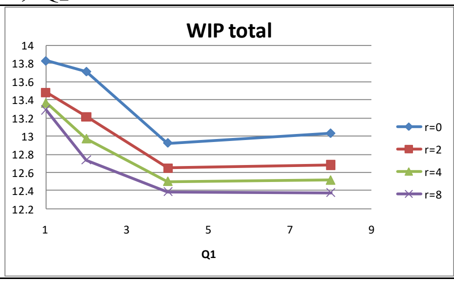
Average inventory at the retailer as a function of replenishment order Q_1

$s=4$, $Q_2=12$	$s=10$, $Q_2=8$
	

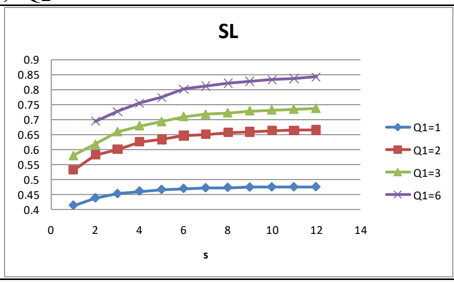
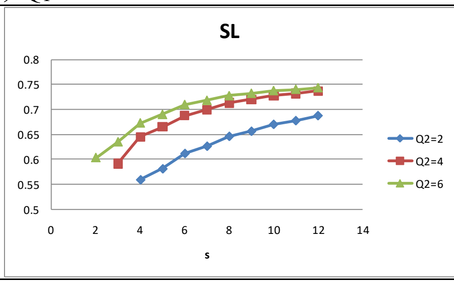
Average inventory at the vendor as a function of replenishment order Q_1

$s=4$, $Q_2=12$	$s=10$, $Q_2=8$
	

Average total inventory as a function of replenishment order Q_1

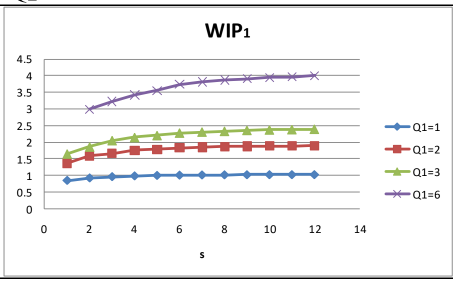
$s=4$, $Q_2=12$	$s=10$, $Q_2=8$
	

Service Level as a function of echelon reorder point s

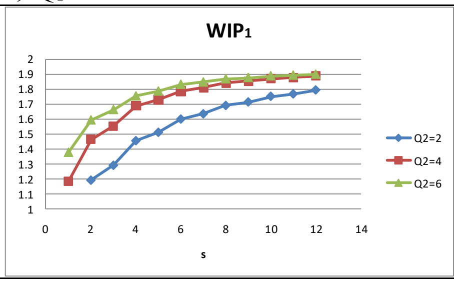
$r=2$, $Q_2=6$	$r=4$, $Q_1=2$
	

Average inventory at the retailer as a function of echelon reorder point *s*

r=2, Q_2=6

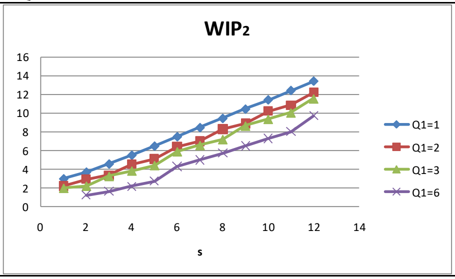


r=2, Q_1=2

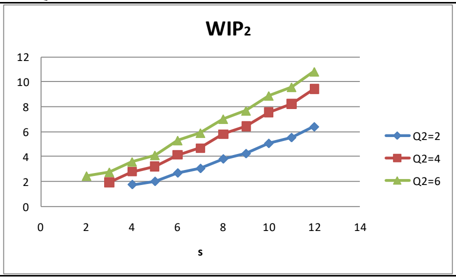


Average inventory at the wholesaler as a function of echelon reorder point *s*

r=2, Q_2=6

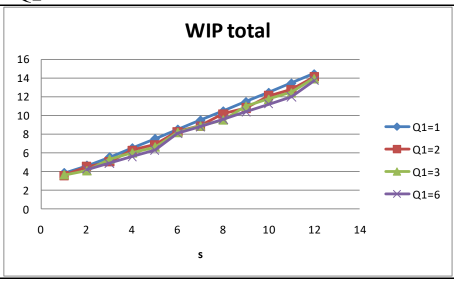


r=4, Q_1=2



Average total inventory as a function of echelon reorder point *s*

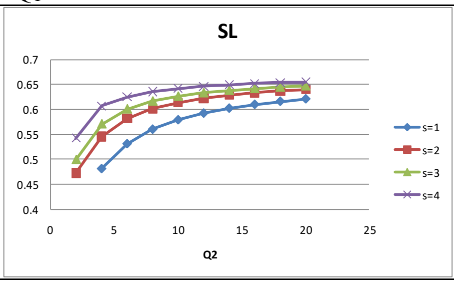
r=2, Q_2=6



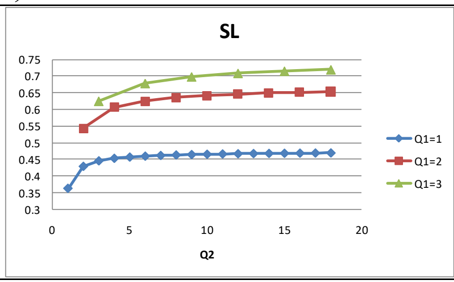
r=2, Q_1=2

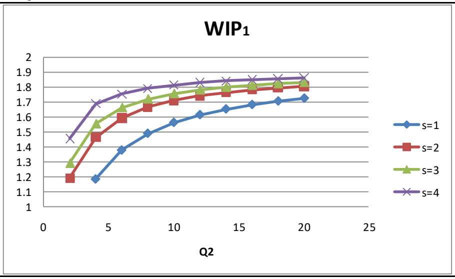
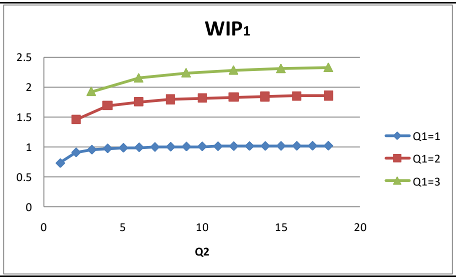
Service level as a function of replenishment order Q_2 (Q_1=const)

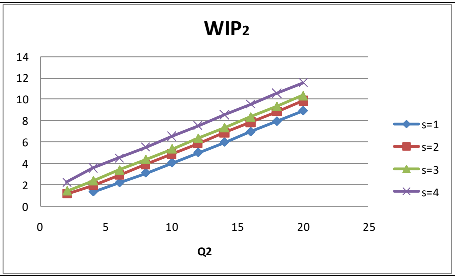
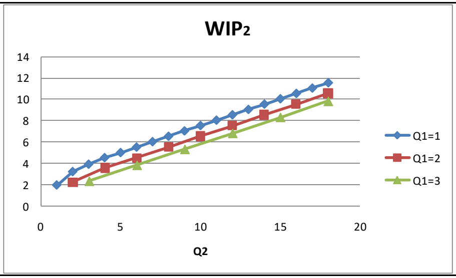
r=2, Q_1=2

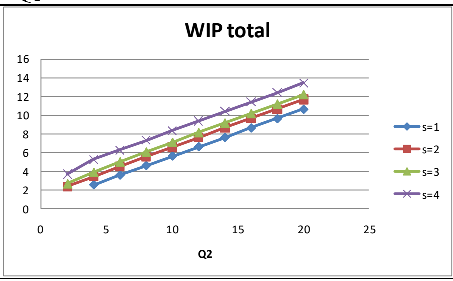
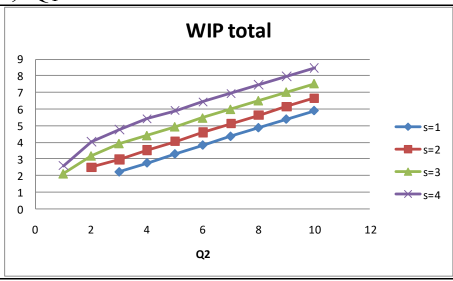


s=4, r=2



Average inventory at the retailer as a function of replenishment order Q_2	
$r=2$, $Q_1=2$	$s=4$, $r=2$
	

Average inventory at the wholesaler as a function of replenishment order Q_2 (Q_1=const)	
$r=2$, $Q_1=2$	$s=4$, $r=2$
	

Total average inventory as a function of replenishment order Q_2	
$r=2$, $Q_1=2$	$r=3$, $Q_1=1$
	

In general, the behavior of the system is similar to that for balanced systems also when the combined effect of the decision variables is investigated. The conclusions outlined in 6.10.1.2 for balanced systems also hold for supply constrained systems.

6.10.2.2 Effect of demand characteristics on the performance measures

Performance measures as a function of external demand variance. $s=8$, $Q_2=8$, $r=4$, $Q_1=4$, $E_x=1.8$, $\mu_{11}=1$, $\mu_{12}=0.2$, $d_{12}=0.3$, $\mu_{21}=1$, $\mu_{22}=0.2$, $d_{22}=0.4$, $\lambda=0.5$

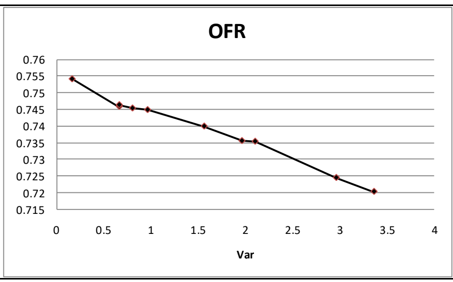
dm(1)	dm(2)	dm(3)	dm(4)	dm(5)	dm(6)	Var	OFR	SL	WIPtotal	WIP_1	WIP_2	$util_1$
0.2	0.8	0	0	0	0	0.16	0.754	0.768	10.637	3.998	6.639	0.432
0.45	0.3	0.25	0	0	0	0.66	0.746	0.757	10.639	3.989	6.650	0.426
0.39	0.45	0.15	0	0	0.01	0.66	0.746	0.757	10.641	3.993	6.647	0.426
0.48	0.28	0.2	0.04	0	0	0.8	0.745	0.755	10.653	4.002	6.651	0.425
0.6	0	0.4	0	0	0	0.96	0.745	0.752	10.667	4.018	6.649	0.423
0.65	0.1	0.1	0.1	0.05	0	1.56	0.740	0.741	10.717	4.055	6.662	0.417
0.7	0.1	0	0.1	0.1	0	1.96	0.736	0.733	10.744	4.073	6.671	0.412
0.7	0.05	0.15	0.02	0.01	0.07	2.1	0.735	0.730	10.755	4.094	6.661	0.410
0.82	0	0	0	0.1	0.08	2.96	0.724	0.713	10.823	4.132	6.691	0.401
0.84	0	0	0	0	0.16	3.36	0.720	0.704	10.863	4.168	6.694	0.396

Performance measures as a function of external demand variance. s=4, Q_2=4, r=4, Q_1=2, E_x=1.8, μ_{11}=1, μ_{12}=0.2, d_{12}=0.3, μ_{21}=1, μ_{22}=0.2, d_{22}= 0.4, λ=0.5												
dm(1)	dm(2)	dm(3)	dm(4)	dm(5)	dm(6)	Var	OFR	SL	WIPtotal	WIP_1	WIP_2	$util_1$
0.2	0.8	0	0	0	0	0.16	0.528	0.548	5.056	1.959	3.097	0.616
0.45	0.3	0.25	0	0	0	0.66	0.511	0.536	5.028	1.991	3.037	0.603
0.39	0.45	0.15	0	0	0.01	0.66	0.517	0.537	5.033	1.993	3.040	0.604
0.48	0.28	0.2	0.04	0	0	0.8	0.514	0.534	5.034	2.006	3.028	0.601
0.6	0	0.4	0	0	0	0.96	0.507	0.530	5.038	2.022	3.016	0.596
0.65	0.1	0.1	0.1	0.05	0	1.56	0.525	0.523	5.056	2.084	2.972	0.588
0.7	0.1	0	0.1	0.1	0	1.96	0.532	0.518	5.065	2.125	2.940	0.582
0.7	0.05	0.15	0.02	0.01	0.07	2.1	0.532	0.517	5.068	2.139	2.929	0.581
0.82	0	0	0	0.1	0.08	2.96	0.542	0.504	5.090	2.233	2.857	0.567
0.84	0	0	0	0	0.16	3.36	0.548	0.499	5.104	2.281	2.823	0.562

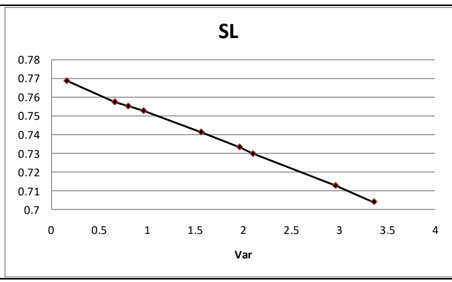
We investigate the effect of external demand variability on the performance measures. Different scenarios are explored where average external demand remains constant, but its variance *Var* is changed. Some results are given both in the form of a table and graphically.

The effect of external demand variance on the performance measures. s=8, Q_2=8 ,r=4, Q_1=4, Average demand E_x=1.8, μ_{11}=1, μ_{12}=0.2, d_{12}=0.3, μ_{21}=1, μ_{22}=0.2, d_{22}= 0.4, λ=0.5

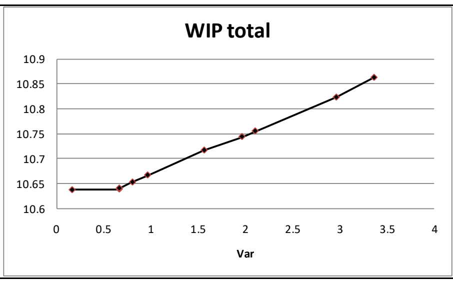
OFR



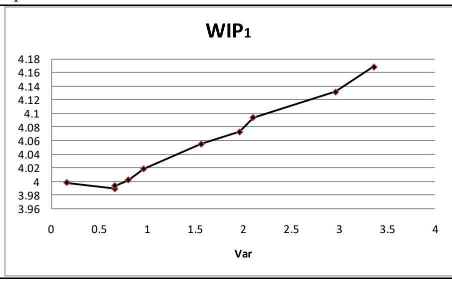
SL



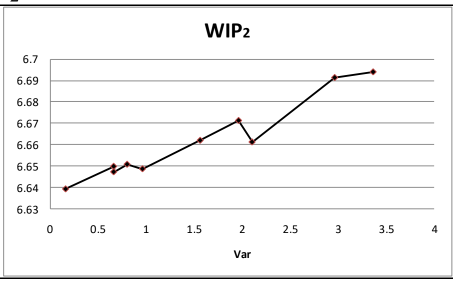
WIP total



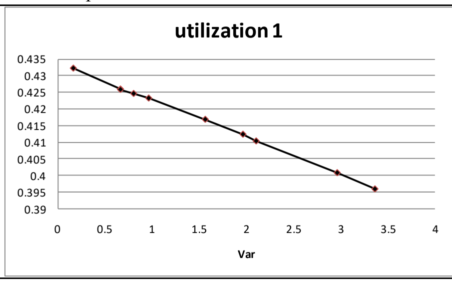
WIP_1



WIP_2



$Utilization_1$



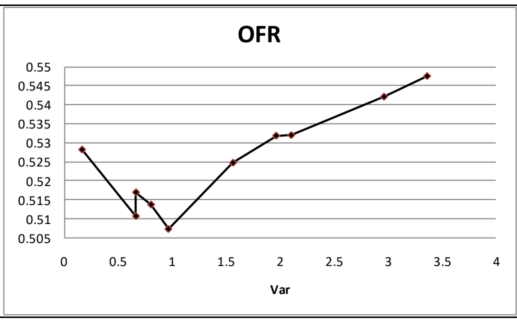
With regard to Order Fill Rate (the percentage of external orders that are fully met), the exact values of the external demand parameters are more important than demand variance, and no safe prediction can be made. For low r and Q_1 values increasing variance may cause an increase in Order Fill Rate, while for higher retailer's policy parameters OFR tends to decrease. On the other hand, service level (the percentage of external demand in terms of product units that is met from inventory on hand) consistently decreases with increasing variance.

Average inventory at the retailer also tends to increase with increasing demand variance, which is somewhat paradoxical as service level also decreases. The effect on average inventory at the wholesaler is less straightforward and depends on the specific values of the system parameters. For the average total inventory, changes in WIP_1 dominate and WIP total tends to increase with increased variance.

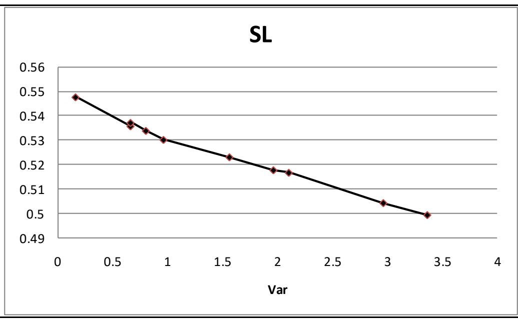
In conclusion, increasing the variance of external demand is detrimental to the performance of the system. The effects are more pronounced at the retailer and include both accumulation of inventory and lower service level and ultimately lower system output. The behavior of the system is similar to that observed in the case of balanced systems.

The effect of external demand variance on the performance measures. s=4, Q_2=4, r=4, Q_1=2, Average demand Ex=1.8, μ_{11}=1, μ_{12}=0.2, d_{12}=0.3, μ_{21}=1, μ_{22}=0.2, d_{22}= 0.4, λ=0.5

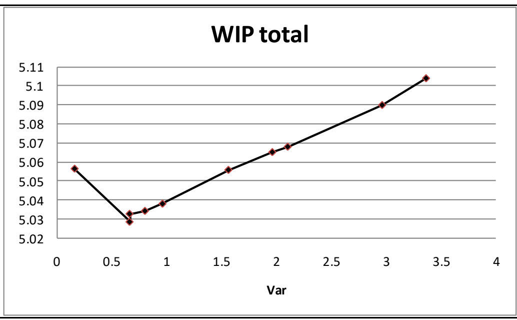
OFR



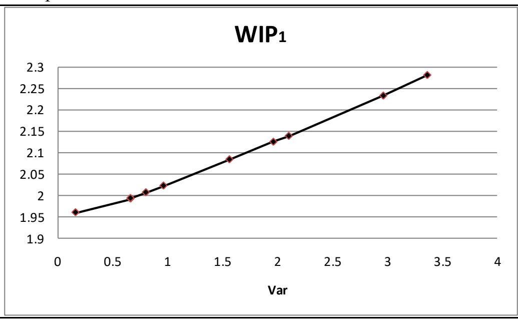
SL



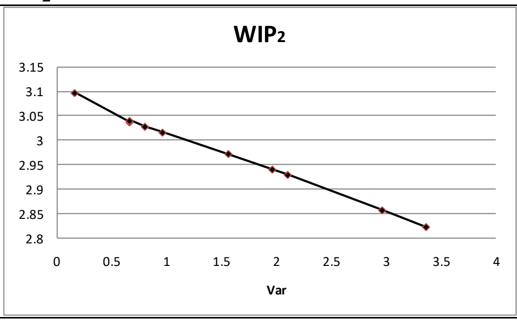
WIP total



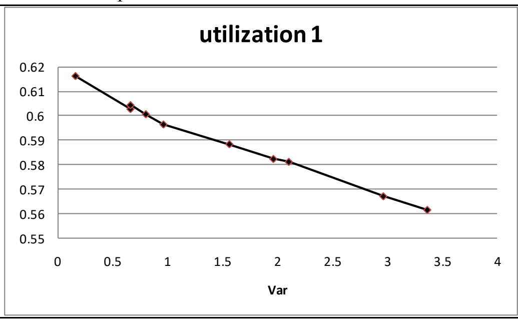
WIP_1



WIP_2



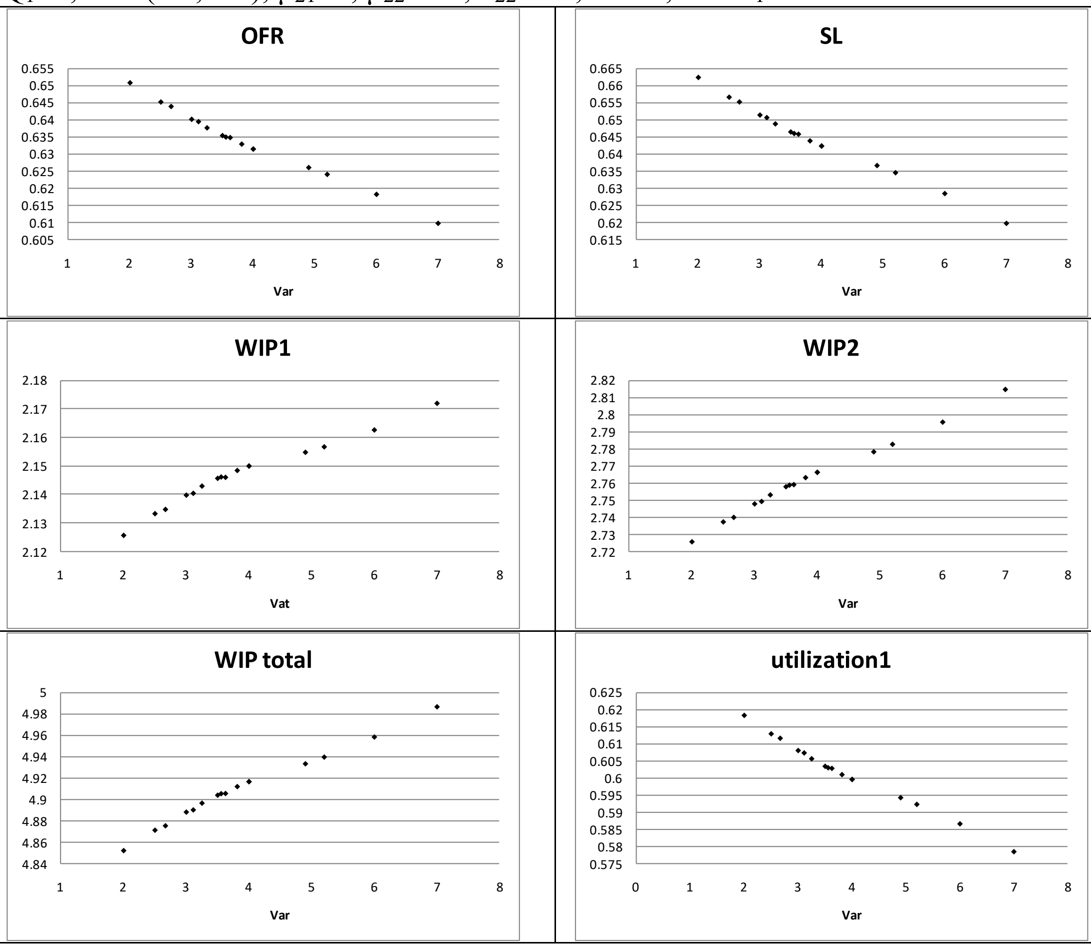
$Utilization_1$



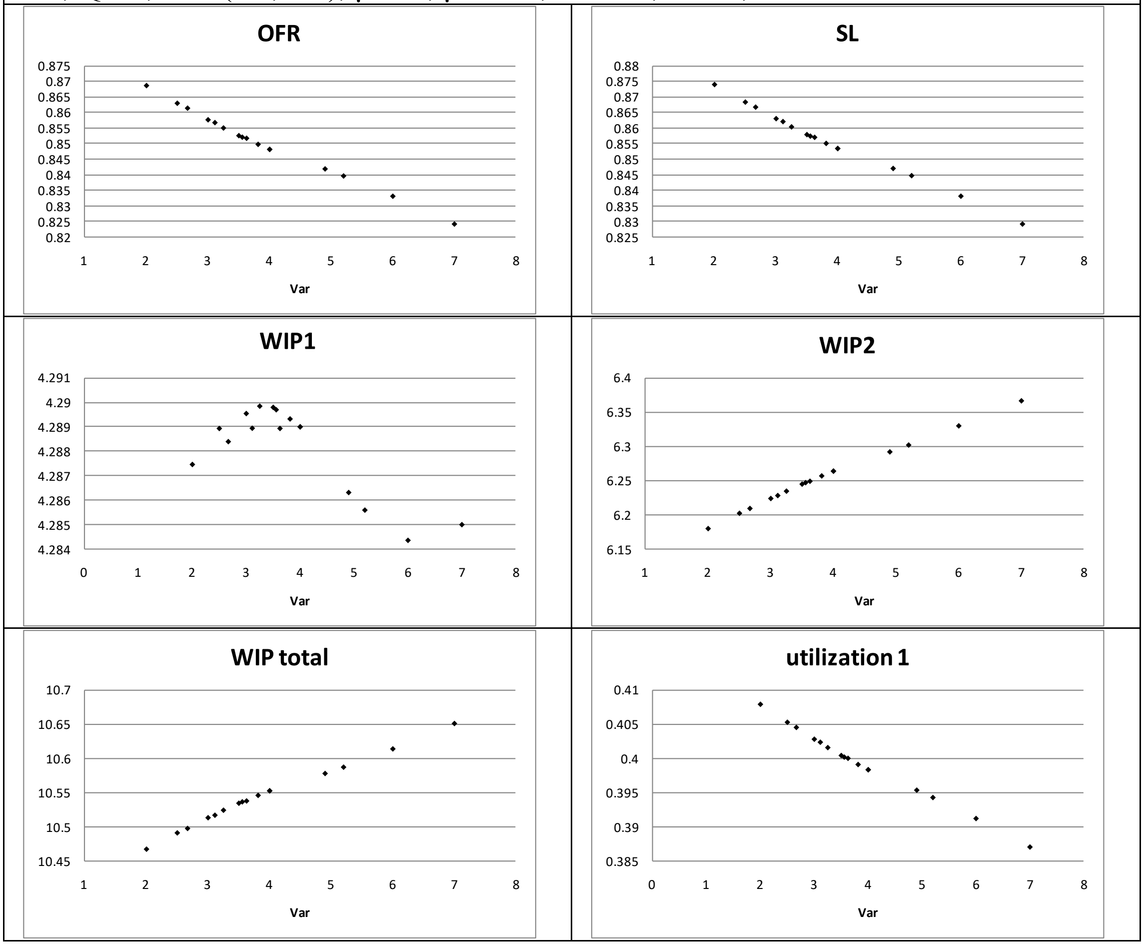
6.10.2.3 Effect of lead time variance on the performance measures

With regard to the effect of lead time variance, the effect on WIP_1 depends on the specific scenario under investigation, and in some scenarios a consistent increase in WIP_1 with increasing variance is observed. For the rest of the performance measures, the general behavior is similar to that observed for balanced systems. The assumption of linear dependence between lead time variance and the performance measures is no longer accurate.

The effect of retailer's Lead time Variance on performance measures. s=4, Q_2=4, r=4, Q_1=2, dm=(0.6, 0.4), μ_{21}=1, μ_{22}=0.2, d_{22}=0.3, λ=2/3, and T_1=2



The effect of Retailer's Lead time Variance on performance measures. s=8, Q_2=8, r=4, Q_1=4, dm=(0.6, 0.4), μ_{21}=1, μ_{22}=0.2, d_{22}=0.3, λ=2/3, T_1=2



6.10.3 Demand constrained systems

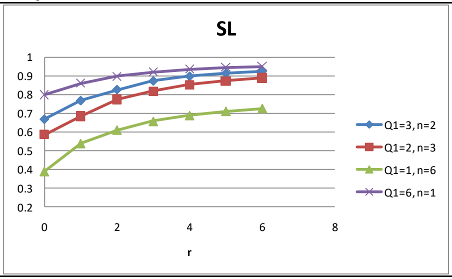
In demand constrained systems the average time between successive arrivals of external customers is longer than the average transportation time from the vendor to the retailer, which in its turn is longer than the average transportation time from the manufacturer to the vendor: $\frac{1}{\lambda} > \frac{1}{\mu_{11}} + d_{12}\frac{1}{\mu_{12}} > \frac{1}{\mu_{21}} + d_{22}\frac{1}{\mu_{22}}$

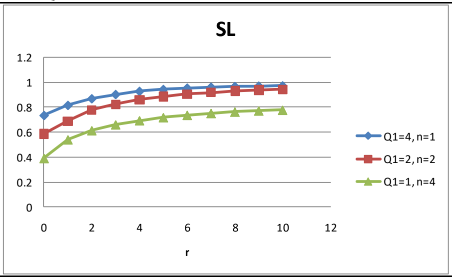
For the examples presented below we use μ_{11}=1, μ_{12}=0.2, d_{12}=0.3, μ_{21}=1, μ_{22}=0.2, d_{22}= 0.2, λ=1/3 and $dm = (0.6, 0.4)$ (there is 0.6 probability that an external customer will ask 1 product unit and 0.4 probability that he will ask 2 units).

6.10.3.1 The effect of the decision variables on the performance measures

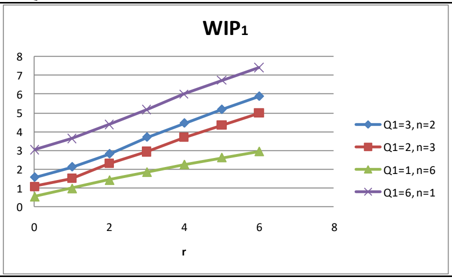
Compared to balanced systems, demand constrained systems present higher Order Fill Rates and Service Levels. However more inventory is accumulated in the system and although inventory at the vendor may decrease, such a decrease is more than offset by an increase in the retailer's inventory (WIP_1).

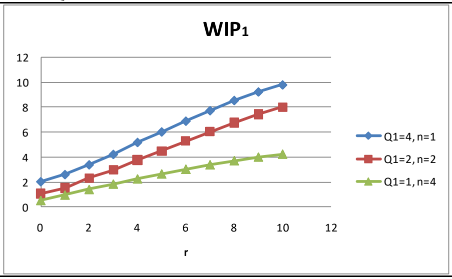
Service Level as a function of reorder point *r*	
s=6, Q_2=6	s=10, Q_2=4



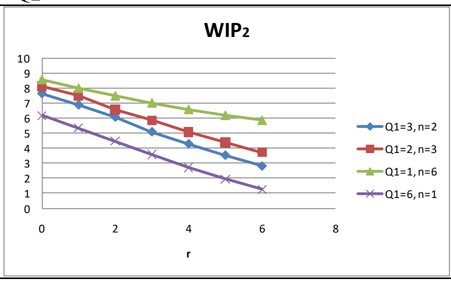


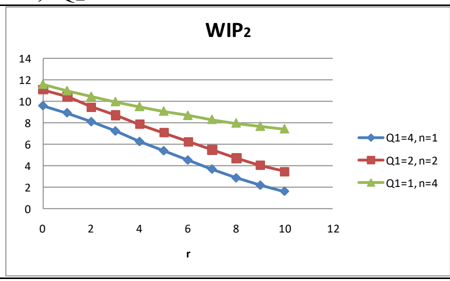
Retailer's WIP as a function of reorder point *r*	
s=6, Q_2=6	s=10 , Q_2=4



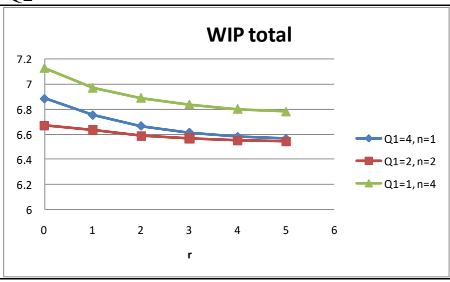


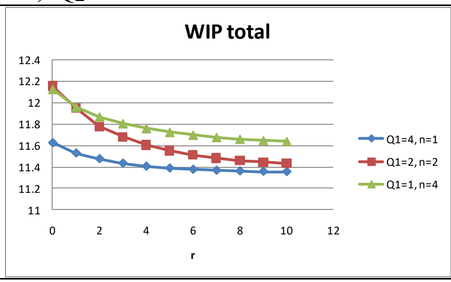
Wholesaler's WIP as a function of reorder point *r*	
s=6, Q_2=6	s=10 , Q_2=4



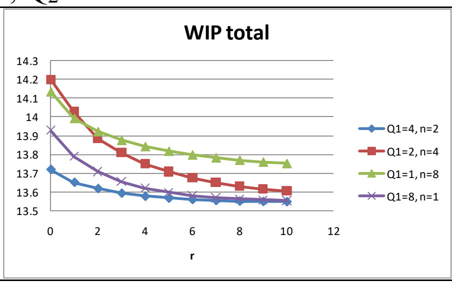


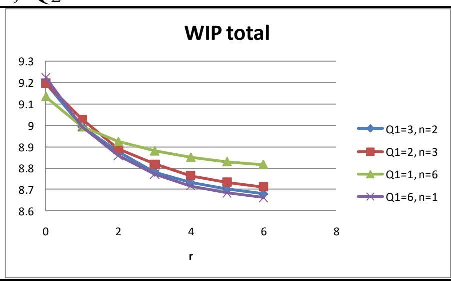
Total average inventory as function of reorder point *r*	
s=5 Q_2=4	s=10 , Q_2=4





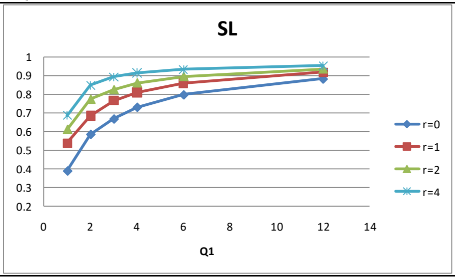
s=10, Q_2=8	s=6, Q_2=6



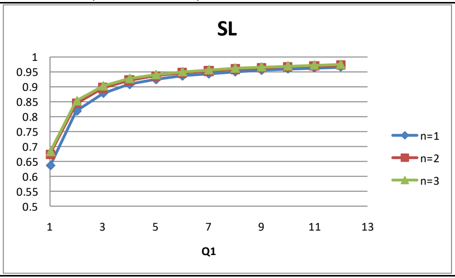


Service level as a function of replenishment order Q_1

s=4, Q_2=12

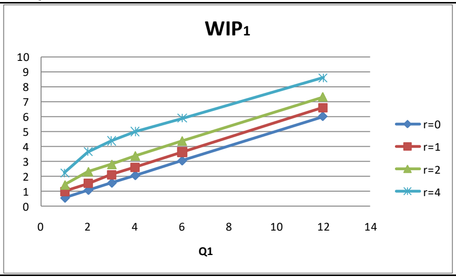


s=6, r=4 (n=const)

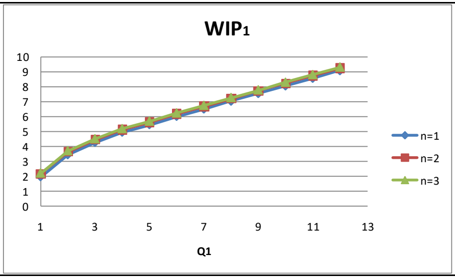


Average inventory at the retailer as a function of replenishment order Q_1

s=4, Q_2=12

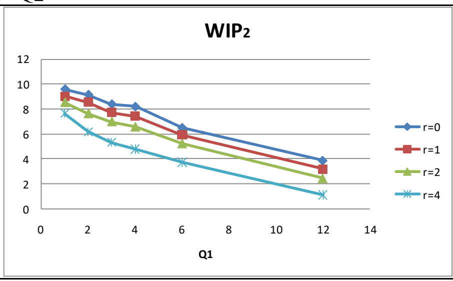


s=6, r=4

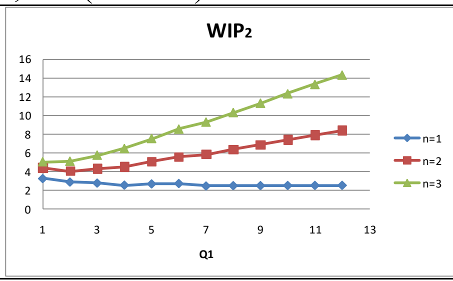


Average inventory at the vendor as a function of replenishment order Q_1

s=4, Q_2=12

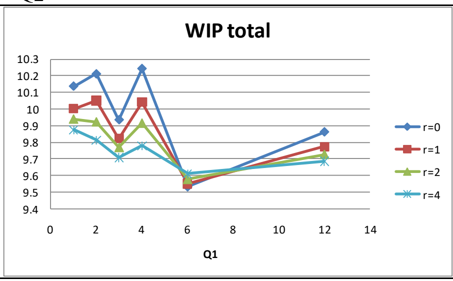


s=6, r=4 (n=const)

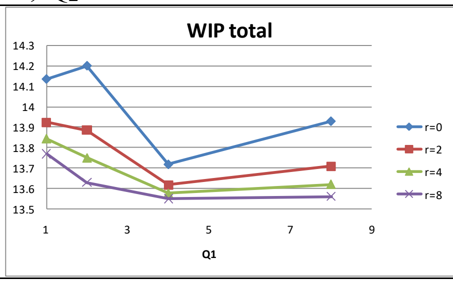


Average total inventory as a function of replenishment order Q_1

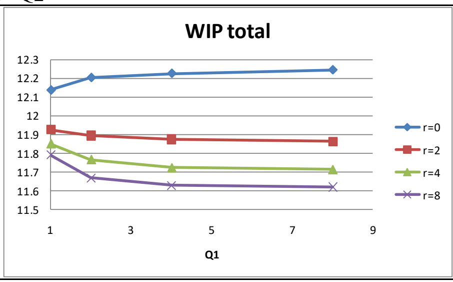
s=4, Q_2=12



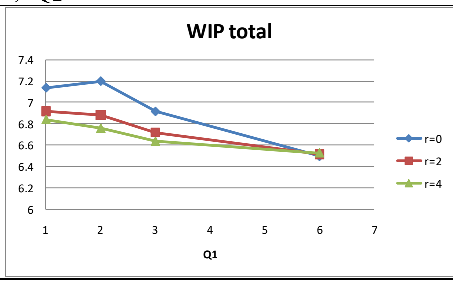
s=10, Q_2=8



s=8, Q_2=8

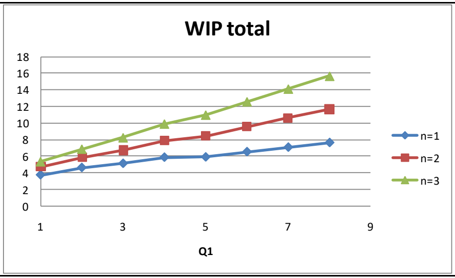


s=4, Q_2=6

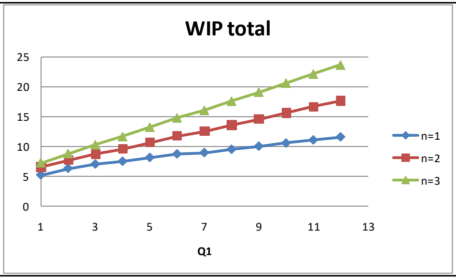


Average total inventory as a function of replenishment order Q_1 (n=const)

s=4, r=2

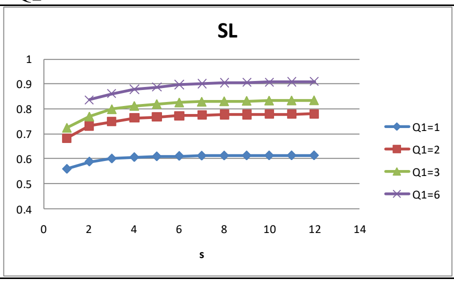


s=6, r=4

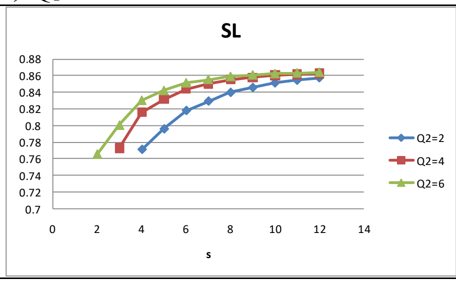


Service Level as a function of echelon reorder point s

r=2, Q_2=6

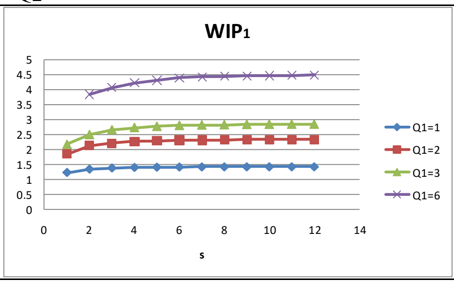


r=4, Q_1=2

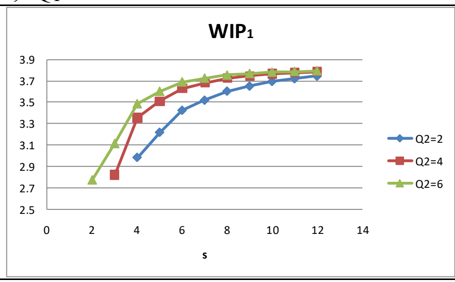


Average inventory at the retailer as a function of echelon reorder point s

r=2, Q_2=6

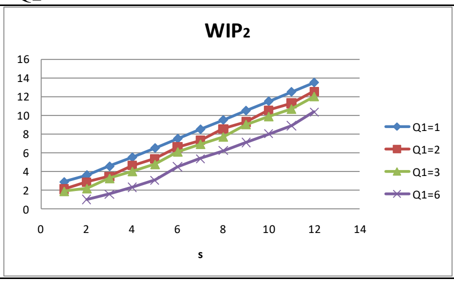


r=4, Q_1=2

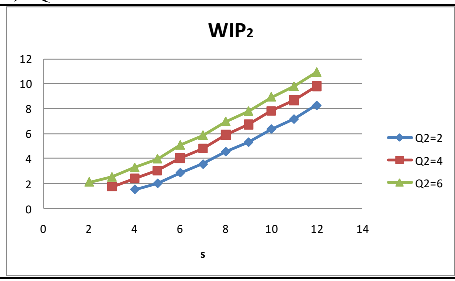


Average inventory at the wholesaler as a function of echelon reorder point *s*

r=2, Q_2=6

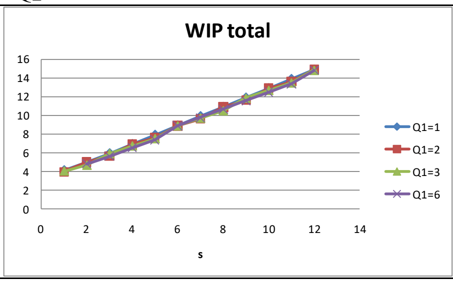


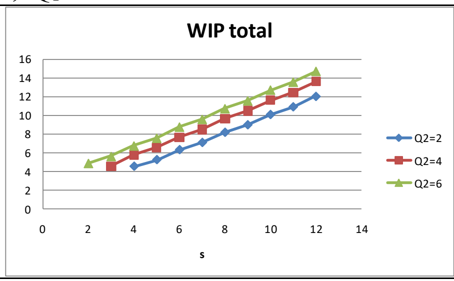
r=4, Q_1=2



Average total inventory as a function of echelon reorder point *s*

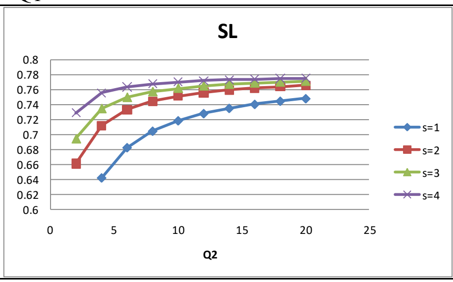
r=2, Q_2=6	r=4, Q_1=2

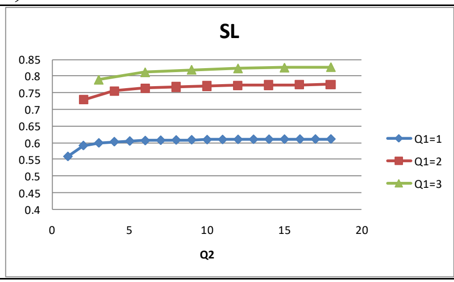




Service level as a function of replenishment order Q_2

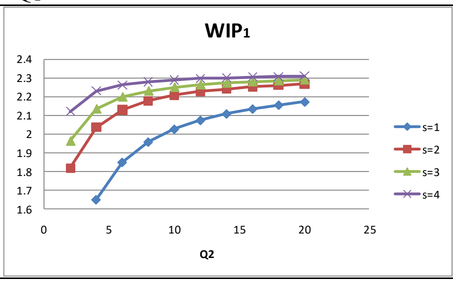
r=2, Q_1=2	s=4, r=2

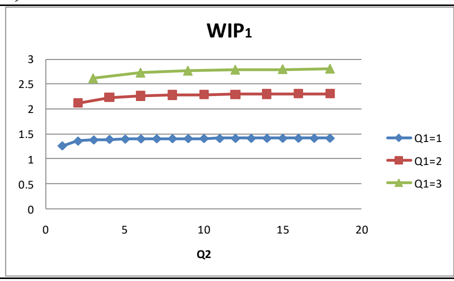




Average inventory at the retailer as a function of replenishment order Q_2

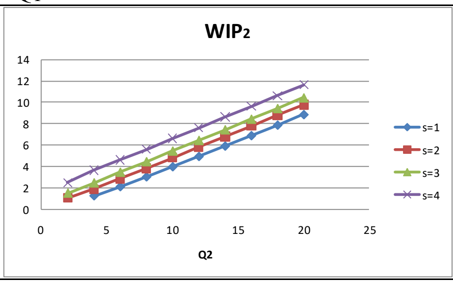
r=2, Q_1=2	s=4, r=2

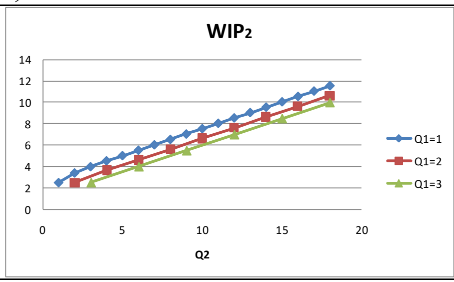




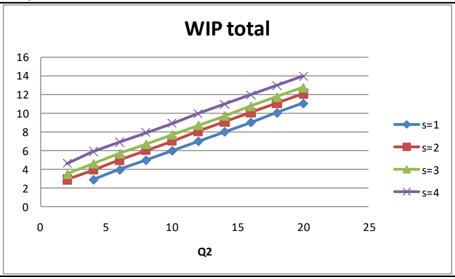
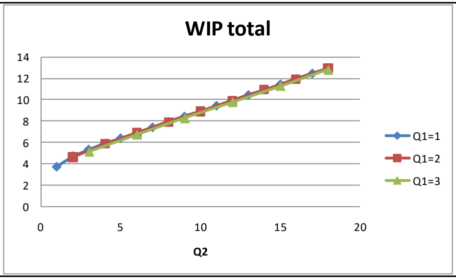
Average inventory at the wholesaler as a function of replenishment order Q_2

r=2, Q_1=2	s=4, r=2





The general behavior of the performance measures with changing values of the decision variables is the same as to that of balanced system. The only exception that was observed concerns total inventory with changing Q_1, but it must be noted that in absolute terms very small numerical differences were concerned. Both r-WIP_2 and s-WIP_2 curves have a greater grade in demand constrained systems.

Total average inventory as a function of replenishment order Q_2	
$r=2$, $Q_1=2$	$s=4$, $r=2$
	

The behavior of demand constrained systems is similar to that of balanced system also when the combined effect of the decision variables is concerned. The conclusions outlined in 6.10.1.2 for balanced systems also hold for demand constrained systems.

6.10.3.2 Effect of demand characteristics on the performance measures

We investigate the effect of external demand variability on the performance measures. Different scenarios are explored where the average external demand remains constant, but its variance is changed. Some results are given below both in the form of a table and graphically.

Both Order Fill rate (the percentage of external orders that are fully met) and Service Level (the percentage of external demand in terms of product units that is met from inventory on hand) tend to decrease with increasing variance.

Average inventory at the retailer as well as average total inventory tends to increase with increasing variance. The effect on vendor's inventory is less straightforward and depends on the specific scenario under consideration.

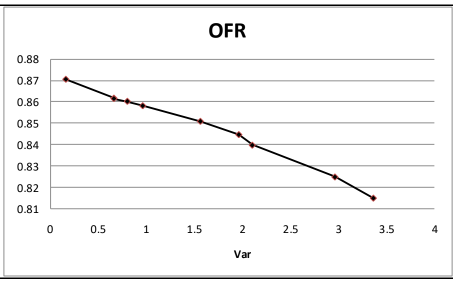
Increased demand variability is detrimental to the performance of system. For most scenarios the behavior of the system is similar to that of balanced systems, with some minor exceptions for OFR.

Performance measures as a function of external demand variance. $s=8$, $Q_2=8$, $r=4$, $Q_1=4$, $E_x=1.8$, $\mu_{11}=1$, $\mu_{12}=0.2$, $d_{12}=0.3$, $\mu_{21}=1$, $\mu_{22}=0.2$, $d_{22}= 0.2$, $\lambda=1/3$

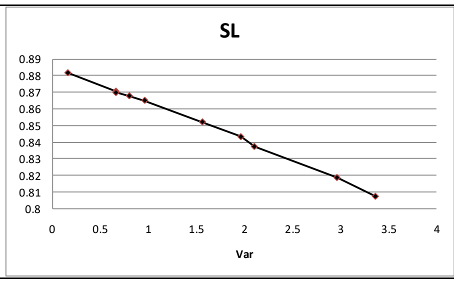
dm(1)	dm(2)	dm(3)	dm(4)	dm(5)	dm(6)	Var	OFR	SL	WIPtotal	WIP_1	WIP_2	$util_1$
0.2	0.8	0	0	0	0	0.16	0.870	0.882	11.622	4.880	6.742	0.331
0.45	0.3	0.25	0	0	0	0.66	0.862	0.871	11.612	4.856	6.756	0.326
0.39	0.45	0.15	0	0	0.01	0.66	0.861	0.870	11.615	4.861	6.754	0.326
0.48	0.28	0.2	0.04	0	0	0.8	0.860	0.868	11.622	4.865	6.758	0.325
0.6	0	0.4	0	0	0	0.96	0.858	0.865	11.632	4.876	6.756	0.324
0.65	0.1	0.1	0.1	0.05	0	1.56	0.851	0.852	11.668	4.894	6.774	0.320
0.7	0.1	0	0.1	0.1	0	1.96	0.845	0.843	11.685	4.898	6.787	0.316
0.7	0.05	0.15	0.02	0.01	0.07	2.1	0.840	0.837	11.697	4.919	6.778	0.314
0.82	0	0	0	0.1	0.08	2.96	0.825	0.819	11.741	4.929	6.812	0.307
0.84	0	0	0	0	0.16	3.36	0.815	0.807	11.775	4.958	6.817	0.303

The effect of external demand variance on the performance measures. $s=8$, $Q_2=8$, $r=4$, $Q_1=4$, Average demand $E_x=1.8$, $\mu_{11}=1$, $\mu_{12}=0.2$, $d_{12}=0.3$, $\mu_{21}=1$, $\mu_{22}=0.2$, $d_{22}=0.2$, $\lambda=1/3$

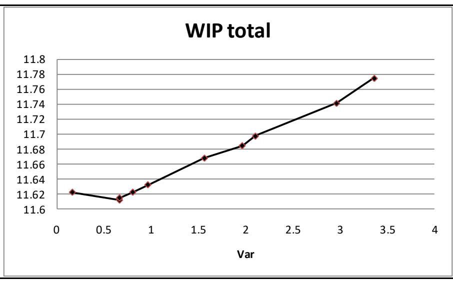
OFR



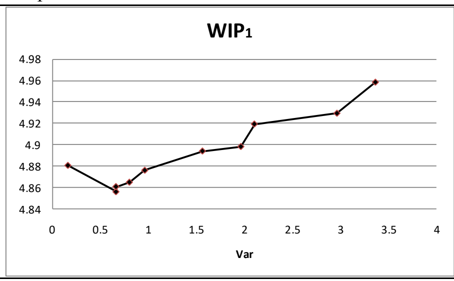
SL



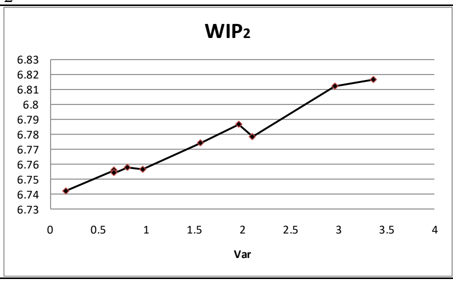
WIP total



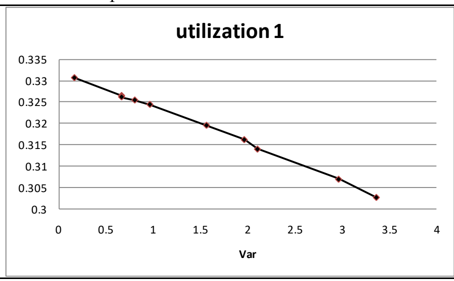
WIP_1



WIP_2

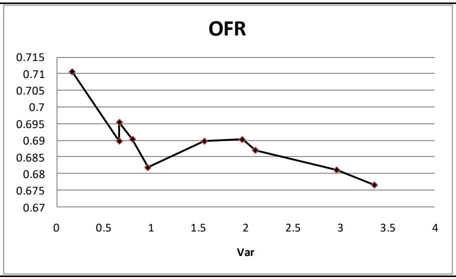
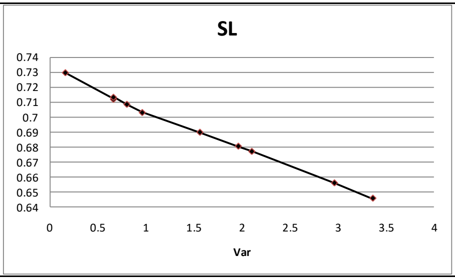
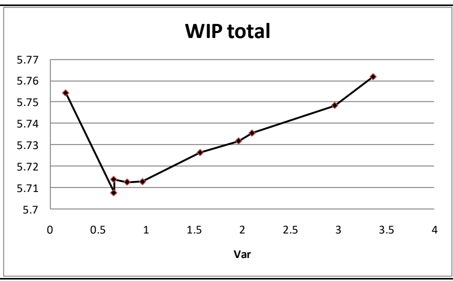
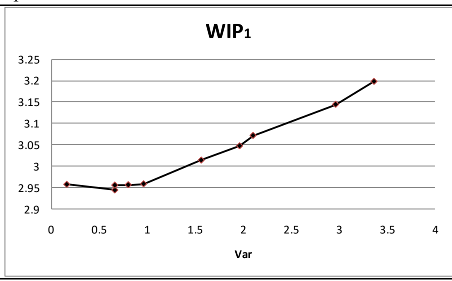
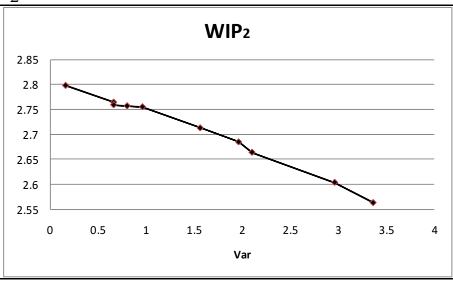
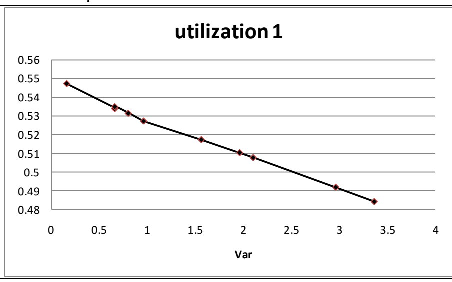


$Utilization_1$



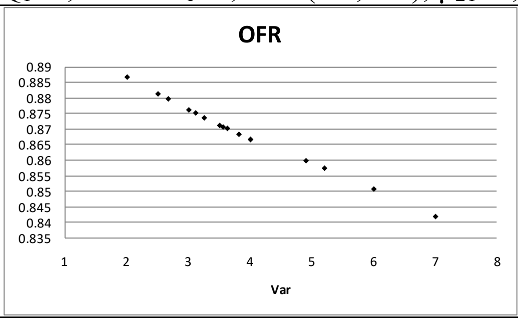
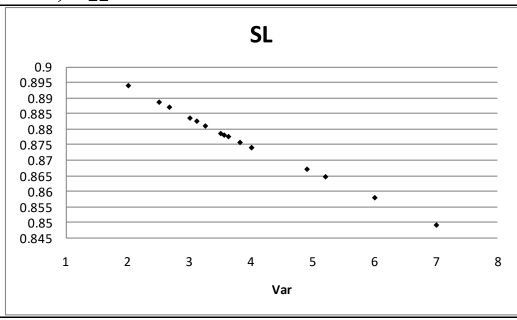
Performance measures as a function of external demand variance. $s=4$, $Q_2=4$, $r=4$, $Q_1=2$, $E_x=1.8$, $\mu_{11}=1$, $\mu_{12}=0.2$, $d_{12}=0.3$, $\mu_{21}=1$, $\mu_{22}=0.2$, $d_{22}=0.2$, $\lambda=1/3$

dm(1)	dm(2)	dm(3)	dm(4)	dm(5)	dm(6)	Var	OFR	SL	WIPtotal	WIP_1	WIP_2	$util_1$
0.2	0.8	0	0	0	0	0.16	0.711	0.730	5.754	2.957	2.797	0.547
0.45	0.3	0.25	0	0	0	0.66	0.690	0.712	5.708	2.944	2.764	0.534
0.39	0.45	0.15	0	0	0.01	0.66	0.695	0.713	5.714	2.955	2.759	0.535
0.48	0.28	0.2	0.04	0	0	0.8	0.690	0.708	5.713	2.956	2.757	0.531
0.6	0	0.4	0	0	0	0.96	0.682	0.703	5.713	2.958	2.755	0.527
0.65	0.1	0.1	0.1	0.05	0	1.56	0.690	0.690	5.726	3.014	2.713	0.517
0.7	0.1	0	0.1	0.1	0	1.96	0.690	0.680	5.732	3.047	2.685	0.510
0.7	0.05	0.15	0.02	0.01	0.07	2.1	0.687	0.677	5.735	3.071	2.664	0.508
0.82	0	0	0	0.1	0.08	2.96	0.681	0.656	5.748	3.144	2.604	0.492
0.84	0	0	0	0	0.16	3.36	0.677	0.646	5.762	3.198	2.564	0.484

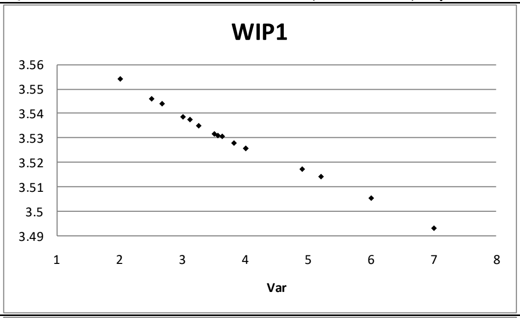
The effect of external demand variance on the performance measures. $s=4$, $Q_2=4$, $r=4$, $Q_1=2$, Average demand $Ex=1.8$, $\mu_{11}=1$, $\mu_{12}=0.2$, $d_{12}=0.3$, $\mu_{21}=1$, $\mu_{22}=0.2$, $d_{22}= 0.2$, $\lambda=1/3$	
OFR	SL
	
WIP total	WIP_1
	
WIP_2	$Utilization_1$
	

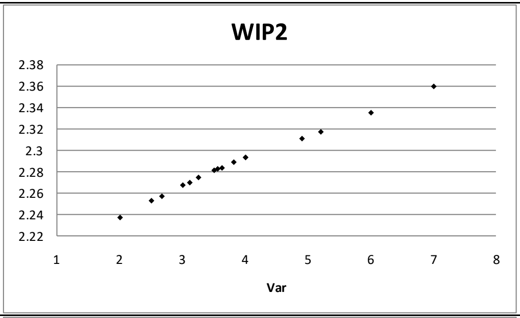
6.10.3.3 Effect of lead time variance on the performance measures

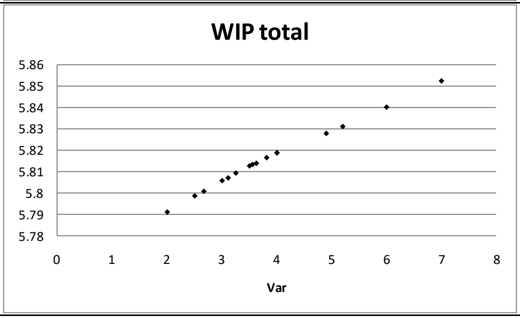
The behavior of the system with changing retailer's lead time variance is similar to that observed for balanced systems. Demand constrained systems are in general more stable for lead time variance changes, and the dependence of the performance measures can be described quite accurately with simple linear relations.

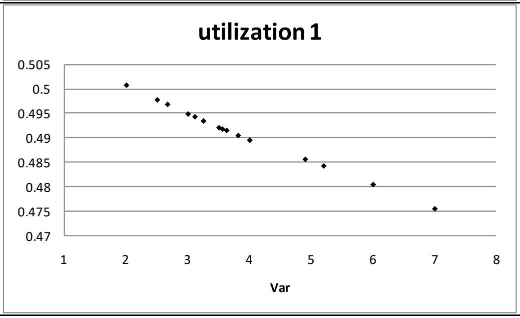
The effect of Retailer's Lead time Variance on performance measures $s=4$, $Q_2=4$ $r=4$, $Q_1=2$, $l=0.4$. $T_1=2$, $dm=(0.6, 0.4)$, $\mu_{21}=2$, $\mu_{22}=0.5$, $d_{22}=0.5$	
	

The effect of Retailer's Lead time Variance on performance measures s=4,Q_2=4 r=4, Q_1=2, l=0.4. T_1=2, dm=(0.6, 0.4), μ_{21}=2, μ_{22}=0.5, d_{22}=0.5

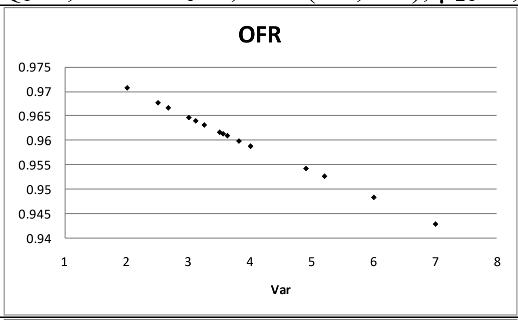


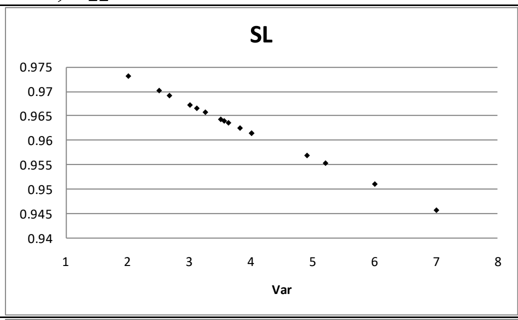


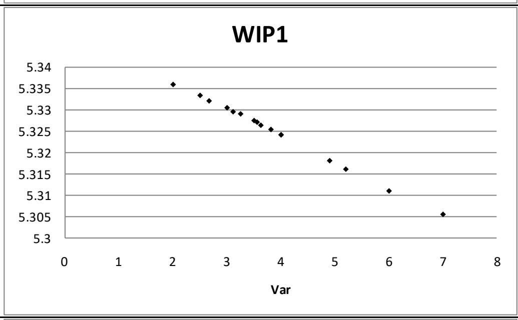


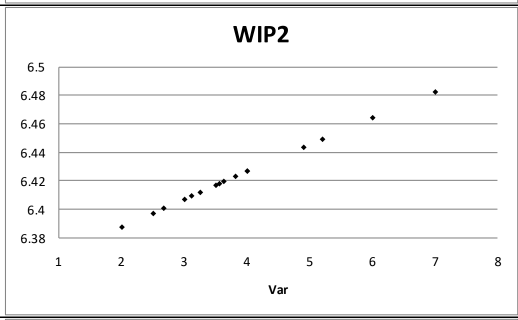


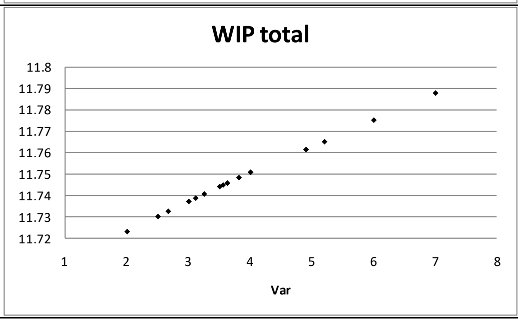
The effect of Retailer's Lead time Variance on performance measures. s=8,Q_2=8 r=4, Q_1=4, l=0.4. T_1=2, dm=(0.6, 0.4), μ_{21}=2, μ_{22}=0.5, d_{22}=0.5

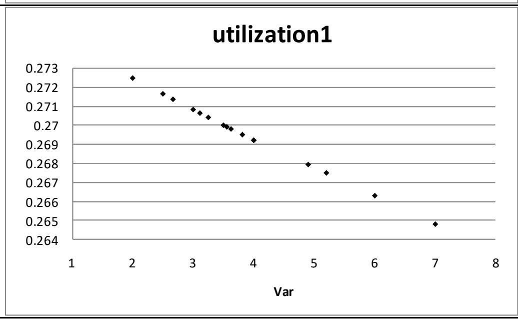












6.11 Conclusions

An algorithm based on matrix analytic methods was presented for the exact numerical performance evaluation of a two echelon inventory system working according to Vendor Managed Inventory Logic. A lost sales regime was assumed and attention was paid to the stochastic characteristics of the system. Uncertainties about lead times were captured using a phase type, Coxian distribution with two phases. Moreover, the stochastic nature of external demand was modeled using a compound Poisson distribution comprised of a Pure Poisson arrival process and a discrete empirical distribution for the demand of individual customers. These assumptions allow for more realistic modeling, permit us to better capture and analyze the uncertainties which are integral to inventory networks, and constitute our basic contribution with regard to VMI models found in the literature.

A model based on the algorithm was developed in Matlab and was used to investigate the effect of different policies under different operating conditions. Our analysis was based on the numerical evaluation of the system performance measures and systems with different relations between lead times and customer inter-arrival times were tested.

In balanced systems all four decision variables (r, Q_1, s, Q_2) affect the performance measures and could under certain conditions be used to guide the system towards the desired targets. The effect of each parameter depends on the values of the others, so the interplay between variables must always be taken into consideration.

To increase customer satisfaction, as it is expressed through OFR and SL, any of the decision variables could be used. Changing the echelon inventory policy by increasing *s* or Q_2 has the side-effect of significantly increasing the inventory accumulated in the system and thus mitigates any gains in system performance. In general *s* has a greater effect than Q_2 on both OFR/SL and average inventory, while Q_2 is important mainly when the lowest Q_2 values are concerned.

On the other hand, by increasing the retailer's policy parameters *r* or Q_1 we can enhance customer satisfaction without any increase in total inventory (actually in some cases a diminished total inventory was observed). In general, Q_1 is more important than *r*, but when Q_1 is changed it must be kept in mind that local minima for total average inventory may be observed for intermediate values.

From the vendors point of view a policy of high *r* and high Q_1 is desirable, but in practice there may be constraints about the inventory at the retailer as many retailers put contractual limits on the inventory at their premises. Our analysis also indicates that high Q_1 and low Q_2 values are preferable, while a high value of *r* in relation to *s* is desirable.

Small changes in the external demand characteristics do not alter the behaviour of the system with changing decision variable values. However, the increased variance of the external demand is detrimental for system performance both in terms of customer satisfaction and inventory accumulation. For rising demand variance the somewhat paradoxical effect of rising retailer's inventory and falling of service level was observed.

Supply constrained systems, in general, exhibit the same behaviour as the balanced systems and the conclusions mentioned above also hold. In comparison with balanced systems, for the same design variables values, supply constrained systems present lower Order Fill Rates and Service Levels, as well as lower average inventories at the retailer. For average inventory at the wholesaler, as well as for average total inventory, both higher and lower values were observed, depending on the specific scenarios under consideration. Average total inventory (WIP total) and average inventory at the retailer (WIP_1) were found to be more sensitive for changing demand variability.

Demand constrained systems are more dynamic and less predictable than balanced systems when more than one decision variables are changed. In general, the performance measures curves are similar in shape with those of balanced systems. In comparison with balanced systems, for the same design variables values, higher OFR and SL are achieved, but on the downside more inventory is accumulated in the system. For demand constrained systems, OFR and SL are more sensitive to changes of external demand variability.

With regard to further research, the expansion of the model for more Coxian phases would allow for greater flexibility with regard to lead times and would offer a better insight in the effect of replenishment time uncertainty. In a further step, the investigation of different topologies, and especially the investigation of an arborescent lay-out with more than one retailers, would also be of interest as it would bring our analysis closer to real practice. Finally, a third approach to expand our research would be the combination of the developed evaluative algorithm with a heuristic algorithm in a greater framework of an optimization model.

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6.13 Appendix

6.13.1 Matlab algorithm

We present the computer code for the described model. The computer algorithm is given for Mathworks' Matlab, version R2018a (9.4.0.813654). The computer code essentially follows the lines of the presented analysis. Some parts of the model that are used repetitively have been modeled as sub-routines (functions) that are stored separately and called whenever necessary. Such functions are given separately from the main body of the algorithm. Comments start with the symbol "%".

Important note: Some lines of the algorithm have been omitted on purpose. Their position is denoted with […]

6.13.1.1 Main body algorithm

```
%-------------------------- Input --------------------------------
% r: Reorder point of the retailer
% Q1:The quantity of orders sent from the wholesaler to the retailer
% s: Reorder point of the wholesaler (refers to echelon stock)
% n: Q2=nQ1, where n a natural number
% mij: the tranportation rate of the jth phase for transportation
towards the ith station
% dij: the probability that transportation towards ith station will
have j phases
% dm(i)= the probability that the external demand will be i units of
1 unit, p2
s=2;
n=2;
r=1;
Q1=1;
Q2=n*Q1;
% when r>s we use the equivalent r value.
r=min(r,s+Q2-Q1);
l=0.3;
m11=2;
m12=0.4;
d12=0.2;
m21=1.25;
m22=0.5;
d22=0.1;
dm=[0.3;0.55;0.0];
%   Useful computations
d11=1-d12;
d21=1-d22;
dmd=length(dm);
% rd(i): the probability that demand will be equal to or more than i
rd=zeros(1,dmd);
for i=1:dmd
    for j=i:dmd
        rd(i)=rd(i)+dm(j);
    end
end
h=floor((s-r)/Q1);
k=floor(s/Q1+1);
f=floor((s-r)/Q1)+n;
```

```
Nl=floor((s+Q2)/Q1);
% dimension(L+1): A vector recording the dimension of the diagonal
submatrix
dimension=zeros(1,Nl+1);
% ---------------------- Creation of sub-matrices ------------
%   * Diagonal sub matrix for level L=0 (
L=0;
Z1=0;
if h>0
    Z2=0;
else
    Z2=r-s+Q1;
end
O1=r+1;
if h>0
    O2=Q1;
else
    O2=s-r;
end
dimension(1)=Z1+Z2+2*(O1+O2);
dimensionZ1(1)=Z1;
dimensionZ2(1)=Z2;
dimensionO1(1)=O1;
dimensionO2(1)=O2;
%Submatrix "D0"
D00=Diagonal0(Z1,Z2,m11,m12,l,d12,k,L,dm,rd,dmd);
%Sub-matrix "D1"
D01=zeros(O1+O2);
D01(1,1)=-m21;
for i=2:O1+O2
    D01(i,i)=-m21-l;
end
for i=2:O1+O2-1
    for j=1:min(O1+O2-i,dmd)
        D01(i+j,i)=dm(j)*l;
    end
end
for i=1:min(O1+O2-1,dmd)
    D01(i+1,1)=rd(i)*l;
end
%Sub-matrix "D2"
D02=zeros(O1+O2);
D02(1,1)=-m22;
for i=2:O1+O2
    D02(i,i)=-m22-l;
end
for i=2:O1+O2-1
    for j=1:min(O1+O2-i,dmd)
        D02(i+j,i)=dm(j)*l;
    end
end
for i=1:min(O1+O2-1,dmd)
    D02(i+1,1)=rd(i)*l;
end
%Sub-matrix U1
U0=Upper(O1,O2,d22,m21) ;
%Sub-sub matrix U0
UZ0=zeros(Z2,O1+O2);
for i=1:min(O1+O2-1,dmd)
    for j=1:min(Z2,dmd-i+1)
        UZ0(j,O1+O2-i+1)=dm(i+j-1)*l;
```

```
    end
end
for i=1:min(Z2, dmd-O1-O2+1)
    UZ0(i,1)=rd(O1+O2+i-1)*l;
end
% Diagonal sub-matrix for level 0
DZ=zeros(Z2+O1+O2);
DZ(1:Z1+Z2,1:Z1+Z2)=D00;
DZ(Z1+Z2+1:Z1+Z2+O1+O2,Z1+Z2+1:Z1+Z2+O1+O2)=D01;
DZ(Z1+Z2+O1+O2+1:Z1+Z2+2*(O1+O2),Z1+Z2+O1+O2+1:Z1+Z2+2*(O1+O2))=D02;
DZ(Z1+Z2+1:Z1+Z2+O1+O2,Z1+Z2+O1+O2+1:Z1+Z2+2*(O1+O2))=U0;
DZ(1:Z1+Z2,Z1+Z2+1:Z1+Z2+O1+O2)=UZ0;
P=DZ;
lp=dimension(1);
LP(1)=lp;
%Iterative process to construct the diagonal sub-matrices for levels
1 to Nl
for L=1:Nl
% inf = an index indicating if L>f
if L<=f
    inf=0;
else
    inf=1;
end
% inh = an index indicating if L>h
if L<=h
    inh=0;
else
    inh=1;
end
Z1=(min(2*(r-s)+2*L*Q1, 2*(r+1))-inf*2*(r-n*Q1-s+L*Q1))*inh;
if h<L && L<f
    Z2=Q1;
elseif L==h
    Z2=r+Q1-s+L*Q1;
elseif L==f
    Z2=(n-L)*Q1+s-r;
else
    Z2=0;
end
% Sub matrix "D0"
D0=Diagonal0(Z1,Z2,m11,m12,l,d12,k,L,dm,rd,dmd);
if L<=h
    O1=2*(r+1);
elseif h<L && L<k
    O1=max(2*(s-L*Q1+1),0);
else
    O1=0;
end
if L<h
    O2=Q1;
elseif L==h
    O2=s-r-L*Q1;
else
    O2=0;
end
% Sub-matrix "D1"
D1=Diagonal1(O1,O2,m11,m12,m21,l,d12,dm,rd,dmd);
% Create sub-matrix "D2"
D2=Diagonal2(O1,O2,m11,m12,m22,l,d12,dm,rd,dmd);
% Create sub-matrix U1
```

```
U=Upper(O1,O2,d22,m21);
% Create sub-matrix U0
UZ=UpperZ(O1,O2,Z1,Z2,l,dm,rd,dmd);
% Create diagonal submatrix D for level L
D=zeros(Z1+Z2+O1+O2);
D(1:Z1+Z2,1:Z1+Z2)=D0;
D(Z1+Z2+1:Z1+Z2+O1+O2,Z1+Z2+1:Z1+Z2+O1+O2)=D1;
D(Z1+Z2+O1+O2+1:Z1+Z2+2*(O1+O2),Z1+Z2+O1+O2+1:Z1+Z2+2*(O1+O2))=D2;
D(Z1+Z2+1:Z1+Z2+O1+O2,Z1+Z2+O1+O2+1:Z1+Z2+2*(O1+O2))=U;
D(1:Z1+Z2,Z1+Z2+1:Z1+Z2+O1+O2)=UZ;
%Record dimensions for submatrices
dimension(L+1)=Z1+Z2+2*(O1+O2);
dimensionZ1(L+1)=Z1;
dimensionZ2(L+1)=Z2;
dimensionO1(L+1)=O1;
dimensionO2(L+1)=O2;
%lp:last position
P(lp+1:lp+dimension(L+1),lp+1:lp+dimension(L+1))=D;
lp=lp+dimension(L+1);
LP(L+1)=lp;
end
%Submatrix UD1 and UD2 for level 0
L=0;
Z1L2=dimensionZ1(n+1);
Z2L2=dimensionZ2(n+1);
O1L2=dimensionO1(n+1);
O2L2=dimensionO2(n+1);
Z1L1=dimensionZ1(1);
Z2L1=dimensionZ2(1);
O1L1=dimensionO1(1);
O2L1=dimensionO2(1);
D1=zeros(O1L1+O2L1,Z1L2+Z2L2+O1L2+O2L2);
D2=zeros(O1L1+O2L1,Z1L2+Z2L2+O1L2+O2L2);
for i=1:O1L2/2
    D1(i,Z1L2+Z2L2+2*i-1)=d21*m21;
    D2(i,Z1L2+Z2L2+2*i-1)=m22;
end
for i=O1L2/2+1:(O1L2/2)+O2L2
    D1(i,Z1L2+Z2L2+(O1L2)/2+i)=d21*m21;
    D2(i,Z1L2+Z2L2+(O1L2)/2+i)=m22;
end
j=(O1L2/2)+O2L2+1;
for i=j:j+Z1L2/2-1
    D1(i,2*(i-j)+1)=d21*m21;
    D2(i,2*(i-j)+1)=m22;
end
for i=j+Z1L2/2:O1L1+O2L1
    D1(i,Z1L2+(i-j-Z1L2/2)+1)=d21*m21;
    D2(i,Z1L2+(i-j-Z1L2/2)+1)=m22;
end
P(Z1L1+Z2L1+1:Z1L1+Z2L1+O1L1+O2L1,LP(n)+1:LP(n)+Z1L2+Z2L2+O1L2+O2L2)=
D1;
P(O1L1+O2L1+Z1L1+Z2L1+1:O1L1+O2L1+Z1L1+Z2L1+O1L1+O2L1,LP(n)+1:LP(n)+Z
1L2+Z2L2+O1L2+O2L2)=D2;
% Iterative process to create the sub-matrices above the diagonal,
% UD1 and UD2
for L=1:k-1
Z1L2=dimensionZ1(L+n+1);
Z2L2=dimensionZ2(L+n+1);
O1L2=dimensionO1(L+n+1);
O2L2=dimensionO2(L+n+1);
```

```
Z1L1=dimensionZ1(L+1);
Z2L1=dimensionZ2(L+1);
O1L1=dimensionO1(L+1);
O2L1=dimensionO2(L+1);
P(LP(L)+Z1L1+Z2L1+1:LP(L)+Z1L1+Z2L1+O1L1+O2L1,LP(L+n)+1:LP(L+n)+Z1L2+
Z2L2+O1L2+O2L2)= UpperD1(Z1L2,Z2L2,O1L2,O2L2,O1L1,O2L1,d21,m21);
P(LP(L)+Z1L1+Z2L1+O1L1+O2L1+1:LP(L)+Z1L1+Z2L1+2*(O1L1+O2L1),LP(L+n)+1
:LP(L+n)+Z1L2+Z2L2+O1L2+O2L2)=
UpperD2(Z1L2,Z2L2,O1L2,O2L2,O1L1,O2L1,m22);
end
%Sub-matrices below the diagonal
L=1;
Z1L2=dimensionZ1(L);
Z2L2=dimensionZ2(L);
O1L2=dimensionO1(L);
O2L2=dimensionO2(L);
Z1L1=dimensionZ1(L+1);
Z2L1=dimensionZ2(L+1);
O1L1=dimensionO1(L+1);
O2L1=dimensionO2(L+1);
if O1L1>0
    D=zeros(O1L1,O1L2+O2L2);
    for i=1:2:O1L1-1
        D(i,(i+1)/2+Q1)=d11*m11;
        D(i+1,(i+1)/2+Q1)=m12;
    end
P(LP(L)+Z1L1+Z2L1+1:LP(L)+Z1L1+Z2L1+O1L1,
Z1L2+Z2L2+1:Z1L2+Z2L2+O1L2+O2L2)=D;
P(LP(L)+Z1L1+Z2L1+O1L1+O2L1+1:LP(L)+Z1L1+Z2L1+O1L1+O2L1+O1L1,
Z1L2+Z2L2+O1L2+O2L2+1:Z1L2+Z2L2+O1L2+O2L2+O1L2+O2L2)=D;
end
if Z1L1>0
    if L<k
        x=0;
    elseif L==k
        x=Q1-(s-(L-1)*Q1+1);
    end
    D=zeros(Z1L1,Z1L2+Z2L2);
    for i=1:2:Z1L1-1
        D(i,x+(i+1)/2)=d11*m11;
        D(i+1,x+(i+1)/2)=m12;
    end
    P(LP(L)+1:LP(L)+Z1L1,1:Z1L2+Z2L2)=D;
end
% Iterative process to create the sub-matrices below the diagonal
for L=2:Nl
Z1L2=dimensionZ1(L);
Z2L2=dimensionZ2(L);
O1L2=dimensionO1(L);
O2L2=dimensionO2(L);
Z1L1=dimensionZ1(L+1);
Z2L1=dimensionZ2(L+1);
O1L1=dimensionO1(L+1);
O2L1=dimensionO2(L+1);
% Submatrices coresponding to transitions from Z1 states
x=0;
z=0;
if Z1L1>0
    if L<k
        x=0;
    elseif L==k
```

```
        x=2*Q1-2*(s-(L-1)*Q1+1);
    elseif L>k
        x=2*Q1;
    end
    if Q1>r
        if L<k
            z=0;
        elseif L==k
            z=min(Q1-(s-(L-1)*Q1+1),Q1-r-1);
        elseif L>k
            z=Q1-r-1;
        end
    end
    P(LP(L)+1:LP(L)+Z1L1, LP(L-1)+1:LP(L-
1)+Z1L2+Z2L2)=LowerD0(Z1L1,Z1L2,Z2L2,d11,m11,m12,x,z);
end
% Submatrices corresponding to transitions from O1 and T1 states
if O1L1>0
[...]
[...]
end
end
%----------- Calculation of stationary Probabilities Vector X ---
Q=P';
ns=LP(Nl+1);
for i=1:ns
    Q(ns,i)=1;
end
Y=zeros(ns,1);
Y(ns,1)=1;
X=linsolve(Q,Y);
%---------------- Calculation of performance measures ----------------
%   * Average inventory at wholesaler - WIP2
WIP2=0;
for L=1:Nl
    for j=LP(L)+1:LP(L+1)
        WIP2=WIP2+L*Q1*X(j);
    end
end
%   * Stockout probability - SO
SO=0;
Z1=dimensionZ1(1);
Z2=dimensionZ2(1);
O1=dimensionO1(1);
O2=dimensionO2(1);
SO=SO+X(Z1+Z2+1);
SO=SO+X(Z1+Z2+O1+O2+1);
for L=1:Nl
    Z1=dimensionZ1(L+1);
    Z2=dimensionZ2(L+1);
    O1=dimensionO1(L+1);
    O2=dimensionO2(L+1);
    if Z1>0 && L>=k
        SO=SO+X(LP(L)+1)+X(LP(L)+2);
    end
    if O1>0
    SO=SO+X(LP(L)+Z1+Z2+1)+X(LP(L)+Z1+Z2+2);
    SO=SO+X(LP(L)+Z1+Z2+O1+O2+1)+X(LP(L)+Z1+Z2+O1+O2+2);
    end
end
%   * WIP retailer
```

```
% IR(i) is the probability of I1 being i units
IR=zeros(r+Q1,1);
    %L=0
for i=1:dimensionZ1(1)+dimensionZ2(1)
    IR(s+i)=IR(s+i)+X(i);
end
for i=2:dimensionO1(1)+dimensionO2(1)
    IR(i-1)=IR(i-1)+X(dimensionZ1(1)+dimensionZ2(1)+i);
end
for i=2:dimensionO1(1)+dimensionO2(1)
    IR(i-1)=IR(i-
1)+X(dimensionZ1(1)+dimensionZ2(1)+dimensionO1(1)+dimensionO2(1)+i);
end
    %L=1 to NL
for L=1:Nl
    if L<k
        for i=1:2:dimensionZ1(L+1)-1
          IR(s-L*Q1+((i+1)/2))=IR(s-L*Q1+((i+1)/2))+
X(LP(L)+i)+X(LP(L)+i+1);
        end
    else
        for i=3:2:dimensionZ1(L+1)-1
           IR((i-1)/2)=IR((i-1)/2)+X(LP(L)+i)+X(LP(L)+i+1);
        end
    end
    if L==h
        for i=1:dimensionZ2(L+1)
            IR(s-L*Q1+i)=IR(s-L*Q1+i)+X(LP(L)+dimensionZ1(L+1)+i);
        end
    else
        for i=1:dimensionZ2(L+1)
            IR(r+i)=IR(r+i)+X(LP(L)+dimensionZ1(L+1)+i);
        end
    end
    for i=3:2:dimensionO1(L+1)
        IR((i-1)/2)=IR((i-
1)/2)+X(LP(L)+dimensionZ1(L+1)+dimensionZ2(L+1)+i)+X(LP(L)+dimensionZ
1(L+1)+dimensionZ2(L+1)+i+1);
    end
    for i=1:dimensionO2(L+1)
IR(r+i)=IR(r+i)+X(LP(L)+dimensionZ1(L+1)+dimensionZ2(L+1)+dimensionO1
(L+1)+i);
    end
    for i=3:2:dimensionO1(L+1)
        IR((i-1)/2)= IR((i-
1)/2)+X(LP(L)+dimensionZ1(L+1)+dimensionZ2(L+1)+dimensionO1(L+1)+dime
nsionO2(L+1)+i)+X(LP(L)+dimensionZ1(L+1)+dimensionZ2(L+1)+dimensionO1
(L+1)+dimensionO2(L+1)+i+1);
    end
    for i=1:dimensionO2(L+1)
IR(r+i)=IR(r+i)+X(LP(L)+dimensionZ1(L+1)+dimensionZ2(L+1)+dimensionO1
(L+1)+dimensionO2(L+1)+dimensionO1(L+1)+i);
    end
end
WIP1=0;
for i=1:r+Q1
    WIP1=WIP1+i*IR(i);
end
% IRO(i):the probability that the inventory at the retailer I1 is
equal to or % greater than i
IRO=zeros(r+Q1,1);
```

```
for i =1:r+Q1
    for j=i:r+Q1
        IRO(i)=IRO(i)+IR(j);
    end
end
% met(i):the probability that d=i and I1>=i
met=zeros(dmd);
for i=1:min(r+Q1,dmd)
    met(i)=dm(i)*IRO(i);
end
% unmet(i):the probability that I1=i and d>i
unmet=zeros(r+Q1);
for i=1:min(r+Q1,dmd)
    unmet(i)=IR(i)*(rd(i)-dm(i));
end
%   Throughput
Output=0;
for i=1:min(r+Q1,dmd)
    Output=Output+i*(met(i)+unmet(i));
end
Throughput=Output*l;
%   Order Fill Rate - OFR
OFR=0;
for i=1:min(r+Q1,dmd)
    OFR=OFR+met(i);
end
%   * Service level II - SL2
Ex=0;
for i=1:dmd
    Ex=Ex+i*dm(i);
end
SL2=Output/Ex;
%   Average lost sales
ALS=Ex-(SL2*Ex);
%   Average lost sales per lost order
Lost=ALS/(1-OFR);
%   WIP in transit to the retailer
utilization1=0;
for L=1:Nl
    for i=1:dimensionZ1(L+1)
        utilization1=utilization1+X(LP(L)+i);
    end
    for i=1:dimensionO1(L+1)

utilization1=utilization1+X(LP(L)+dimensionZ1(L+1)+dimensionZ2(L+1)+i
);
    end
    for i=1:dimensionO1(L+1)
utilization1=utilization1+X(LP(L)+dimensionZ1(L+1)+dimensionZ2(L+1)+d
imensionO1(L+1)+dimensionO2(L+1)+i);
    end
 end
Intransit1=utilization1*Q1;
% WIP in transit to the Wholesaler
utilization2=0;
    for i=dimensionZ1(1)+dimensionZ2(1)+1:dimension(1)
        utilization2=utilization2+X(i);
    end
for L=1:Nl
    for i=dimensionZ1(L+1)+dimensionZ2(L+1)+1:dimension(L+1)
        utilization2=utilization2+X(LP(L)+i);
```

```
    end
end
Intransit2=utilization2*Q2;
```

6.13.1.2 Functions

Diagonal0(Z1,Z2,m11,m12,l,d12,k,L,dm,rd,dmd)

```
function [D] = D0(Z1,Z2,m11,m12,l,d12,k,L,dm,rd,dmd)
D=zeros(Z1+Z2);
for i=1:2:Z1-1
    D(i,i)=-m11-l;
    D(i+1,i+1)=-m12-l;
    D(i,i+1)=d12*m11;
end
if L>=k && Z1>0
    D(1,1)=D(1,1)+l;
    D(2,2)=D(2,2)+l;
end
for i=1:Z1-2
    for j=1:min((Z1-i)/2,dmd)
        D(2*j+i,i)=dm(j)*l;
    end
end
if L>=k && Z1>0
    for i=1:2
        for j=1:min((Z1-i)/2,dmd)
            D(2*j+i,i)=rd(j)*l;
        end
    end
end
for i=Z1+1:Z1+Z2
    D(i,i)=-l;
end
for i=1:Z2-1
    for j=1:min((Z2-i),dmd)
        D(Z1+j+i,Z1+i)=dm(j)*l;
    end
end
for i=1:2:Z1-1
    for j=1:min(Z2,dmd-(Z1-i-1)/2)
        D(Z1+j,i)=dm((Z1-i-1)/2+j)*l;
    end
end
if L>=k
    for j=1:min(Z2,dmd-(Z1-2)/2)
        D(Z1+j,1)=rd((Z1-2)/2+j)*l;
    end
end
end
```

Upper(O1,O2,d22,m21)

```
function [U] = U( O1,O2,d22,m21 )
U=zeros(O1+O2);
for i=1:O1+O2
    U(i,i)=d22*m21;
end
end
```

Diagonal1(O1,O2,m11,m12,m21,l,d12,dm,rd,dmd)

```
function [D] = D1(O1,O2,m11,m12,m21,l,d12,dm,rd,dmd)
```

```
D=zeros(O1+O2);
if O1>0
    D(1,1)=-m21-m11;
    D(2,2)=-m21-m12;
    D(1,2)=d12*m11;
    for i=3:2:O1-1
        D(i,i)=-m21-m11-l;
        D(i+1,i+1)=-m21-m12-l;
        D(i,i+1)=d12*m11;
    end
    for i=1:2
        [...]
    end
    for i=3:O1-2
        for j=1:min((O1-i)/2,dmd)
            D(2*j+i,i)=dm(j)*l;
        end
    end
    for i=O1+1:O1+O2
        D(i,i)=-m21-l;
    end
    for i=1:O2-1
        for j=1:min((O2-i),dmd)
            D(O1+j+i,O1+i)=dm(j)*l;
        end
    end
    for j=1:min(O2,dmd-(O1-2)/2)
        D(O1+j,1)=rd((O1-2)/2+j)*l;
    end
    for i=3:2:O1-1
        for j=1:min(O2,dmd-(O1-i-1)/2)
            D(O1+j,i)=dm((O1-i-1)/2+j)*l;
        end
    end
end
end
```

Diagonal2(O1,O2,m11,m12,m22,l,d12,dm,rd,dmd)

```
function [D] = D2(O1,O2,m11,m12,m22,l,d12,dm,rd,dmd)
D=zeros(O1+O2);
if O1>0
    D(1,1)=-m22-m11;
    D(2,2)=-m22-m12;
    D(1,2)=d12*m11;
    for i=3:2:O1-1
        D(i,i)=-m22-m11-l;
        D(i+1,i+1)=-m22-m12-l;
        D(i,i+1)=d12*m11;
    end
    for i=1:2
        for j=1:min((O1-i)/2,dmd)
            D(2*j+i,i)=rd(j)*l;
        end
    end
    for i=3:O1-2
        for j=1:min((O1-i)/2,dmd)
            D(2*j+i,i)=dm(j)*l;
        end
    end
    for i=O1+1:O1+O2
        D(i,i)=-m22-l;
```

```
    end
    for i=1:O2-1
        for j=1:min((O2-i),dmd)
            D(O1+j+i,O1+i)=dm(j)*l;
        end
    end
    for j=1:min(O2,dmd-(O1-2)/2)
        D(O1+j,1)=rd((O1-2)/2+j)*l;
    end
    for i=3:2:O1-1
        for j=1:min(O2,dmd-(O1-i-1)/2)
            [...]
        end
    end
end
end
```

UpperZ(O1,O2,Z1,Z2,l,dm,rd,dmd)

[…]

UpperD1(Z1L2,Z2L2,O1L2,O2L2,O1L1,O2L1,d21,m21);

```
function [D] = UpperD1(Z1L2,Z2L2,O1L2,O2L2,O1L1,O2L1,d21,m21);
D=zeros(O1L1+O2L1,Z1L2+Z2L2+O1L2+O2L2);
for i=1:O1L2+O2L2
    D(i,Z1L2+Z2L2+i)=d21*m21;
end
for i=O1L2+O2L2+1:O1L1+O2L1
    D(i,i-O1L2-O2L2)=d21*m21;
end
end
```

UpperD2(Z1L2,Z2L2,O1L2,O2L2,O1L1,O2L1,m22)

```
function [D] = UpperD2(Z1L2,Z2L2,O1L2,O2L2,O1L1,O2L1,m22);
D=zeros(O1L1+O2L1,Z1L2+Z2L2+O1L2+O2L2);
for i=1:O1L2+O2L2
    D(i,Z1L2+Z2L2+i)=m22;
end
for i=O1L2+O2L2+1:O1L1+O2L1
    D(i,i-O1L2-O2L2)=m22;
end
end
```

LowerD0(Z1L1,Z1L2,Z2L2,d11,m11,m12,x,z);

```
function [D] = LowerD0(Z1L1,Z1L2,Z2L2,d11,m11,m12,x,z)
D=zeros(Z1L1,Z1L2+Z2L2);
for i=x+1:2:Z1L2-1
    D(i-x,i)=d11*m11;
    D(i+1-x,i)=m12;
end
for i=1:(Z1L1-max((Z1L2-x)*(Z1L2>0),0))/2
    D(max(Z1L2-x,0)+2*i-1,Z1L2+z+i)=d11*m11;
    D(max(Z1L2-x,0)+2*i,Z1L2+z+i)=m12;
end
end
```

LowerD1(O1L1,O1L2,O2L2,Q1,d11,m11,m12,r)

```
function [D] = LowerD1(O1L1,O1L2,O2L2,Q1,d11,m11,m12,r)
[...]
```

6.13.2 Validation data

λ=0.5, μ_{11}=2, μ_{12}=0.4, d_{12}=0.2, μ_{21}=1.25, μ_{22}=0.5, d_{22}=0.1, dm(1)=0.4, dm(2)=03, dm(3)=0.2, dm(4)=0.1. One replication of 1600000 time units.

	Input				Matlab							Arena						
	s	n	r	Q1	OFR	SL2	WIP1	WIP2	Intransit1	Intransit2	Throughput	OFR	SL2	WIP1	WIP2	Intransit1	Intransit2	Throughput
1	0	1	0	1	0.20000	0.25000	0.50000	0.25000	0.25000	0.25000	0.25000	0.200	0.250	0.499	0.251	0.251	0.250	0.250
2	0	1	0	2	0.35833	0.41667	1.00000	0.41667	0.41667	0.41667	0.41667	0.358	0.416	0.999	0.418	0.418	0.416	0.416
3	0	1	0	3	0.47692	0.52448	1.48951	0.52448	0.52448	0.52448	0.52448	0.476	0.524	1.487	0.527	0.527	0.524	0.524
4	0	1	0	4	0.55993	0.59453	1.96908	0.59453	0.59453	0.59453	0.59453	0.561	0.594	1.966	0.596	0.596	0.594	0.595
5	0	1	0	5	0.61145	0.64373	2.44814	0.64373	0.64373	0.64373	0.64373	0.612	0.644	2.450	0.643	0.643	0.644	0.644
6	0	2	0	1	0.22857	0.28571	0.57143	0.71429	0.28571	0.28571	0.28571	0.228	0.286	0.571	0.714	0.286	0.287	0.286
7	0	2	0	2	0.40000	0.46512	1.11628	1.34884	0.46512	0.46512	0.46512	0.401	0.466	1.115	1.350	0.467	0.464	0.466
8	0	2	0	3	0.52261	0.57471	1.63218	1.93103	0.57471	0.57471	0.57471	0.523	0.575	1.634	1.931	0.573	0.573	0.574
9	0	2	0	4	0.60488	0.64226	2.12717	2.48170	0.64226	0.64226	0.64226	0.605	0.643	2.127	2.483	0.645	0.641	0.643
10	0	2	0	5	0.65352	0.68802	2.61658	3.01602	0.68802	0.68802	0.68802	0.653	0.687	2.616	3.016	0.688	0.689	0.689
11	0	3	0	1	0.24000	0.30000	0.60000	1.20000	0.30000	0.30000	0.30000	0.240	0.300	0.600	1.202	0.301	0.299	0.300
12	0	3	0	2	0.41613	0.48387	1.16129	2.32258	0.48387	0.48387	0.48387	0.418	0.485	1.161	2.322	0.483	0.485	0.485
13	0	3	0	3	0.53984	0.59367	1.68602	3.39578	0.59367	0.59367	0.59367	0.540	0.594	1.687	3.396	0.593	0.592	0.593
14	0	3	0	4	0.62151	0.65992	2.18566	4.43995	0.65992	0.65992	0.65992	0.622	0.660	2.186	4.439	0.657	0.663	0.659
15	0	3	0	5	0.66886	0.70417	2.67800	5.46945	0.70417	0.70417	0.70417	0.670	0.706	2.679	5.466	0.703	0.702	0.704
16	0	4	0	1	0.24615	0.30769	0.61538	1.69231	0.30769	0.30769	0.30769	0.247	0.308	0.615	1.692	0.308	0.307	0.308
17	0	4	0	2	0.42469	0.49383	1.18519	3.30864	0.49383	0.49383	0.49383	0.425	0.494	1.185	3.312	0.494	0.494	0.494
18	0	4	0	3	0.54889	0.60362	1.71429	4.87726	0.60362	0.60362	0.60362	0.548	0.602	1.713	4.884	0.606	0.602	0.603
19	0	4	0	4	0.63018	0.66912	2.21613	6.41820	0.66912	0.66912	0.66912	0.630	0.669	2.217	6.416	0.670	0.667	0.669
20	0	4	0	5	0.67681	0.71253	2.70980	7.94533	0.71253	0.71253	0.71253	0.677	0.713	2.710	7.943	0.713	0.710	0.711
21	0	5	0	1	0.25000	0.31250	0.62500	2.18750	0.31250	0.31250	0.31250	0.250	0.313	0.625	2.187	0.313	0.309	0.313
22	0	5	0	2	0.43000	0.50000	1.20000	4.30000	0.50000	0.50000	0.50000	0.431	0.500	1.200	4.295	0.499	0.501	0.501
23	0	5	0	3	0.55447	0.60976	1.73171	6.36585	0.60976	0.60976	0.60976	0.554	0.610	1.732	6.367	0.609	0.608	0.610
24	0	5	0	4	0.63549	0.67476	2.23482	8.40486	0.67476	0.67476	0.67476	0.635	0.675	2.234	8.401	0.675	0.675	0.674
25	0	5	0	5	0.68166	0.71765	2.72925	10.43059	0.71765	0.71765	0.71765	0.682	0.718	2.728	10.412	0.719	0.717	0.718
26	1	1	0	1	0.24860	0.31075	0.62151	1.00000	0.31075	0.31075	0.31075	0.249	0.311	0.621	1.001	0.312	0.311	0.311
27	1	1	0	2	0.37862	0.44025	1.05660	0.67925	0.44025	0.44025	0.44025	0.379	0.440	1.056	0.680	0.441	0.441	0.441
28	1	1	0	3	0.50292	0.55307	1.57071	0.89834	0.55307	0.55307	0.55307	0.504	0.554	1.570	0.900	0.555	0.552	0.554
29	1	1	0	4	0.58826	0.62461	2.06871	1.05184	0.62461	0.62461	0.62461	0.588	0.624	2.068	1.055	0.626	0.625	0.625
30	1	1	0	5	0.63893	0.67266	2.55815	1.14700	0.67266	0.67266	0.67266	0.639	0.673	2.558	1.146	0.674	0.673	0.673
31	1	1	1	1	0.32878	0.38657	0.90328	0.54249	0.38657	0.38657	0.38657	0.329	0.386	0.902	0.544	0.387	0.387	0.387
32	1	1	1	2	0.42350	0.48088	1.23632	0.48088	0.48088	0.48088	0.48088	0.423	0.480	1.233	0.485	0.485	0.480	0.481
33	1	1	1	3	0.55549	0.59896	1.82458	0.59896	0.59896	0.59896	0.59896	0.555	0.598	1.824	0.600	0.600	0.599	0.599

34	1	1	1	4	0.63882	0.67035	2.37492	0.67035	0.67035	0.67035	0.67035	0.639	0.671	2.373	0.672	0.672	0.669	0.671
35	1	1	1	5	0.68694	0.71582	2.89044	0.71582	0.71582	0.71582	0.71582	0.687	0.716	2.889	0.717	0.717	0.716	0.717
36	1	2	0	1	0.25732	0.32165	0.64330	1.51753	0.32165	0.32165	0.32165	0.258	0.322	0.643	1.517	0.322	0.323	0.322
37	1	2	0	2	0.41233	0.47945	1.15068	1.65068	0.47945	0.47945	0.47945	0.414	0.480	1.149	1.653	0.480	0.481	0.480
38	1	2	0	3	0.53784	0.59147	1.67976	2.35657	0.59147	0.59147	0.59147	0.538	0.592	1.681	2.350	0.589	0.594	0.591
39	1	2	0	4	0.62104	0.65941	2.18398	2.99901	0.65941	0.65941	0.65941	0.621	0.659	2.184	2.996	0.662	0.658	0.659
40	1	2	0	5	0.66890	0.70421	2.67813	3.58355	0.70421	0.70421	0.70421	0.670	0.704	2.679	3.584	0.703	0.707	0.705
41	1	2	1	1	0.36850	0.43078	1.02071	0.95498	0.43078	0.43078	0.43078	0.368	0.430	1.021	0.955	0.431	0.431	0.431
42	1	2	1	2	0.47905	0.54086	1.41308	1.39574	0.54086	0.54086	0.54086	0.480	0.541	1.413	1.400	0.543	0.539	0.541
43	1	2	1	3	0.61314	0.65802	2.03732	1.97759	0.65802	0.65802	0.65802	0.614	0.658	2.037	1.979	0.659	0.657	0.659
44	1	2	1	4	0.69173	0.72386	2.60434	2.52188	0.72386	0.72386	0.72386	0.692	0.724	2.603	2.517	0.723	0.722	0.723
45	1	2	1	5	0.73487	0.76393	3.12667	3.04937	0.76393	0.76393	0.76393	0.735	0.764	3.126	3.054	0.768	0.762	0.764
46	1	3	0	1	0.26036	0.32545	0.65090	2.02365	0.32545	0.32545	0.32545	0.261	0.326	0.651	2.023	0.326	0.325	0.325
47	1	3	0	2	0.42494	0.49412	1.18588	2.64000	0.49412	0.49412	0.49412	0.426	0.494	1.184	2.643	0.495	0.493	0.495
48	1	3	0	3	0.55058	0.60548	1.71956	3.84133	0.60548	0.60548	0.60548	0.551	0.606	1.721	3.839	0.605	0.603	0.605
49	1	3	0	4	0.63279	0.67189	2.22531	4.98007	0.67189	0.67189	0.67189	0.633	0.672	2.224	4.980	0.673	0.671	0.672
50	1	3	0	5	0.67952	0.71539	2.72066	6.06105	0.71539	0.71539	0.71539	0.681	0.717	2.725	6.057	0.710	0.711	0.715
51	1	3	1	1	0.38903	0.45406	1.08000	1.48452	0.45406	0.45406	0.45406	0.389	0.454	1.079	1.487	0.454	0.456	0.453
52	1	3	1	2	0.50075	0.56427	1.48229	2.36183	0.56427	0.56427	0.56427	0.502	0.565	1.484	2.366	0.565	0.563	0.564
53	1	3	1	3	0.63494	0.68033	2.11793	3.43093	0.68033	0.68033	0.68033	0.636	0.681	2.117	3.430	0.681	0.681	0.680
54	1	3	1	4	0.71128	0.74362	2.68920	4.46665	0.74362	0.74362	0.74362	0.711	0.743	2.685	4.465	0.747	0.746	0.743
55	1	3	1	5	0.75233	0.78148	3.21257	5.48949	0.78148	0.78148	0.78148	0.754	0.783	3.220	5.474	0.775	0.781	0.780
56	1	4	0	1	0.26191	0.32739	0.65477	2.52676	0.32739	0.32739	0.32739	0.263	0.328	0.655	2.528	0.327	0.328	0.327
57	1	4	0	2	0.43154	0.50179	1.20430	3.63441	0.50179	0.50179	0.50179	0.432	0.502	1.203	3.638	0.502	0.500	0.502
58	1	4	0	3	0.55718	0.61274	1.74017	5.33343	0.61274	0.61274	0.61274	0.557	0.612	1.738	5.337	0.614	0.613	0.612
59	1	4	0	4	0.63883	0.67831	2.24657	6.97032	0.67831	0.67831	0.67831	0.639	0.679	2.247	6.970	0.677	0.678	0.678
60	1	4	0	5	0.68496	0.72112	2.74243	8.54954	0.72112	0.72112	0.72112	0.686	0.722	2.741	8.564	0.719	0.719	0.722
61	1	4	1	1	0.39813	0.46425	1.10669	1.96514	0.46425	0.46425	0.46425	0.399	0.465	1.105	1.968	0.465	0.465	0.465
62	1	4	1	2	0.51233	0.57674	1.51919	3.34372	0.57674	0.57674	0.57674	0.514	0.577	1.520	3.343	0.577	0.574	0.576
63	1	4	1	3	0.64640	0.69206	2.16032	4.90636	0.69206	0.69206	0.69206	0.647	0.692	2.159	4.911	0.693	0.690	0.692
64	1	4	1	4	0.72146	0.75392	2.73339	6.43788	0.75392	0.75392	0.75392	0.722	0.754	2.732	6.444	0.754	0.754	0.754
65	1	4	1	5	0.76136	0.79055	3.25702	7.95848	0.79055	0.79055	0.79055	0.762	0.791	3.256	7.967	0.792	0.788	0.790
66	1	5	0	1	0.26285	0.32856	0.65712	3.02865	0.32856	0.32856	0.32856	0.263	0.329	0.656	3.027	0.329	0.329	0.329
67	1	5	0	2	0.43560	0.50651	1.21563	4.63097	0.50651	0.50651	0.50651	0.437	0.508	1.217	4.625	0.505	0.504	0.506
68	1	5	0	3	0.56122	0.61718	1.75278	6.82860	0.61718	0.61718	0.61718	0.560	0.617	1.752	6.825	0.619	0.616	0.618
69	1	5	0	4	0.64252	0.68222	2.25952	8.96439	0.68222	0.68222	0.68222	0.643	0.682	2.260	8.962	0.680	0.686	0.682
70	1	5	0	5	0.68826	0.72460	2.75567	11.04254	0.72460	0.72460	0.72460	0.689	0.725	2.756	11.053	0.726	0.717	0.725
71	1	5	1	1	0.40444	0.47137	1.12502	2.46895	0.47137	0.47137	0.47137	0.405	0.472	1.124	2.468	0.472	0.471	0.471
72	1	5	1	2	0.51952	0.58450	1.54212	4.33246	0.58450	0.58450	0.58450	0.519	0.584	1.541	4.334	0.586	0.585	0.584

73	1	5	1	3	0.65347	0.69929	2.18647	6.39121	0.69929	0.69929	0.69929	0.655	0.700	2.189	6.395	0.698	0.695	0.698
74	1	5	1	4	0.72771	0.76023	2.76050	8.42023	0.76023	0.76023	0.76023	0.727	0.760	2.759	8.419	0.760	0.759	0.759
75	1	5	1	5	0.76689	0.79610	3.28419	10.43953	0.79610	0.79610	0.79610	0.766	0.795	3.279	10.447	0.799	0.805	0.795
76	2	1	0	1	0.26085	0.32606	0.65213	1.91338	0.32606	0.32606	0.32606	0.261	0.326	0.651	1.914	0.327	0.327	0.327
77	2	1	0	2	0.43370	0.50430	1.21033	2.00000	0.50430	0.50430	0.50430	0.434	0.505	1.209	1.999	0.505	0.506	0.506
78	2	1	0	3	0.52112	0.57308	1.62754	1.28077	0.57308	0.57308	0.57308	0.521	0.573	1.626	1.281	0.575	0.574	0.573
79	2	1	0	4	0.60897	0.64660	2.14154	1.54292	0.64660	0.64660	0.64660	0.609	0.647	2.143	1.541	0.647	0.647	0.647
80	2	1	0	5	0.66012	0.69497	2.64300	1.71974	0.69497	0.69497	0.69497	0.661	0.695	2.643	1.722	0.695	0.695	0.694
81	2	1	1	1	0.37880	0.44340	1.04736	1.19634	0.44340	0.44340	0.44340	0.378	0.443	1.045	1.196	0.444	0.444	0.444
82	2	1	1	2	0.49831	0.56170	1.47417	1.58766	0.56170	0.56170	0.56170	0.499	0.562	1.473	1.586	0.562	0.564	0.562
83	2	1	1	3	0.58514	0.62956	1.93239	0.98425	0.62956	0.62956	0.62956	0.585	0.630	1.931	0.984	0.632	0.630	0.630
84	2	1	1	4	0.67055	0.70269	2.50843	1.15285	0.70269	0.70269	0.70269	0.671	0.703	2.508	1.154	0.705	0.702	0.703
85	2	1	1	5	0.71817	0.74743	3.03952	1.26306	0.74743	0.74743	0.74743	0.718	0.747	3.038	1.266	0.748	0.749	0.747
86	2	1	2	1	0.42390	0.47562	1.27975	0.84539	0.47562	0.47562	0.47562	0.423	0.475	1.280	0.845	0.475	0.475	0.476
87	2	1	2	2	0.59119	0.62963	2.03704	0.84443	0.62963	0.62963	0.62963	0.591	0.630	2.035	0.845	0.630	0.630	0.630
88	2	1	2	3	0.62833	0.66422	2.26852	0.66422	0.66422	0.66422	0.66422	0.628	0.663	2.265	0.667	0.667	0.664	0.665
89	2	1	2	4	0.71097	0.73585	2.90851	0.73585	0.73585	0.73585	0.73585	0.711	0.736	2.905	0.740	0.740	0.734	0.736
90	2	1	2	5	0.75633	0.77846	3.47811	0.77846	0.77846	0.77846	0.77846	0.755	0.778	3.478	0.781	0.781	0.779	0.777
91	2	2	0	1	0.26416	0.33020	0.66040	2.49530	0.33020	0.33020	0.33020	0.265	0.331	0.660	2.495	0.330	0.330	0.331
92	2	2	0	2	0.44296	0.51508	1.23618	3.02136	0.51508	0.51508	0.51508	0.444	0.515	1.235	3.022	0.517	0.517	0.516
93	2	2	0	3	0.54807	0.60272	1.71172	2.76943	0.60272	0.60272	0.60272	0.549	0.604	1.713	2.769	0.602	0.603	0.603
94	2	2	0	4	0.63239	0.67147	2.22390	3.52534	0.67147	0.67147	0.67147	0.632	0.671	2.223	3.525	0.671	0.670	0.672
95	2	2	0	5	0.68033	0.71624	2.72390	4.19585	0.71624	0.71624	0.71624	0.681	0.717	2.726	4.186	0.715	0.719	0.717
96	2	2	1	1	0.40701	0.47458	1.13146	1.86346	0.47458	0.47458	0.47458	0.408	0.475	1.132	1.862	0.474	0.476	0.475
97	2	2	1	2	0.52180	0.58701	1.54914	2.57041	0.58701	0.58701	0.58701	0.523	0.588	1.549	2.570	0.588	0.586	0.587
98	2	2	1	3	0.63084	0.67627	2.10184	2.42588	0.67627	0.67627	0.67627	0.631	0.677	2.101	2.427	0.677	0.677	0.676
99	2	2	1	4	0.70998	0.74245	2.68135	3.07689	0.74245	0.74245	0.74245	0.710	0.743	2.679	3.080	0.744	0.743	0.742
100	2	2	1	5	0.75236	0.78163	3.21027	3.67315	0.78163	0.78163	0.78163	0.753	0.782	3.208	3.673	0.785	0.780	0.783
101	2	2	2	1	0.48015	0.53266	1.47729	1.41658	0.53266	0.53266	0.53266	0.481	0.533	1.478	1.419	0.533	0.531	0.532
102	2	2	2	2	0.64374	0.68007	2.25510	1.64330	0.68007	0.68007	0.68007	0.644	0.680	2.254	1.649	0.682	0.677	0.681
103	2	2	2	3	0.69252	0.72623	2.57252	2.00809	0.72623	0.72623	0.72623	0.692	0.726	2.571	2.011	0.730	0.724	0.726
104	2	2	2	4	0.76608	0.78886	3.22480	2.54143	0.78886	0.78886	0.78886	0.767	0.789	3.224	2.534	0.787	0.794	0.788
105	2	2	2	5	0.80435	0.82431	3.79431	3.05839	0.82431	0.82431	0.82431	0.805	0.824	3.797	3.051	0.822	0.824	0.823
106	2	3	0	1	0.26499	0.33124	0.66247	3.00000	0.33124	0.33124	0.33124	0.266	0.332	0.662	3.001	0.332	0.331	0.331
107	2	3	0	2	0.44614	0.51877	1.24504	4.02868	0.51877	0.51877	0.51877	0.447	0.519	1.245	4.025	0.520	0.522	0.520
108	2	3	0	3	0.55769	0.61329	1.74175	4.26538	0.61329	0.61329	0.61329	0.559	0.613	1.742	4.273	0.614	0.610	0.613

6.13.3 Numerical Results data

Effect of r: λ=0.5, μ_{11}=1, μ_{12}=0.2, d_{12}=0.2, μ_{21}=1, μ_{22}=0.2, d_{22}=0.2, dm=(0.6, 0.4)

s	n	r	Q_1	Q_2	OFR	SL	WIP retailer	WIP Wholealer	WIP total	utilization1	utilization2	Lost saler per lost customer
6	1	0	6	6	0.751173	0.765698	2.90909012	6	8.90909012	0.17866294	0.178662945	1.31827649
6	1	1	6	6	0.814576	0.821721	3.433781123	5.221206244	8.654987367	0.19173497	0.191734966	1.346047642
6	1	2	6	6	0.850963	0.856472	4.105624179	4.395175858	8.500800037	0.19984354	0.199843535	1.348243353
6	1	3	6	6	0.874851	0.878757	4.821469213	3.580326563	8.401795776	0.20504339	0.205043393	1.356299571
6	1	4	6	6	0.890226	0.893443	5.55286381	2.78428425	8.33714806	0.20847015	0.20847015	1.358971848
6	1	5	6	6	0.89939	0.902202	6.180990115	2.117723661	8.298713776	0.2105137	0.210513705	1.360876806
6	1	6	6	6	0.905606	0.90819	6.773691141	1.498848939	8.27254008	0.21191106	0.21191106	1.36166886
6	2	0	3	6	0.605229	0.630797	1.460085821	7.483635593	8.943721414	0.29437214	0.147186071	1.309325499
6	2	1	3	6	0.707407	0.719796	1.949928162	6.754459112	8.704387273	0.33590465	0.167952327	1.340724659
6	2	2	3	6	0.764895	0.77463	2.567558759	5.991346245	8.558905003	0.36149403	0.180747014	1.342031053
6	2	3	3	6	0.813172	0.819486	3.381883415	5.060479508	8.442362923	0.38242671	0.191213357	1.352683756
6	2	4	3	6	0.837344	0.842532	4.033470701	4.347753125	8.381223826	0.39318167	0.196590836	1.35534447
6	2	5	3	6	0.8538	0.858232	4.655194598	3.686471844	8.341666441	0.40050848	0.20025424	1.357556125
6	2	6	3	6	0.865359	0.869351	5.246652483	3.06817457	8.314827053	0.40569726	0.202848631	1.358489493
6	3	0	2	6	0.519423	0.545612	0.993013639	7.986027278	8.979040917	0.38192832	0.127309441	1.323706124
6	3	1	2	6	0.618574	0.63308	1.377199803	7.401011375	8.778211178	0.44315578	0.147718593	1.346757252
6	3	2	2	6	0.705505	0.716503	2.089483677	6.510141892	8.599625569	0.50155212	0.167184039	1.347714958
6	3	3	2	6	0.747349	0.755588	2.633054024	5.880717681	8.513771705	0.52891146	0.176303821	1.354348192
6	3	4	2	6	0.780492	0.78729	3.278329557	5.168673082	8.447002639	0.55110296	0.183700988	1.356643448
6	3	5	2	6	0.80045	0.80639	3.814181655	4.593878058	8.408059714	0.56447285	0.188157617	1.358325977
6	3	6	2	6	0.81511	0.820519	4.329556197	4.051464865	8.381021062	0.57436308	0.191454359	1.359048313
6	6	0	1	6	0.299631	0.356704	0.499385088	8.496925441	8.996310529	0.49938509	0.083230848	1.285914967
6	6	1	1	6	0.457674	0.484843	0.867803343	7.938938819	8.806742162	0.67878078	0.11313013	1.329864074
6	6	2	1	6	0.521354	0.54525	1.226002331	7.490390948	8.716393279	0.76335065	0.127225108	1.330104761
6	6	3	1	6	0.561519	0.581915	1.552041266	7.109703145	8.661744411	0.81468044	0.135780073	1.334880673
6	6	4	1	6	0.58716	0.605984	1.845783137	6.780280497	8.626063634	0.84837698	0.141396163	1.336165414
6	6	5	1	6	0.604452	0.622246	2.099125216	6.503622583	8.602747799	0.87114506	0.145190843	1.337019761
6	6	6	1	6	0.615493	0.632701	2.299981719	6.289328243	8.589309962	0.88578165	0.147630275	1.337344171

Effect of Q_1: $\lambda=0.5$, $\mu_{11}=1$, $\mu_{12}=0.2$, $d_{12}=0.2$, $\mu_{21}=1$, $\mu_{22}=0.2$, $d_{22}=0.2$, dm=(0.6, 0.4)

s	n	r	Q_1	Q_2	OFR	SL	WIP retailer	WIP Wholealer	WIP total	utilization1	utilization2	Lost saler per lost customer
4	1	0	12	12	0.845512	0.853642	5.798378969	3.732025525	9.530404494	0.099591588	0.099591588	1.326326096
4	2	0	6	12	0.748111	0.762576	2.897228736	6.2941966	9.191425336	0.177934473	0.088967237	1.319599354
4	3	0	4	12	0.674938	0.694266	1.947833684	8.057677933	10.00551162	0.242993224	0.080997741	1.316755192
4	4	0	3	12	0.604063	0.629582	1.457272142	8.227545504	9.684817646	0.293804867	0.073451217	1.30976682
4	6	0	2	12	0.518894	0.545057	0.9920038	9.023988601	10.0159924	0.381539923	0.063589987	1.323867391
4	12	0	1	12	0.299356	0.356376	0.498926964	9.50536518	10.00429214	0.498926964	0.041577247	1.286064343
4	1	1	12	12	0.882115	0.886331	6.334311772	3.098423465	9.432735237	0.10340524	0.10340524	1.349935018
4	2	1	6	12	0.81294	0.820169	3.423711731	5.770234947	9.193946678	0.19137279	0.095686395	1.345894459
4	3	1	4	12	0.757338	0.767011	2.443277673	7.341639211	9.784916884	0.268453773	0.089484591	1.344195211
4	4	1	3	12	0.707007	0.719428	1.948279161	7.597451929	9.54573109	0.335732907	0.083933227	1.34065164
4	6	1	2	12	0.618072	0.63259	1.375796439	8.460190064	9.835986502	0.442812992	0.073802165	1.346780932
4	12	1	1	12	0.456964	0.484126	0.866392814	8.966488383	9.832881197	0.677776329	0.056481361	1.329972676
4	1	2	12	12	0.901375	0.904753	6.959821133	2.418873896	9.37869503	0.105554535	0.105554535	1.352050123
4	2	2	6	12	0.850415	0.856018	4.092511044	5.116391048	9.208902092	0.199737571	0.099868785	1.347560761
4	3	2	4	12	0.808988	0.816408	3.124998743	6.514804675	9.639803418	0.285742661	0.095247554	1.345617786
4	4	2	3	12	0.764521	0.77432	2.562813662	6.910987436	9.473801098	0.361349169	0.090337292	1.341743321
4	6	2	2	12	0.705384	0.716413	2.087690856	7.59114042	9.678831276	0.501489449	0.083581575	1.347587231
4	12	2	1	12	0.520424	0.544333	1.223227223	8.531513946	9.754741169	0.76206678	0.063505565	1.330202577
4	1	3	12	12	0.912345	0.915026	7.544442463	1.803950253	9.348392716	0.106753015	0.106753015	1.357181156
4	2	3	6	12	0.874388	0.878443	4.783873454	4.438255639	9.222129093	0.204970028	0.102485014	1.354810794
4	3	3	4	12	0.841941	0.84717	3.822506318	5.724941038	9.547447356	0.296509617	0.098836539	1.353677409
4	4	3	3	12	0.812512	0.818952	3.365286263	6.051359279	9.416645542	0.382177707	0.095544427	1.351908225
4	6	3	2	12	0.746739	0.755069	2.622583542	6.982465659	9.605049201	0.528548599	0.088091433	1.353948427
4	12	3	1	12	0.560177	0.58063	1.545833378	8.163030396	9.708863773	0.812881735	0.067740145	1.334895187
4	1	4	12	12	0.918188	0.920616	8.043133163	1.288862379	9.331995541	0.107405198	0.107405198	1.358453943
4	2	4	6	12	0.889544	0.892939	5.474841697	3.762050693	9.23689239	0.208352386	0.104176193	1.356975416
4	3	4	4	12	0.866186	0.870346	4.626311762	4.85489691	9.481208672	0.304621061	0.101540354	1.356474383
4	4	4	3	12	0.836354	0.841707	3.993212561	5.395940788	9.389153349	0.392796662	0.098199165	1.354199575
4	6	4	2	12	0.778962	0.785912	3.247083281	6.300522391	9.547605672	0.550138619	0.09168977	1.355979832
4	12	4	1	12	0.584736	0.603697	1.828041907	7.851395856	9.679437763	0.845175666	0.070431305	1.336075576

Effect of *s*: λ=0.5, μ_{11}=1, μ_{12}=0.2, d_{12}=0.2, μ_{21}=1, μ_{22}=0.2, d_{22}=0.2, dm=(0.6, 0.4)

s	n	r	Q1	Q2	OFR	SL	WIP retailer	WIP Wholealer	WIP total	utilization1	utilization2	Lost saler per lost customer
1	6	2	1	6	0.465704	0.490623	1.045117363	2.929495236	3.974612599	0.686871836	0.114478639	1.334707059
2	6	2	1	6	0.493346	0.51765	1.142884924	3.661372248	4.804257173	0.724710069	0.120785012	1.332843498
3	6	2	1	6	0.507109	0.531097	1.184849388	4.553513444	5.738362832	0.743535828	0.123922638	1.331864359
4	6	2	1	6	0.514339	0.538253	1.206155675	5.506335114	6.712490789	0.753553835	0.125592306	1.331064772
5	6	2	1	6	0.518614	0.542512	1.218326131	6.486594844	7.704920975	0.7595168	0.126586133	1.330499337
6	6	2	1	6	0.521354	0.54525	1.226002331	7.490390948	8.716393279	0.76335065	0.127225108	1.330104761
7	6	2	1	6	0.523093	0.546992	1.230840648	8.468294955	9.699135603	0.765788108	0.127631351	1.329844497
8	6	2	1	6	0.524257	0.548157	1.23406809	9.45549658	10.68956467	0.767419928	0.127903321	1.329667208
9	6	2	1	6	0.525039	0.54894	1.236234028	10.44819348	11.68442751	0.768516663	0.128086111	1.329547022
10	6	2	1	6	0.525565	0.549468	1.237690629	11.44406856	12.68175919	0.769254662	0.12820911	1.329465777
11	6	2	1	6	0.525919	0.549822	1.238670856	12.44186122	13.68053208	0.76975142	0.128291903	1.329410949
12	6	2	1	6	0.526157	0.550061	1.239330772	13.44105547	14.68038624	0.770085884	0.128347647	1.329373976
1	3	2	2	6	0.600875	0.613893	1.631632025	2.133215638	3.764847663	0.42972539	0.143241797	1.354336172
2	3	2	2	6	0.652703	0.664628	1.880979571	2.877713129	4.758692701	0.465239851	0.15507995	1.351926816
3	3	2	2	6	0.671366	0.682824	1.954808635	3.416404675	5.37121331	0.477976868	0.159325623	1.351188088
4	3	2	2	6	0.690294	0.701452	2.031905971	4.569233961	6.601139932	0.491016487	0.163672162	1.349561742
5	3	2	2	6	0.697282	0.708356	2.058736671	5.216441158	7.27517783	0.495849515	0.165283172	1.348785337
6	3	2	2	6	0.705505	0.716503	2.089483677	6.510141892	8.599625569	0.501552116	0.167184039	1.347714958
7	3	2	2	6	0.70871	0.719684	2.101212192	7.167504461	9.268716653	0.503778499	0.167926166	1.347256646
8	3	2	2	6	0.712637	0.723585	2.115437533	8.424919014	10.54035655	0.506509166	0.168836389	1.346665384
9	3	2	2	6	0.714346	0.725283	2.121590323	9.104895828	11.22648615	0.507698114	0.169232705	1.346399134
10	3	2	2	6	0.716473	0.727396	2.12922205	10.38787254	12.51709459	0.509177348	0.169725783	1.346061233
11	3	2	2	6	0.717405	0.728323	2.132561934	11.07696353	13.20952547	0.509825812	0.169941937	1.345910976
12	3	2	2	6	0.718571	0.729482	2.136737711	12.37384228	14.51057999	0.510637196	0.170212399	1.345721267
1	2	2	3	6	0.647189	0.659962	1.932349234	1.91373333	3.846082564	0.307982404	0.153991202	1.349315787
2	2	2	3	6	0.690494	0.702088	2.203295796	2.175192619	4.378488415	0.327640931	0.163820465	1.347557919
3	2	2	3	6	0.726905	0.737399	2.38288373	3.240267016	5.623150746	0.344119522	0.172059761	1.346204344
4	2	2	3	6	0.743706	0.753818	2.469599994	3.916414757	6.386014752	0.351781645	0.175890823	1.344762846
5	2	2	3	6	0.754185	0.764084	2.519309299	4.592305439	7.111614738	0.35657253	0.178286265	1.343622767
6	2	2	3	6	0.764895	0.77463	2.567558759	5.991346245	8.558905003	0.361494028	0.180747014	1.342031053
7	2	2	3	6	0.769754	0.77942	2.589064719	6.743156295	9.332221013	0.363729454	0.181864727	1.341225758
8	2	2	3	6	0.773291	0.782912	2.604519777	7.463339899	10.06785968	0.365358727	0.182679363	1.340591856
9	2	2	3	6	0.777284	0.786857	2.621749788	8.843738069	11.46548786	0.367199869	0.183599935	1.339826923
10	2	2	3	6	0.779304	0.788853	2.630443279	9.638152782	12.26859606	0.368131522	0.184065761	1.339426562
11	2	2	3	6	0.780805	0.790337	2.636886154	10.38554931	13.02243546	0.368823728	0.184411864	1.339122613
12	2	2	3	6	0.782539	0.792051	2.644316279	11.79943024	14.44374652	0.369623658	0.184811829	1.33876448

Effect of Q_2: λ=0.5, μ_{11}=1, μ_{12}=0.2, d_{12}=0.2, μ_{21}=1, μ_{22}=0.2, d_{22}=0.2, dm=(0.6, 0.4)

s	n	r	Q1	Q2	OFR	SL	WIP retailer	WIP Wholealer	WIP total	utilization1	utilization2	Lost saler per lost customer
1	2	2	2	4	0.554036	0.567876	1.430687797	1.280433318	2.711121115	0.397513393	0.198756696	1.356553191
1	3	2	2	6	0.600875	0.613893	1.631632025	2.133215638	3.764847663	0.42972539	0.143241797	1.354336172
1	4	2	2	8	0.627149	0.639703	1.744006798	3.050453276	4.794460075	0.447792439	0.11194811	1.352859066
1	5	2	2	10	0.643997	0.656254	1.81605776	3.99736616	5.81342392	0.459378143	0.091875629	1.351797464
1	6	2	2	12	0.655722	0.667773	1.866198485	4.960421456	6.826619941	0.467440766	0.077906794	1.35099738
1	7	2	2	14	0.664352	0.67625	1.903104524	5.933228292	7.836332816	0.473375254	0.067625036	1.350372766
1	8	2	2	16	0.67097	0.682751	1.931404933	6.91237594	8.843780873	0.477925958	0.059740745	1.349871601
1	9	2	2	18	0.676206	0.687895	1.953795379	7.895878175	9.849673555	0.48152634	0.053502927	1.349460578
1	10	2	2	20	0.680452	0.692066	1.971952101	8.882499911	10.85445201	0.48444594	0.048444594	1.349117384
2	1	2	2	2	0.557481	0.570656	1.50877134	1.118839647	2.627610986	0.399459227	0.399459227	1.358318484
2	2	2	2	4	0.62285	0.635221	1.765020589	1.914045156	3.679065745	0.444654747	0.222327374	1.354076491
2	3	2	2	6	0.652703	0.664628	1.880979571	2.877713129	4.758692701	0.465239851	0.15507995	1.351926816
2	4	2	2	8	0.668601	0.680287	1.942538245	3.85643848	5.798976724	0.476201055	0.119050264	1.350630826
2	5	2	2	10	0.678502	0.690039	1.980868379	4.843092913	6.823961293	0.483027224	0.096605445	1.349759195
2	6	2	2	12	0.685261	0.696697	2.007037616	5.833977046	7.841014662	0.487687708	0.081281285	1.349132606
2	7	2	2	14	0.69017	0.701532	2.026041229	6.827357085	8.853398314	0.491072067	0.070153152	1.348660453
2	8	2	2	16	0.693896	0.705202	2.040467933	7.822331494	9.862799427	0.493641322	0.061705165	1.348291903
2	9	2	2	18	0.696822	0.708083	2.051793496	8.818386196	10.87017969	0.495658294	0.055073144	1.347996229
2	10	2	2	20	0.699179	0.710405	2.060920802	9.815206669	11.87612747	0.497283779	0.049728378	1.347753759
3	1	2	2	2	0.590021	0.602345	1.639455739	1.46067734	3.100133079	0.421641586	0.421641586	1.357915543
3	2	2	2	4	0.64839	0.66011	1.866253096	2.414486544	4.28073964	0.462076985	0.231038493	1.353333088
3	3	2	2	6	0.671366	0.682824	1.954808635	3.416404675	5.37121331	0.477976868	0.159325623	1.351188088
3	4	2	2	8	0.683177	0.694502	2.000169203	4.41129239	6.411461593	0.486151108	0.121537777	1.349958096
3	5	2	2	10	0.690451	0.701693	2.02809876	5.407844057	7.435942817	0.491185139	0.098237028	1.349153716
3	6	2	2	12	0.695384	0.70657	2.047040743	6.405492042	8.452532784	0.494599278	0.082433213	1.348586307
3	7	2	2	14	0.698949	0.710096	2.06073271	7.403791352	9.464524062	0.497067146	0.071009592	1.348164585
3	8	2	2	16	0.701647	0.712763	2.071091688	8.402504631	10.47359632	0.498934269	0.062366784	1.347838822
3	9	2	2	18	0.70376	0.714852	2.079202631	9.401497145	11.48069978	0.500396202	0.055599578	1.347579613
3	10	2	2	20	0.705458	0.716531	2.08572569	10.4006869	12.48641259	0.501571931	0.050157193	1.347368451
4	1	2	2	2	0.630375	0.642054	1.806139119	2.37947922	4.185618339	0.449438083	0.449438083	1.355763307
4	2	2	2	4	0.676741	0.68801	1.981926564	3.606809407	5.588735971	0.481606811	0.240803406	1.351195379
4	3	2	2	6	0.690294	0.701452	2.031905971	4.569233961	6.601139932	0.491016487	0.163672162	1.349561742
4	4	2	2	8	0.697767	0.708861	2.059532789	5.580909136	7.640441926	0.496202752	0.124050688	1.348608616
4	5	2	2	10	0.702318	0.713373	2.076356603	6.587047705	8.663404308	0.499361427	0.099872285	1.348004649
4	6	2	2	12	0.705384	0.716413	2.087690856	7.59114042	9.678831276	0.501489449	0.083581575	1.347587231
4	7	2	2	14	0.70759	0.718601	2.095846037	8.594083383	10.68992942	0.503020596	0.071860085	1.347281476
4	8	2	2	16	0.709253	0.72025	2.101995177	9.59630235	11.69829753	0.504175106	0.063021888	1.347047864
4	9	2	2	18	0.710552	0.721538	2.106797391	10.59803526	12.70483266	0.505076729	0.056119637	1.346863556
4	10	2	2	20	0.711595	0.722572	2.110651547	11.59942607	13.71007761	0.505800352	0.050580035	1.346714433

Effect of Lead time Variance on system performance – Balanced systems: s=8,Q_2=8 r=4, Q_1=4, l=0.5. T_1=2, dm=(0.6, 0.4), μ_{21}=2/3, μ_{22}=2/3, d_{22}=1/3, λ=0.5, T_1=2

μ_{11}	d_{12}	μ_{12}	μ_{21}	d_{22}	μ_{22}	Var	OFR	SL	WIP total	WIP_1	WIP_2	$util_1$
0.666667	0.5	1	0.666667	0.333333	0.666667	3	0.941007	0.944603	11.25982	5.011106	6.248709	0.330611
2	0.5	0.333333	0.666667	0.333333	0.666667	7	0.911808	0.915514	11.3438	4.98323	6.360569	0.32043
0.666667	0.4	0.8	0.666667	0.333333	0.666667	3.25	0.938856	0.942483	11.26598	5.009498	6.256481	0.329869
0.533333	0.25	2	0.666667	0.333333	0.666667	3.625	0.935927	0.93955	11.27441	5.006371	6.268039	0.328843
0.75	0.4	0.6	0.666667	0.333333	0.666667	3.555556	0.936351	0.939997	11.27317	5.007273	6.2659	0.328999
0.666667	0.333333	0.666667	0.666667	0.333333	0.666667	3.5	0.936783	0.940428	11.27193	5.007707	6.264225	0.32915
1	0.333333	0.333333	0.666667	0.333333	0.666667	6	0.919312	0.922926	11.32226	4.989105	6.333157	0.323024
0.555556	0.1	0.5	0.666667	0.333333	0.666667	4	0.932927	0.93658	11.28303	5.003878	6.279147	0.327803
0.533333	0.1	0.8	0.666667	0.333333	0.666667	3.8125	0.93435	0.938001	11.27893	5.005312	6.273618	0.3283
1.5	0.8	0.6	0.666667	0.333333	0.666667	3.111111	0.940145	0.943728	11.2623	5.009949	6.252355	0.330305
0.8	0.75	1	0.666667	0.333333	0.666667	2.5	0.945337	0.948871	11.2474	5.014346	6.233056	0.332105
0.666667	0.75	1.5	0.666667	0.333333	0.666667	2.666667	0.943933	0.947462	11.25144	5.012773	6.23867	0.331612
0.5	0	1	0.666667	0.333333	0.666667	4	0.932927	0.93658	11.28303	5.003878	6.279147	0.327803
1	1	1	0.666667	0.333333	0.666667	2	0.949807	0.953248	11.23461	5.017096	6.217515	0.333637
0.714286	0.2	0.333333	0.666667	0.333333	0.666667	5.2	0.925054	0.928658	11.30574	4.994828	6.310911	0.32503
0.645161	0.15	0.333333	0.666667	0.333333	0.666667	4.9	0.927111	0.930722	11.29981	4.997074	6.302738	0.325753

Effect of Lead time Variance on system performance – Supply constrained systems: s=8,Q_2=8 r=4, Q1=4, l=2/3. T_1=2, dm=(0.6, 0.4), μ_{21}=1, μ_{22}=0.2, d_{22}=0.3

m11	d12	m12	m21	d22	m22	Tavg	Var1	OFR	SL	WIP total	WIP1	WIP2	util1
0.666667	0.5	1	1	0.3	0.2	2	3	0.857678	0.86307	10.51302	4.289549	6.223472	0.402766
2	0.5	0.333333	1	0.3	0.2	2	7	0.824142	0.829212	10.65115	4.285	6.366155	0.386965
0.666667	0.4	0.8	1	0.3	0.2	2	3.25	0.85504	0.860432	10.52404	4.289851	6.234188	0.401535
0.533333	0.25	2	1	0.3	0.2	2	3.625	0.851735	0.85707	10.5376	4.288939	6.248662	0.399966
0.75	0.4	0.6	1	0.3	0.2	2	3.555556	0.85208	0.85745	10.53631	4.289705	6.246606	0.400144
0.666667	0.333333	0.666667	1	0.3	0.2	2	3.5	0.852573	0.85795	10.53428	4.289803	6.244482	0.400376
1	0.333333	0.333333	1	0.3	0.2	2	6	0.833087	0.838162	10.61393	4.284357	6.329575	0.391142
0.555556	0.1	0.5	1	0.3	0.2	2	4	0.848155	0.853478	10.55246	4.289005	6.26346	0.39829
0.533333	0.1	0.8	1	0.3	0.2	2	3.8125	0.849778	0.855121	10.5458	4.289328	6.256467	0.399057
1.5	0.8	0.6	1	0.3	0.2	2	3.111111	0.856767	0.862127	10.5167	4.288947	6.227755	0.402326
0.8	0.75	1	1	0.3	0.2	2	2.5	0.862999	0.868392	10.49079	4.288938	6.201855	0.40525
0.666667	0.75	1.5	1	0.3	0.2	2	2.666667	0.861413	0.866773	10.4973	4.288399	6.208902	0.404494
0.5	0	1	1	0.3	0.2	2	4	0.848155	0.853478	10.55246	4.289005	6.26346	0.39829
1	1	1	1	0.3	0.2	2	2	0.868676	0.874034	10.46694	4.287464	6.179474	0.407882
0.714286	0.2	0.333333	1	0.3	0.2	2	5.2	0.839602	0.844756	10.58716	4.285595	6.301561	0.394219
0.645161	0.15	0.333333	1	0.3	0.2	2	4.9	0.841876	0.847068	10.57788	4.286315	6.291567	0.395298

Effect of Lead time Variance on system performance – Demand constrained systems: s=8,Q_2=8 r=4, Q1=4, l=0.4. T_1=2, dm=(0.6, 0.4), μ_{21}=2, μ_{22}=0.5, d_{22}=0.5

m11	d12	m12	m21	d22	m22	Tavg	Var1	OFR	SL	WIP total	WIP1	WIP2	util1
0.666667	0.5	1	2	0.5	0.5	2	3	0.964606	0.967152	11.73731	5.330525	6.40679	0.270803
2	0.5	0.333333	2	0.5	0.5	2	7	0.942837	0.945681	11.7878	5.305529	6.482268	0.264791
0.666667	0.4	0.8	2	0.5	0.5	2	3.25	0.963097	0.965683	11.7408	5.329087	6.41171	0.270391
0.533333	0.25	2	2	0.5	0.5	2	3.625	0.960912	0.963519	11.74587	5.326421	6.419449	0.269785
0.75	0.4	0.6	2	0.5	0.5	2	3.555556	0.961289	0.963907	11.74498	5.327142	6.417842	0.269894
0.666667	0.333333	0.666667	2	0.5	0.5	2	3.5	0.961608	0.964223	11.74424	5.327519	6.416723	0.269983
1	0.333333	0.333333	2	0.5	0.5	2	6	0.948293	0.951015	11.77518	5.311017	6.464163	0.266284
0.555556	0.1	0.5	2	0.5	0.5	2	4	0.958749	0.961401	11.75087	5.324178	6.426695	0.269192
0.533333	0.1	0.8	2	0.5	0.5	2	3.8125	0.95981	0.962449	11.74841	5.325432	6.42298	0.269486
1.5	0.8	0.6	2	0.5	0.5	2	3.111111	0.96394	0.966486	11.73886	5.329585	6.409279	0.270616
0.8	0.75	1	2	0.5	0.5	2	2.5	0.967637	0.970105	11.73032	5.333416	6.396899	0.27163
0.666667	0.75	1.5	2	0.5	0.5	2	2.666667	0.966601	0.969079	11.73272	5.33213	6.400589	0.271342
0.5	0	1	2	0.5	0.5	2	4	0.958749	0.961401	11.75087	5.324178	6.426695	0.269192
1	1	1	2	0.5	0.5	2	2	0.970688	0.973057	11.72329	5.335976	6.38731	0.272456
0.714286	0.2	0.333333	2	0.5	0.5	2	5.2	0.952613	0.955284	11.76516	5.316118	6.449039	0.26748
0.645161	0.15	0.333333	2	0.5	0.5	2	4.9	0.954192	0.956853	11.76149	5.318111	6.443374	0.267919

7. Analysis of a three stages arborescent system

7.1 Research rationale

Real life supply chains may deviate from the linear configuration and more general topologies, such as assembly systems and arborescent networks can be found in practice. The presence of more than one member in a given echelon increases the complexity of the analysis. The behavior of the systems is more difficult to predict and the choice of optimal policies becomes a challenging task. Modeling becomes even harder when multi echelon inventory systems are concerned: Different installations cannot be treated separately, but must be analyzed together as an integrated system; the effect of each member on the performance of the others must be understood and evaluated; and the global optimal policies may not (and usually not) coincide with the local optimal policies of each separate member, creating conflicting interests within the network.

Multi-echelon inventory systems of an arborescent structure have practical importance. Divergent configurations are common at the end of a supply network where one central wholesaler supplies multiple local retailers, or at the end of a production line where a semi-finished product is sent to different processes. Understanding the dynamics of such systems is important for effective decision making, but realistic models are usually highly complex, sometimes even to the point of mathematical intractability. In general, to tackle complexity either simplifying assumptions must be made (deterministic parameters, no stock-outs, nested policies), or approximate methods have to be employed. Both approaches have their respective drawbacks, and there is an ongoing need for more realistic models that could capture to a greater extent the characteristics of the real systems and that would allow us to assess the dynamics of longer and more complex supply chains.

7.2 Literature review

A great part of the literature on divergent systems is concerned with two echelon systems with one wholesaler and multiple retailers (OWMR). An introduction to the evaluation of such systems is given in Axsäter (2015). Amongst others are included a generalization of the Scarf-Clark model for distribution systems, a METRIC approximation approach, and a recursive procedure to determine the inventory level distribution when (S-1, S) policies are concerned.

In the simplest cases, single cycle models are investigated. Yang and Wee (2001) formulate a mathematical model to compute the optimal number of deliveries in a single cycle for an OWMR system. They provide a closed form solution for a system with two retailers under the assumptions of constant demand with no shortages allowed, and instantaneous replenishment times. Hsiao (2018) investigates the optimal single cycle policies of a system with one warehouse and multiple retailers under deterministic assumptions, negligible lead times, and no shortages allowed. Similar assumptions are also used by Solyalı and Süral (2011) who analyze different integer programming formulations for the OWMR problem on a finite horizon basis.

Most models in the literature are concerned with periodic review policies. A more systematic review of such systems can be found at Agrawal and Smith (2015). Geng et al. (2010) compare the performance of an OWMR system for different operating scenarios concerning information sharing. They consider a system that is observed periodically and is operated for a finite planning horizon and they assume zero lead times and lost sales. System costs are obtained using approximate dynamic programming (ADP) and stochastic dynamic programming.

Helper et al. (2010) and Panda et al. (2010) investigate more complex demand patterns. Helper et al. analyze a system with one supplier and two retailers with correlated demand and lost sales. They describe their problem as a restricted observation Markov Decision Process and use their model to evaluate the effect of different information sharing schemes. Panda et al. (2010) analyze an OWMR system in the context of an imprecise environment where demand is correlated with available inventory and price. They formulate a model for a basic period policy with no shortages and they employ a genetic algorithm and fuzzy simulation to propose near optimal solutions.

Gayon et al. (2016) analyze the OWMR problem in a deterministic setting. They use decomposition into single echelon sub-systems and propose an approximation approach to give a solution along with lower bounds for both back-orders and lost sales systems. For the lost sales case, the proposed policy gives total cost of up to two times the optimal cost.

In many cases stricter assumptions are made about the inventory policies of the various installations. Yao and Wang (2006) and Abdul-Jalbar et al. (2006) assume stationary nested policies (Each facility orders at equally spaced points in time and in equal amounts, while each facility orders every time any of each immediate supplier does and perhaps at other times as well). Yao and Wang analyze the characteristics of the optimal cost curve and they propose a search algorithm for the optimal solution of the respective lot sizing problem, while Abdul-Jalbar et al. develop a heuristic for a deterministic system with no shortages and negligible lead times.

Abdul Jalbar et al. (2010) and Wang (2013) investigate integer-ratio policies (the replenishment interval at the wholesaler is a multiple of the replenishment interval at the retailer or vice versa). Abdul Jalbar et al. (2010) propose an iterative procedure for a system with deterministic demand, no shortages, and negligible lead times. Wang (2013) develops an exact optimal inventory control policy assuming identical retailers and synchronized ordering activities. The retailers face Poisson external demand with backordering in case of stock-out, while the wholesaler follows an echelon-stock, order-up-to policy. The analysis is based on the fact that each warehouse replenishment forms a regenerative cycle for the system. The author investigates the structural properties of the system cost function and proposes a procedure to identify an optimal solution.

Tayebi et al. (2018) propose an heuristic procedure for the optimal policies of a system with one warehouse which works as a cross-docking terminal and multiple non-identical retailers which follow a (1,T) policy (replenishment orders of one unit every period T). They assume constant lead times, Poisson external demand and lost sales for stock-outs.

Tempelmeier (2013) studies a system with one wholesaler, n retailers and a factory working according to make-to-order logic. The retailers follow a base stock policy with a daily review and a service level constraint, while the wholesaler follows a periodic (s,nq) inventory control policy. Partial orders from the wholesaler are allowed but unmet demand is backordered. The system is analyzed through decomposition and the analysis is based on the waiting times between the different stages of supply network. The author formulates the overall optimization problem and stresses the importance of the upstream stages of the network on overall performance.

Ahire and Schmidt (1996) investigate a two echelon system, with one Wholesaler and multiple retailers, that follows a mixed Continuous-Periodic review policy. The retailers follow continuous review policies but their replenishment orders are reviewed by the wholesaler periodically. The model assumes backordering, Poisson external demand and deterministic lead times. Under their assumptions the authors establish the equivalence of the continuous policy with a periodic review policy and propose an approximation method to estimate system performance measures. The approximation is found to be accurate for high values of Fill rate. The authors propose a closed form expression for warehouse demand variance based on renewal theory.

With regard to continuous review policies, Huang and Iravani (2006) analyze the effect of different production and rationing policies in a system with one manufacturer and two retailers that follow (R,Q) policies and face pure Poisson external demand. Exponentially distributed production times and negligible lead times are assumed. The manufacturer's production and rationing decisions are formulated as a Markov Decision Process.

Guan and Zhao (2011) investigate a OWMR system where the retailers follow continuous review (R,Q) policies and face stochastic demand in the form of a Poisson process. Their assumptions include constant lead times for the replenishment orders, and backlogging of the unmet demand. The authors focus on both inventory management and pricing decisions and they investigate scenarios for centralized and decentralized decision making based on approximate optimal solutions.

Seifbarghy et al. (2013) propose a statistical method and simulation for the estimation of the optimal policies of a two-echelon inventory system with one central warehouse and multiple non-identical retailers. All nodes follow continuous review (R,Q) inventory control policies. Unmet demand at the wholesaler is back-ordered. The retailers face external demand with pure Poisson characteristics and lost sales, while all transportation times are assumed to be constant.

Divergent systems with more than two echelons are less common. Gonzalez et al. (1995) analyze a multi echelon system with general arborescent topology. They assume backordering, demand with Poisson characteristics and zero lead times. They define the system as a Markov process and characterize the optimal cost applying dynamic programming that provides numerical approximations for global optimization.

Yang et al. (2006) consider replenishment and pricing policies in an arborescent network with one producer, one distributor and multiple retailers. They investigate a system following periodic review policies and they assume that the demand is constant and the replenishment times are negligible. The authors formulate a model for different levels of integration and they apply a genetic algorithm for optimization.

Wu et al. (2012) investigate a system with multiple stocking echelons and multiple retailers. They formulate a model for a mixed produce-to-order and produce-in-advance inventory system and seek to determine the optimal inventory at each installation on a single period basis. They analyze the system for uniform and normal external demand with allowed transshipments between the retailers, and they conclude that in both cases the problem can be solved as a constrained optimization problem.

Islam et al. (2017) develop a model for a three tier system with one supplier, one manufacturer and multiple retailers. The retailers' demands are random variables with generic probability density functions and lost sales are allowed. The model is based on a single cycle basis and a procedure for its solution is proposed. The authors focus on the comparison of traditional policies, where every node acts independently, to consignment policies where the retailers and the supplier are subordinates of the manufacturer.

In this thesis we propose an algorithm for the exact numerical evaluation of a three echelons system consisting of a distribution center, a wholesaler, and multiple retailers. Taking into consideration the widespread implementation of information systems providing real time information, as well as the general trend towards lean systems and just in time practices, we assume continuous review inventory policies. Each installation follows a separate (r,Q) policy and no restrictions are assumed about policy parameters. To make the model more realistic, both demand and transportation times are assumed to be stochastic, while demand that cannot be met from inventory on hand is assumed to be lost.

The system is modeled as a continuous time Markov chain and the analysis is based on the infinitesimal generator matrix. Although the model does not offer optimal policies, its value is based on the fact that it can offer an insight into the dynamics of divergent systems. Optimal policies may not always be applicable and sub-optimal choices may have to be made so it is important to understand the effect of small changes in structural and operational parameters on the overall system performance. The proposed algorithm could be used as the evaluative tool in the context of a more general optimization model.

7.3 Description of the system

We investigate an arborescent pull system of 3 tiers. A Distribution Centre (DC) orders from a saturated plant and supplies a Wholesaler. In its turn, the Wholesaler supplies n independent retailers (fig. 7.1). The retailers hold inventory (I_r) and face external demand with pure Poisson characteristics. At each retailer, inter-arrival times for external customers are exponentially distributed and each customer asks for exactly one product unit. Demand that cannot be met from inventory on hand at each retailer is lost. The retailers follow a continuous review inventory control policy with parameters (s_i, Q_i). Whenever inventory on hand at the retailer i becomes equal to or less than s_i, a replenishment order of Q_i units is ordered from the Wholesaler. When the inventory at the Wholesaler is not enough, then a partial order is sent.

The Wholesaler holds inventory and follows a continuous review inventory control policy based on local information with parameters (s_w, Q_w). Whenever inventory on hand at the Wholesaler becomes equal to or less than s_w, a replenishment order of Q_w units is ordered upstream from the Distribution Centre (DC). When inventory at the DC is less than Q_w, then a partial order is sent. When there is a stock-out at the Wholesaler ($I_w^t = 0$) and the wholesaler faces demand from multiple retailers, on replenishment order arrival, the highest indexed retailer has always priority (retailer i has priority over retailer i-1). Only when the highest priority demand is fully met, is the next highest priority demand addressed with the remaining inventory. There is no restriction on how many retailers may be served at the same moment.

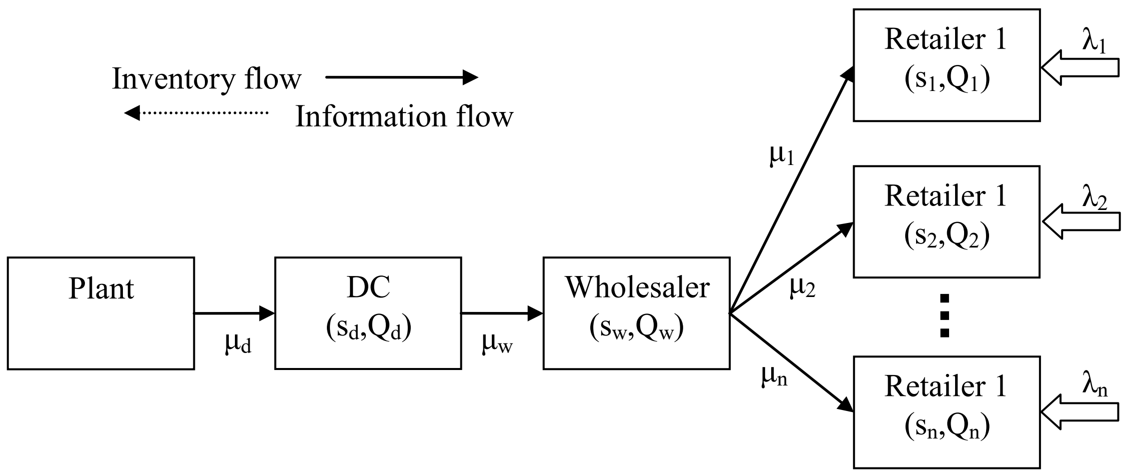
The DC also follows a continuous review inventory control police with parameters (s_d, Q_d) and based on local information. Whenever the inventory on hand at the DC becomes less than or equal to s_d, a replenishment order is requested upstream from the Plant. As the plant is assumed to be saturated, always complete orders of Q_d units are sent towards the DC on request.

The following assumptions are also made:

- Transportation times are exponentially distributed and independent of replenishment order quantity.

- At any given time at most one order can be in-transit towards any given node (one outstanding order assumption). Such an assumption is common in analytic models and is necessary in order to maintain a tractable level of complexity. (Bijvank & Vis, 2011).
- The transportation process is modeled as an independent station. On replenishment order initiation, the respective inventory is subtracted from the upstream node and remains "in transit" until its delivery to the downstream node.
- Transportation is reliable and there are no extra loading and unloading times.
- Lead time for the information flow is zero

Figure 7.1: System layout



7.3.1 Model variables

We denote as the decision variables the parameters concerning the topology of the system and the inventory control policies at each node. The decisions variables in our model include:

n: The number of retailers
s_d: The reorder point at the Distribution Centre
Q_d: The replenishment order quantity at the Distribution Centre
s_w: The reorder point at the Wholesaler
Q_w: The replenishment order quantity at the Wholesaler
s_i: The reorder point at Retailer (i)
Q_i: the replenishment order quantity at Retailer (i)

The other parameters that are necessary to completely define the system are:
μ_d: The transportation rate for orders from the Plant towards the Distribution Centre
μ_w: The transportation rate for orders from the Distribution Centre towards the Wholesaler
μ_i: The transportation rate for orders from the Wholesaler towards retailer (i)
λ_i: The arrival rate of external customers at Retailer (i)

7.4 States definition and state transitions

7.4.1. States definition

The whole system is a (2n+3) dimensional continuous time Markov chain

$\{I_d^t, T_w^t, I_w^t, T_n^t, I_n^t, T_{n-1}^t, I_{n-1}^t, ..., T_1^t, I_1^t, t \geq 0\}$

At any given time, the state of the system can be defined by a *2n+3* dimensional vector

$$\bar{S}_t = \left(I_d^t, T_{w,}^t I_w^t, T_n^t, I_n^t, T_{n-1}^t, I_{n-1}^t, \dots, T_1^t, I_1^t\right)$$

,where:

I_d^t: The inventory on hand at the Distribution Centre at time *t*. $0 \leq I_d^t \leq s_d + Q_d$

T_w^t: The inventory in transit towards the Wholesaler at time t. $0 \leq T_w^t \leq Q_w$

I_w^t: The inventory on hand at the Wholesaler at time *t*. $0 \leq I_w^t \leq s_w + Q_w$

T_i^t: The inventory in transit towards retailer i at time t. $0 \leq T_i^t \leq Q_i$

I_i^t: The inventory on hand at retailer i at time t. $0 \leq I_i^t \leq s_i + Q_i$

By the definition of the system, certain restrictions hold about the relations between the decision variables. If $s_d < Q_d \leq Q_w$, the inventory on hand at the Distribution Centre cannot take values between 0 and $Q_d - 1$. To exclude the associated transient states and save computational power, the equivalent value $s_d = 0$ can be used. Moreover, Q_w cannot exceed $s_d + Q_d$, while Q_i cannot exceed $s_w + Q_w$, $1 \leq i \leq n$.

The possible values for the inventory at the Distribution Centre (I_d) depend on the specific values of Q_d and Q_w. More specifically, I_d will be a multiple of the Greatest Common Divisor of Q_d and Q_w. If *bsd* the basic incremental step for inventory at the DC:

$bsd = GCD(Q_d, Q_w)$

Inventory in transit towards the wholesaler will also be a multiple of *bsd.*

Possible Inventory on hand at the Wholesaler (I_w) values will be multiples of the Greatest Common Divisor of *bsd*, Q_1, Q_2,…,Q_n. If *bsw* the basic incremental step for Wholesaler inventory values:

$bsw = GCD(bsd, Q_1, Q_2, .., Q_n)$

Inventory in transit towards the retailers will also be a multiple of *bsw*.

The state space Ω of the Markov process is comprised of all permissible $\bar{S}_t$ vectors. Its dimension depends on the values of the decision variables and can be calculated recursively as will be explained in section 7.5. It must be noted that transient states may still be included in Ω. Transient states have zero stationary probabilities and although their inclusion costs in terms of computational power, it also significantly simplifies the modeling approach.

The states are ordered linearly using the lexicographical ordering. The subset of all states corresponding to fixed inventory at the Distribution Centre (I_d) is taken as a basic level and the basic levels are ordered from lower to higher. Within each basic level, the states are grouped according to the inventory in transit towards the Wholesaler (T_w). Again the sublevels are ordered from lower to higher values. For fixed basic level and fixed inventory in transit towards the Wholesaler, the states are ordered by inventory at the wholesaler (I_w). Lower sub-levels concern the states at the retailers. They are ordered according T_i and I_i with higher priority retailers preceding lower priority ones, and lower inventory values preceding higher ones.

7.4.2. State transitions

The state of the system can be altered instantaneously by four kinds of events. For methodology reasons and without posing any restrictions to our model, it is assumed that no two events can occur at exactly the same time. In infinitesimal time *dt* only one event can occur. The four classes of the events are:

1. The arrival of an outstanding order from the Plant to the Distribution Centre (DC). As it is assumed that the Plant is saturated, always Q_d units are delivered at the DC. If there is no outstanding demand from the Wholesaler, I_d increases by Q_d units. When $I_d^t = 0$ and there is demand from the Wholesaler, part or all of the incoming order is forwarded to the Wholesaler. In infinitesimal time *dt* the possibility of the event occurring is $\mu_d \cdot dt + O(dt)$, where $O(dt)$ is an unspecified function such that $\lim_{dt \to 0} \frac{O(dt)}{dt} = 0$. $O(dt)$ stands for the probability that a second event will occur in infinitesimal time *dt*.

2. The arrival of an outstanding order from the DC to the Wholesaler. 1 to Q_w units may be delivered at the wholesaler, depending on the availability of inventory at the DC at the time of replenishment order initiation ($1 \le T_w^t \le Q_w$). If there is no outstanding demand from the retailers, I_w increases by T_d^t units. When I_w=0 and there is demand from the retailers, part or all of the incoming order is forwarded to the respective retailers, starting from the highest priority retailer. In infinitesimal time *dt* the possibility of the event occurring is $\mu_w \cdot dt + O(dt)$.

3. The arrival of an outstanding order at retailer *i*. In this case the inventory on hand at the retailer increases by T_i^t units ($I_i^{t+dt} = I_i^t + T_i^t$). If the new inventory at the retailer *i* is not above the reorder point s_i, then a new replenishment order is asked from the wholesaler. T_i^{t+dt} takes the value of this new order and I_w decreases accordingly. In infinitesimal time *dt*, the possibility of the event occurring is $\mu_i \cdot dt + O(dt)$.

4. The occurrence of external demand at retailer i. Since we assume unitary demand, inventory on hand of the retailer decreases by one unit ($I_i^{t+dt} = I_i^t - 1$). If the new inventory is less than or equal to the reorder point s_i and $T_i^t = 0$, a replenishment order is asked from the wholesaler. T_i^{t+dt} takes the new value of inventory in transit and I_w decreases correspondingly. In infinitesimal time dt, the possibility of external demand occurring is $\lambda_i \cdot dt + O(dt)$

7.5 The infinitesimal Generator Matrix

The infinitesimal generator matrix P is a matrix such that p_{ij} is the instantaneous transition rate from state i to state j, i≠j, and $p_{ii} = -\sum_{\forall j \neq i} p_{ij}$.

We take as basic level the Inventory at the Distribution Centre (DC) I_d^t. As explained in 7.4.1, the inventory on hand at the DC, as well as the inventory in transit towards the wholesaler, are multiples of *bsd*:

$$bsd = GCD(Q_d, Q_w)$$

If *NLd* the number of basic levels where $I_d > 0$, and nQ_w the number of levels for inventory in transit towards the wholesaler when $T_w^t > 0$:

$$NLd = floor\left(\frac{s_d + Q_d}{bsd}\right)$$

$$nQ_w = floor\left(\frac{Q_w}{bsd}\right)$$

$floor(x)$: The integer produced by rounding x downwards.

The maximum value of I_d^t will be $NLd \cdot bsd$.

Inventory on hand at the Wholesaler I_w^t and inventory in transit towards the retailers T_i^t are both multiples of *bsw*:

$$bsw = GCD(bsd, Q_1, Q_2, .., Q_n)$$

If *NLw* is the number of levels of inventory on hand at the Wholesaler for $I_w^t > 0$, and nQ_i the number of the permissible values for inventory in transit towards retailer i when $T_i^t > 0$:

$$NLw = floor\left(\frac{s_w + Q_w}{bsw}\right)$$

$$nQ_i = floor\left(\frac{Q_i}{bsw}\right)$$

The maximum value of inventory at retailer i will be $I_i^{\max} = s_i + nQ_i \cdot bsw$.

We also define *nsw* as the greatest I_w^t level where the wholesaler asks for a replenishment order from the DC:

$$nsw = floor\left(\frac{s_w}{bsw}\right)$$

The state at each retailer at time *t* is defined by two variables, T_i^t: the inventory in transit towards retailer *i*, and I_i^t:the inventory on hand at retailer *i*.

7.5.1 Diagonal sub-matrices

We use as basic building blocks for the diagonal sub-matrices the blocks that describe transitions in the retailers for a given state of the upstream part of the system. The structure of these blocks can be defined recursively. We use as "seed" the sub-matrix describing transitions for the lowest priority retailer (Stage 1 - Retailer 1) and its associated transport station. Each stage (retailer) sub-matrix is constructed using as "building block" the sub-matrix of the previous stage and according to rules that hold for all stages. The specific structure for each such matrix depends on whether it corresponds to $I_w^t = 0$ or $I_w^t > 0$.

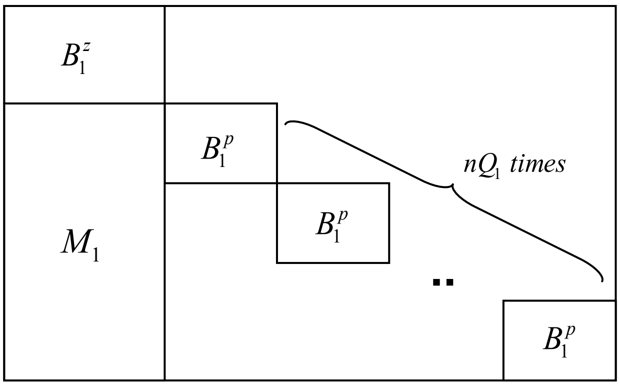
7.5.1.1 I_w=0

<u>Retailer 1</u>

Block B_1 comprises the possible states of Retailer 1 (lowest priority) for a given state of the rest of the system. B_1 is a square matrix and describes transitions where only the state of retailer 1 is changed (I_1^t or T_1^t). The dimension of the block is

$$Bl_1 = (s_1 + Q_1 + 1) + nQ_1 \cdot (s_1 + 1)$$

In the first (s_1+Q_1+1) states there is no inventory in transit towards retailer 1 ($T_1^t = 0$). In the next $nQ_1 \cdot (s_1 + 1)$ states there is inventory in transit towards retailer 1 ($T_1^t > 0$). B_1 can be further reduced into smaller blocks:



B_1^z is a $(s_1+Q_1+1)\times(s_1+Q_1+1)$ block corresponding to $T_1^t=0$. Omitting the zero elements:

$$B_1^z=\begin{bmatrix} -\mu_d-\mu_w & & & & \\ \lambda_1 & -\mu_d-\mu_w-\lambda_1 & & & \\ & \lambda_1 & -\mu_d-\mu_w-\lambda_1 & & \\ & & \cdots & \cdots & \\ & & & \lambda_1 & -\mu_d-\mu_w-\lambda_1 \end{bmatrix}$$

B_1^p is a $(s_1+1)\times(s_1+1)$ matrix corresponding to $T_1^t>0$. There are nQ_1 blocks of B_1^p, each one corresponding to a different value of T_1^t:

$$B_1^p=\left\{\begin{matrix} -\mu_d-\mu_w-\mu_1 & & & & \\ \lambda_1 & -\mu_d-\mu_w-\mu_1-\lambda_1 & & & \\ & \lambda_1 & -\mu_d-\mu_w-\mu_1-\lambda_1 & & \\ & & \cdots & \cdots & \\ & & & \lambda_1 & -\mu_d-\mu_w-\mu_1-\lambda_1 \end{matrix}\right\}$$

M_1 describes the arrival of an outstanding replenishment order at Retailer 1 and is a $nQ_1\cdot(s_1+1)\times(s_1+Q_1+1)$ matrix. It can be divided into nQ_1 $(s_1+1)\times(s_1+Q_1+1)$ matrices, each one corresponding to a different T_1^t level:

$$M_1=\left[\begin{array}{l} \underbrace{\cdots\ 0}_{bsw}\ \ \mu_1\ \ 0 \\ \qquad\quad 0\ \ \mu_1\ \ 0 \\ \qquad\qquad\quad \cdots \\ \qquad\qquad\quad 0\ \ \mu_1\ \ 0 \\ \hdashline \underbrace{0\ \cdots\ 0}_{2\cdot bsw}\ \ \mu_1 \\ \qquad\qquad 0\ \ \mu_1\ \ 0 \\ \qquad\qquad\qquad \cdots \\ \qquad\qquad\qquad 0\ \ \mu_1\ \ 0 \\ \hdashline \qquad\qquad \vdots \qquad \vdots \\ \hdashline \underbrace{0\ \cdots\ \cdots\ 0}_{nQ_1\cdot bsw}\ \ \mu_1 \\ \qquad\qquad\qquad 0\ \ \mu_1\ \ 0 \\ \qquad\qquad\qquad\qquad \cdots \\ \qquad\qquad\qquad\qquad 0\ \ \mu_1 \end{array}\right]$$

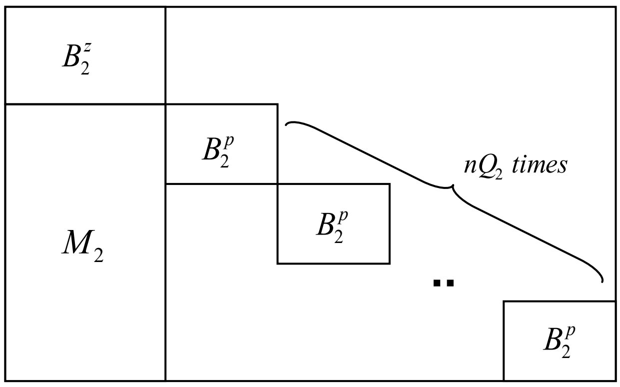
Retailer 2

Block B_2 comprises the possible states for retailers with priority up to 2 (lowest priority retailer 1 and retailer 2) for a given state of the rest of the system. B_2

describes transitions where only the state of retailers 1 and 2 is changed (T_2^t, I_2^t, T_1^t, I_1^t). For every state of Retailer 2 (T_2^t, I_2^t), there are Bl_1 states of Retailer 1. The dimension of block B_2 is

$$Bl_2 = (s_2 + Q_2 + 1) \cdot Bl_1 + nQ_2 \cdot (s_2 + 1) \cdot Bl_1$$

In the first $(s_2+Q_2+1) \cdot Bl_1$ states there is no inventory in transit towards retailer 2 $(T_2^t = 0)$. In the next $nQ_2 \cdot (s_2+1) \cdot Bl_1$ states there is inventory in transit towards retailer 2 ($T_2^t > 0$). B_2 can be further analyzed into smaller blocks:



We denote $\mathbf{I_1}$: the identity matrix of Bl_1 dimension, and $\mathbf{O_1}$: a zero square matrix of Bl_1 dimension.

B_2^z is a $(s_2 + Q_2 + 1) \cdot Bl_1 \times (s_2 + Q_2 + 1) \cdot Bl_1$ matrix corresponding to $T_2^t = 0$:

$$B_2^z = \begin{bmatrix} B_1 & & & & \\ \lambda_2 \cdot I_1 & B_1 - \lambda_2 \cdot I_1 & & & \\ & \lambda_2 \cdot I_1 & B_1 - \lambda_2 \cdot I_1 & & \\ & & \cdots & \cdots & \\ & & & \lambda_2 \cdot I_1 & B_1 - \lambda_2 \cdot I_1 \end{bmatrix}$$

B_2^p is a $(s_2 + 1) \cdot Bl_1 \times (s_2 + 1) \cdot Bl_1$ matrix corresponding to $T_2^t > 0$. Each B_2^p block corresponds to a different value of T_2^t:

$$B_2^p = \begin{bmatrix} B_1 - \mu_2 \cdot I_1 & & & & \\ \lambda_2 \cdot I_1 & B_1 - (\mu_2 - \lambda_2) \cdot I_1 & & & \\ & \lambda_2 \cdot I_1 & B_1 - (\mu_2 - \lambda_2) \cdot I_1 & & \\ & & \cdots & \cdots & \\ & & & \lambda_2 \cdot I_1 & B_1 - (\mu_2 - \lambda_2) \cdot I_1 \end{bmatrix}$$

M_2 is a $nQ_2 \cdot (s_2 + 1) \cdot Bl_1 \times (s_2 + Q_2 + 1) \cdot Bl_1$ matrix. It can be divided into nQ_2 blocks of $(s_2 + 1) \cdot Bl_1 \times (s_2 + Q_2 + 1) \cdot Bl_1$ dimensions, each one corresponding to a different T_2^t value:

$$M_2 = \left[\begin{array}{l}
\underbrace{\cdots\ \ O_1}_{bsw\cdot Bl_1}\quad \mu_2\cdot I_1 \quad O_1 \\
\qquad\qquad O_1 \quad \mu_2\cdot I_1 \quad O_1 \\
\qquad\qquad\qquad\qquad\quad \cdots \\
\qquad\qquad\qquad\qquad\quad O_1 \quad \mu_2\cdot I_1 \quad O_1 \\ \hdashline
\underbrace{O_1\ \ \cdots\ \ O_1}_{2\cdot bsw\cdot Bl_1}\quad \mu_2\cdot I_1 \\
\qquad\qquad\qquad O_1 \quad \mu_2\cdot I_1 \quad O_1 \\
\qquad\qquad\qquad\qquad\qquad\quad \cdots \\
\qquad\qquad\qquad\qquad\qquad\quad O_1 \quad \mu_2\cdot I_1 \quad O_1 \\ \hdashline
\qquad\qquad\quad \vdots \qquad\qquad\qquad \vdots \\ \hdashline
\underbrace{O_1\ \ \cdots\ \ \cdots\ \ O_1}_{nQ_2\cdot bsw\cdot Bl_1}\quad \mu_2\cdot I_1 \\
\qquad\qquad\qquad\qquad O_1 \quad \mu_2\cdot I_1 \quad O_1 \\
\qquad\qquad\qquad\qquad\qquad\qquad\quad \cdots \\
\qquad\qquad\qquad\qquad\qquad\qquad\quad O_1 \quad \mu_2\cdot I_1
\end{array}\right]$$

<u>Retailer i</u>

Following the same approach, the structure of block B_i can be defined (i>2). Block B_i comprises the possible states for retailers with priority up to i (retailers 1 to i) for a given state of the rest of the system. B_i describes transitions where only the state of retailers 1, 2,..,i is changed (T_i^t, I_i^t, T_{i-1}^t, I_{i-1}^t,…, T_1^t, I_1^t). For every state of retailer *i* (T_i^t, I_i^t), correspond Bl_{i-1} states of lower priority retailers:

$$Bl_{i-1} = (s_{i-1} + Q_{i-1} + 1)\cdot Bl_{i-2} + nQ_{i-1}(s_{i-1} + 1)\cdot Bl_{i-2}$$

The dimension of block B_i is

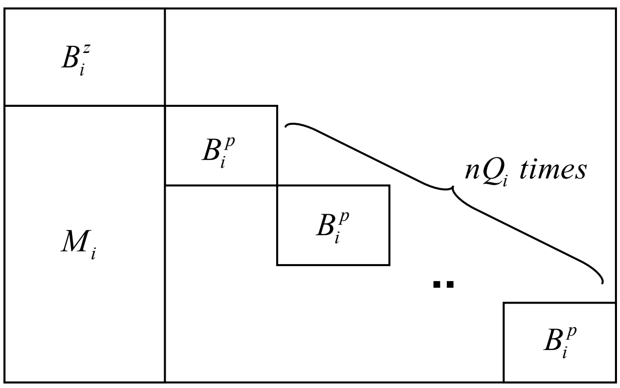
$$Bl_i = (s_i + Q_i + 1)\cdot Bl_{i-1} + nQ_i\cdot(s_i + 1)\cdot Bl_{i-1}$$

In the first $(s_i + Q_i + 1)\cdot Bl_{i-1}$ states there is no inventory in transit towards retailer i $(T_i^t = 0)$. In the next $nQ_i\cdot(s_i + 1)\cdot Bl_{i-1}$ states there is inventory in transit towards retailer *i* ($T_i^t > 0$). We denote:

I_{i-1} :the identity matrix of Bl_{i-1} dimension

O_{i-1} :a zero square matrix of Bl_{i-1} dimension,

The structure of B_i is:



B_i^z is a $(s_i + Q_i + 1) \cdot Bl_{i-1} \times (s_i + Q_i + 1) \cdot Bl_{i-1}$ matrix corresponding to $T_i^t = 0$:

$$B_i^z = \begin{bmatrix} B_{i-1} & & & & \\ \lambda_i \cdot I_{i-1} & B_{i-1} - \lambda_i \cdot I_{i-1} & & & \\ & \lambda_i \cdot I_{i-1} & B_{i-1} - \lambda_i \cdot I_{i-1} & & \\ & & \cdots & \cdots & \\ & & & \lambda_i \cdot I_{i-1} & B_{i-1} - \lambda_i \cdot I_{i-1} \end{bmatrix}$$

B_i^p is a $(s_i + 1) \cdot Bl_{i-1} \times (s_i + 1) \cdot Bl_{i-1}$ matrix corresponding to $T_i^t > 0$. Each B_i^p block corresponds to a different level of T_i^t:

$$B_i^p = \begin{bmatrix} B_{i-1} - \mu_i \cdot I_{i-1} & & & & \\ \lambda_i \cdot I_{i-1} & B_{i-1} - (\mu_i - \lambda_i) \cdot I_{i-1} & & & \\ & \lambda_i \cdot I_{i-1} & B_{i-1} - (\mu_i - \lambda_i) \cdot I_{i-1} & & \\ & & \cdots & \cdots & \\ & & & \lambda_i \cdot I_{i-1} & B_{i-1} - (\mu_i - \lambda_i) \cdot I_{i-1} \end{bmatrix}$$

M_i describes the arrival of an outstanding replenishment order at retailer *i* and it is a $nQ_i \cdot (s_i + 1) \cdot Bl_{i-1} \times (s_i + Q_i + 1) \cdot Bl_{i-1}$ matrix. It can be divided into nQ_i blocks of $(s_i + 1) \cdot Bl_{i-1} \times (s_i + Q_i + 1) \cdot Bl_{i-1}$ dimensions, each one corresponding to a different T_i^t value:

$$M_i = \left[\begin{array}{ccccccc}
\underbrace{\cdots \quad O_{i-1}}_{bsw \cdot Bl_{i-1}} & \mu_i \cdot I_{i-1} & O_{i-1} & & & & \\
 & O_{i-1} & \mu_i \cdot I_{i-1} & O_{i-1} & & & \\
 & & & \cdots & & & \\
 & & & O_{i-1} & \mu_i \cdot I_{i-1} & O_{i-1} & \\
\hdashline
\underbrace{O_{i-1} \quad \cdots \quad O_{i-1}}_{2 \cdot bsw \cdot Bl_{i-1}} & & \mu_i \cdot I_{i-1} & & & & \\
 & & O_{i-1} & \mu_i \cdot I_{i-1} & O_{i-1} & & \\
 & & & & \cdots & & \\
 & & & & O_{i-1} & \mu_i \cdot I_{i-1} & O_{i-1} \\
\hdashline
 & & \vdots & & \vdots & & \\
\hdashline
\underbrace{O_{i-1} \quad \cdots \quad \cdots \quad O_{i-1}}_{nQ_i \cdot bsw \cdot Bl_{i-1}} & & & \mu_i \cdot I_{i-1} & & & \\
 & & & O_{i-1} & \mu_i \cdot I_{i-1} & O_{i-1} & \\
 & & & & & \cdots & \\
 & & & & & O_{i-1} & \mu_i \cdot I_{i-1}
\end{array} \right]$$

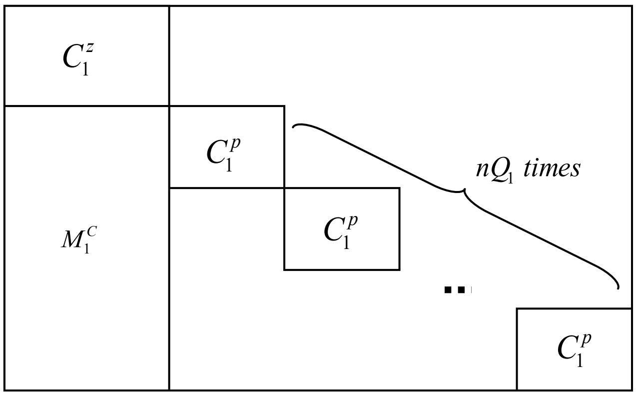
7.5.1.2 I_w>0

<u>Retailer 1</u>

Block C_1 comprises the possible states of retailer 1 (lowest priority retailer) for a given state of the rest of the system. C_1 describes transitions where only the state of retailer 1 is changed (I_1^t or T_1^t). It is a square matrix of Cl_1 dimension

$$Cl_1 = Q_1 + nQ_1 \cdot (s_1 + 1)$$

C_1 can be further reduced into smaller blocks of states:



In the first Q_1 states of C_1 there is no inventory in transit towards retailer 1 ($T_1^t = 0$, $I_1^t > s_1$). C_1^z is a $Q_1 \times Q_1$ matrix corresponding to these states:

$$C_1^z = \begin{bmatrix} -\mu_d - \mu_w - \sum_1^n \lambda_i & & & & \\ \lambda_1 & -\mu_d - \mu_w - \sum_1^n \lambda_i & & & \\ & \lambda_1 & -\mu_d - \mu_w - \sum_1^n \lambda_i & & \\ & & \cdots & \cdots & \\ & & & \lambda_1 & -\mu_d - \mu_w - \sum_1^n \lambda_i \end{bmatrix}$$

C_1^p is a $(s_1 + 1) \times (s_1 + 1)$ matrix that corresponds to states where $T_1^t > 0$. Each C_1^p block corresponds to a different T_1^t value.

$$C_1^p = \begin{bmatrix} -\mu_d - \mu_w - \mu_1 + \lambda_1 - \sum_1^n \lambda_i & & & & \\ \lambda_1 & -\mu_d - \mu_w - \mu_1 - \sum_1^n \lambda_i & & & \\ & \lambda_1 & -\mu_d - \mu_w - \mu_1 - \sum_1^n \lambda_i & & \\ & & \cdots & \cdots & \\ & & & \lambda_1 & -\mu_d - \mu_w - \mu_1 - \sum_1^n \lambda_i \end{bmatrix}$$

M_1^C is a $nQ_1 \cdot (s_1+1) \times Q_1$ matrix and describes the arrival of an outstanding replenishment order at Retailer 1 when there is no need for a new replenishment order ($I_1^{t+dt} > s_1$). It can be divided into nQ$_1$ sub-matrices of $(s_1+1) \times Q_1$ dimension (M_1^{Cs}), each one corresponding to a different T_1^t value:

$$M_1^{Cs} = \begin{array}{l} \overbrace{\qquad}^{y_0} \\ \left\{ \begin{matrix} & \mu_1 & & & \\ & & \mu_1 & & \\ & & & \cdots & \\ & & & & \mu_1 \\ & & & & \\ & & & & \end{matrix} \right\} \end{array}, \; y_0 = T_1^t - s_1 - 1, \text{ if } T_1^t > s_1$$

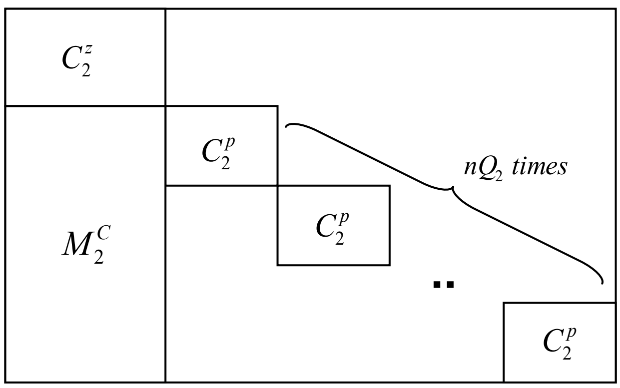
$$M_1^{Cs} = \begin{array}{r} x_0 \left\{ \vphantom{\begin{matrix} a \\ a \end{matrix}} \right. \\ \\ \\ \\ \\ \end{array} \left\{ \begin{matrix} & & & \\ & & & \\ \mu_1 & & & \\ & \mu_1 & & \\ & & \cdots & \\ & & & \mu_1 \end{matrix} \right\}, \; x_0 = s_1 - T_1^t + 1, \text{ if } s_1 \geq T_1^t$$

<u>Retailer 2</u>

Block C$_2$ comprises the possible states for the retailers with up to 2 priority (retailers 1 and 2) for a given state of the rest of the system. It describes transitions where only the state of retailer 1 or retailer 2 is changed (T_2^t, I_2^t, T_1^t, I_1^t). For every state of retailer 2 (T_2^t, I_2^t fixed) correspond Cl_1 possible states of retailer 1. C$_2$ is a square matrix with dimension

$$Cl_2 = Q_2 \cdot Cl_1 + nQ_2 \cdot (s_2+1) \cdot Cl_1$$

In the first $Q_2 \cdot Cl_1$ states there is no inventory in transit towards retailer 2, while in the rest $nQ_2 \cdot (s_2+1) \cdot Cl_1$ states $T_2>0$. C$_2$ can be analyzed into smaller blocks of states:



If $\mathbf{I}_1$: the identity matrix of Cl_1 dimension, and $\mathbf{O}_1$: a zero square matrix of Cl_1 dimension:

C_2^z is a $Q_2 \cdot Cl_1 \times Q_2 \cdot Cl_1$ matrix corresponding to states where $T_2^t = 0$, $I_2^t > s_2$:

$$C_2^z = \begin{bmatrix} C_1 & & & & \\ \lambda_2 \cdot I_1 & C_1 & & & \\ & \lambda_2 \cdot I_1 & C_1 & & \\ & & \cdots & \cdots & \\ & & & \lambda_2 \cdot I_1 & C_1 \end{bmatrix}$$

C_2^p is a square matrix of $(s_2+1) \cdot Cl_1$ dimension corresponding to $T_2^t > 0$. Each C_2^p block corresponds to a different T_2^t level.

$$C_2^p = \begin{bmatrix} C_1 + (\lambda_2 - \mu_2) \cdot I_1 & & & & \\ \lambda_2 \cdot I_1 & C_1 - \mu_2 \cdot I_1 & & & \\ & \lambda_2 \cdot I_1 & C_1 - \mu_2 \cdot I_1 & & \\ & & \cdots & \cdots & \\ & & & \lambda_2 \cdot I_1 & C_1 - \mu_2 \cdot I_1 \end{bmatrix}$$

M_2^C describes the arrival of an outstanding replenishment order at retailer 2, when there is no need for a new replenishment order $(I_2^{t+dt} > s_2)$. It is a $nQ_2 \cdot (s_2+1) \cdot Cl_1 \times Q_2 \cdot Cl_1$ matrix and can be divided into nQ_2 sub-matrices of $(s_2+1) \cdot Cl_1 \times Q_2 \cdot Cl_1$ dimensions $\left(M_2^{Cs}\right)$, each one corresponding to a different T_2^t value:

$$M_2^{Cs} = \begin{bmatrix} \overbrace{\quad}^{y_0} & \mu_2 \cdot I_1 & & & \\ & & \mu_2 \cdot I_1 & & \\ & & & \cdots & \\ & & & & \mu_2 \cdot I_1 \\ & & & & \\ & & & & \end{bmatrix}, y_0 = (T_2^t - s_2 - 1) \cdot Cl_1, \text{ if } T_2^t > s_2$$

$$M_2^{Cs} = \begin{bmatrix} \left.\begin{matrix} \\ \end{matrix}\right\} x_0 & & & \\ \mu_2 \cdot I_1 & & & \\ & \mu_2 \cdot I_1 & & \\ & & \cdots & \\ & & & \mu_2 \cdot I_1 \end{bmatrix}, x_0 = (s_2 - T_2^t + 1) \cdot Cl_1, \text{ if } s_2 \geq T_2^t$$

Retailer i

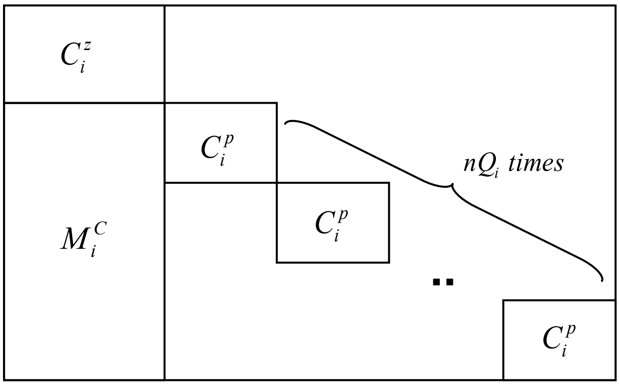
In general, block C_i (i>2) comprises the possible states for the retailers with priority up to i (retailers 1,2,…,i) for a given state of the rest of the system. C_i describes transitions where only the state at retailers 1 to *i* is changed (T_i^t, I_i^t , T_{i-1}^t, I_{i-1}^t,…, T_1^t, I_1^t). For every state of retailer i (T_i^t, I_i^t fixed) correspond Cl_{i-1} states of the retailers with lower priority.

$$Cl_{i-1} = Q_{i-1} \cdot Cl_{i-2} + nQ_{i-1}(s_{i-1}+1) \cdot Cl_{i-2}$$

C_i will be a square matrix with dimension

$$Cl_i = Q_i \cdot Cl_{i-1} + nQ_i(s_i+1) \cdot Cl_{i-1}$$

and general structure



If I_{i-1} : the identity matrix of Cl_{i-1} dimension, and O_{i-1} : a zero square matrix of Cl_{i-1} dimension:

C_i^z is a $Q_i \cdot Cl_{i-1} \times Q_i \cdot Cl_{i-1}$ matrix corresponding to states where $T_i^t = 0$, $I_i^t > s_i$.

$$C_i^z = \begin{bmatrix} C_{i-1} & & & & \\ \lambda_i \cdot I_{i-1} & C_{i-1} & & & \\ & \lambda_i \cdot I_{i-1} & C_{i-1} & & \\ & & \cdots & \cdots & \\ & & & \lambda_i \cdot I_{i-1} & C_{i-1} \end{bmatrix}$$

C_i^p is a square matrix of $(s_i+1) \cdot Cl_{i-1}$ dimension corresponding to $T_i^t > 0$. Each C_i^p sub-matrix corresponds to a different T_i^t level.

$$C_i^p = \begin{bmatrix} C_{i-1} + (\lambda_i - \mu_i) \cdot I_{i-1} & & & & \\ \lambda_i \cdot I_{i-1} & C_{i-1} - \mu_i \cdot I_{i-1} & & & \\ & \lambda_i \cdot I_{i-1} & C_{i-1} - \mu_i \cdot I_{i-1} & & \\ & & \cdots & \cdots & \\ & & & \lambda_i \cdot I_{i-1} & C_{i-1} - \mu_i \cdot I_{i-1} \end{bmatrix}$$

M_i^C describes the arrival of an outstanding replenishment order at retailer i, when there is no need for a new replenishment order $\left(I_i^{t+dt} > s_i\right)$. M_i^C is a $nQ_i \cdot (s_i + 1) \cdot Cl_{i-1} \times Q_i \cdot Cl_{i-1}$ matrix. It can be further divided into nQ_i sub-matrices of $(s_i + 1) \cdot Cl_{i-1} \times Q_i \cdot Cl_{i-1}$ dimensions (M_i^{Cs}), each one corresponding to a different T_i^t level:

$$M_i^s = \begin{array}{c} \overbrace{\hphantom{000}}^{y_0} \\ \left\{ \begin{matrix} & \mu_i \cdot I_{i-1} & & & \\ & & \mu_i \cdot I_{i-1} & & \\ & & & \cdots & \\ & & & & \mu_i \cdot I_{i-1} \\ & & & & \\ & & & & \end{matrix} \right\} \end{array}, y_0 = (T_i^t - s_i - 1) \cdot Cl_{i-1}, \text{ if } T_i^t > s_i$$

$$M_i^s = \left\{ \begin{matrix} \left.\begin{matrix} \\ \\ \end{matrix}\right\} x_0 & & & \\ \mu_i \cdot I_{i-1} & & & \\ & \mu_i \cdot I_{i-1} & & \\ & & \cdots & \\ & & & \mu_i \cdot I_{i-1} \end{matrix} \right\}, x_0 = (s_i - T_i^t + 1) \cdot Cl_{i-1}, \text{ if } s_i \geq T_i^t$$

7.5.1.3 General Structure of the Diagonal Tier

The sub-matrices along the diagonal describe transitions within a given level of I_d (basic level). The first basic level diagonal sub-matrix D_0 corresponds to the boundary conditions where I_d=0, and then follow *NLd* blocks D_1 corresponding to different values of I_d.

Basic Level: L=0

Sub-matrix D_0 is a square block with dimensions:

$$L_0 = \left(Bl_n + NLw \cdot Cl_n\right) + nQ_w \cdot (Bl_n + nsw \cdot Cl_n)$$

and it can be further analyzed into smaller blocks:

$$D_0 = \begin{bmatrix} D_0^z & & & & \\ & D_0^p & & & \\ & & D_0^p & & \\ & & & \cdots & \\ & & & & D_0^p \end{bmatrix} \text{nQ}_\text{w} \text{ times}$$

D_0^z is a $\left(Bl_n + NLw \cdot Cl_n\right)$- dimension block corresponding to states where $I_d^t = 0$ and $T_w^t = 0$:

$$D_0^z = \begin{bmatrix} B_n + \mu_w \cdot I_n & & & & \\ & C_n + \mu_w \cdot I_n & & & \\ & & C_n + \mu_w \cdot I_n & & \\ & & & \cdots & \\ & & & & C_n + \mu_w \cdot I_n \end{bmatrix}$$

I_n is the identity matrix of Bl_n or Cl_n dimension, as appropriate.

D_0^p is a $\left(Bl_n + nsw \cdot Cl_n\right)$- dimension block corresponding to states where $I_d^t = 0$ and $T_w^t > 0$. In each D_0 matrix there are nQ_w blocks of D_0^p, each one corresponding to a different T_w^t value.

$$D_0^p = \begin{bmatrix} B_n & & & & \\ & C_n & \text{nsw times} & & \\ & & C_n & & \\ & & & \cdots & \\ & & & & C_n \end{bmatrix}$$

<u>Basic Level: L>0</u>

D_1 is a square block with dimensions $L_1 = (NLw - nsw) \cdot Cl_n + nQ_w \cdot (Bl_n + nsw \cdot Cl_n)$ and can be further analyzed into smaller blocks:

$$D_1 = \begin{bmatrix} D_1^z & & & & \\ & D_1^p & \text{nQ}_\text{w}\text{ times} & & \\ & & D_1^p & & \\ & & & \cdots & \\ & & & & D_1^p \end{bmatrix}$$

D_1^z is a $(NLw - nsw) \cdot Cl_n$- dimension block corresponding to states where $I_d^t > 0$ and $T_w^t = 0$:

$$D_1^z = \begin{bmatrix} C_n + \mu_w \cdot I_n & & & \\ & C_n + \mu_w \cdot I_n & & \\ & & \cdots & \\ & & & C_n + \mu_w \cdot I_n \end{bmatrix}$$

D_1^p is a $(Bl_n + nsw \cdot Cl_n)$- dimension block corresponding to states where $I_d^t > 0$ and $T_w^t > 0$. In each D_1 matrix there are nQ_w blocks of D_1^p, each one corresponding to a different T_w^t value.

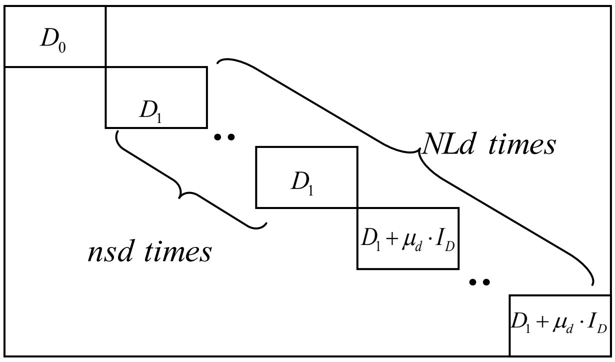
$$D_1^p = \begin{bmatrix} B_n & & & & \\ & C_n & & & \\ & & C_n & & \\ & & & \cdots & \\ & & & & C_n \end{bmatrix} \quad \text{nsw times}$$

General diagonal structure of the infinitesimal generator

Replenishment orders from the Plant occur only as long as

$$I_d^t \le s_d \Rightarrow Level \cdot bsd \le s_d \Rightarrow Level \le \frac{s_d}{bsd}$$

If $nsd = floor(s_d / bsd)$ and I_D the identity matrix of L_1 dimension, the general structure of the diagonal sub-matrices of the Infinitesimal Generator Matrix will be:



7.5.2 Upper-diagonal elements

The elements above the diagonal describe the arrival of a replenishment order at the Distribution Centre (DC). Since we assume that the Plant is saturated, always Q_d units are delivered. An outstanding replenishment order from the plant to the DC occurs as long as $I_d^t \le s_d$, so there are elements above the diagonal from basic level "0" to basic level *nsd*. We define $b = Bl_n + nsw \cdot Cl_n$ the dimension of blocks D_0^p and D_1^p.

7.5.2.1 I_d>0

Here there is no demand from the wholesaler and the incoming order increases the available inventory on hand at the DC:

$$I_d^{t+dt} = I_d^t + Q_d$$

The position of the upper-diagonal elements (μ_d) is $nQ_d \cdot L_1$ states to the right with respect to the diagonal elements,

$$nQ_d = \frac{Q_d}{bsd}$$

7.5.2.2 I_d=0

Here there is a possibility of DC stock-out, when there is demand for a replenishment order from the Wholesaler, but inventory on hand at the DC is zero ($T_w^t = 0$, $I_w^t \le s_w$). In such a case part or all of the incoming replenishment order Q_d will be immediately forwarded towards the Wholesaler. The block corresponding to these transitions U_0 is a $b \times b$ diagonal matrix of μ_d. The position of the upper left element of U_0 in the infinitesimal generator is (1, $step_0$), where:

$$step_0 = L_0 + \left(\frac{(Q_d - Q_w)}{bsd} - 1 \right) \cdot L_1 + (NLw - nsw) \cdot Cl_n + (nQ_w - 1) \cdot b + 1 \text{ , if } Q_d > Q_w$$

$$step_0 = Bl_n + NLw \cdot Cl_n + \left(\frac{Q_d}{bsd} - 1 \right) \cdot b + 1 \text{, if } Q_d \le Q_w.$$

To the states where $T_w^t = 0$ and $I_w^t > s_w$, corresponds a $(NLw - nsw) \cdot Cl_n \times (NLw - nsw) \cdot Cl_n$ diagonal matrix of μ_d (U_1). Its upper left element will be at position (*b+1, step₁*) of the infinitesimal generator, where

$$step_1 = L_0 + \left(\frac{Q_d}{bsd} - 1 \right) \cdot L_1 + 1$$

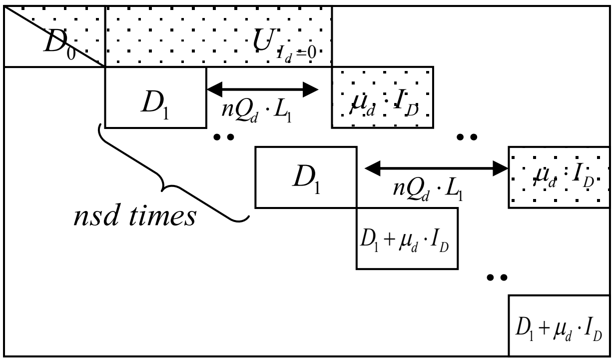
Finally, to the states where $T_w^t > 0$ corresponds a $b \times b$ diagonal matrix of μ_d (U_3). There are nQ_w such blocks, each one corresponding to a different T_w^t Level. The upper left element of each such matrix is at position (x, y) of the infinitesimal generator where:

$$x = Bl_n + NL_w \cdot Cl_n + (Level - 1) \cdot b + 1$$

$$y = L_0 + \left(\frac{Q_d}{bsd} - 1 \right) \cdot L_1 + (NLw - nsw) \cdot Cl_n + (Level - 1) \cdot b + 1$$

$$1 \le Level \le nQ_w$$

The structure of the infinitesimal generator matrix, taking account of diagonal and upper diagonal blocks will be:



7.5.3 Below the diagonal sub-matrices

7.5.3.1 Arrival of a replenishment order at the Wholesaler

These elements (μ_w) describe the arrival of a replenishment order at the wholesaler and occur as long as $I_w^t \le s_w$. A full or a partial order may be delivered at the wholesaler $(1 \le T_w^t \le bsd \cdot nQ_w)$. We work state-wise to define the exact position of each element in the infinitesimal generator matrix.

I_d=0, I_w=0

When $I_w^t = 0$ there is the possibility of wholesaler stock-out, when there is demand for a replenishment order from one or multiple retailers, but inventory on hand at the wholesaler is zero. Part, or all of the incoming replenishment order T_w^t may be forwarded towards the retailers, according to the priority of each retailer. The inventory that is not sent immediately to the retailers increases the inventory on hand at the wholesaler. We code this information in a matrix D.

For every T_w^t value we define a $Bl_n \times (n+1)$ matrix D such that:

$D(i,j)=1$, 1≤j≤n, if at state i, product units from a replenishment order will be forwarded towards retailer j. For this to happen, retailer j must ask for a replenishment order ($I_j^t \le s_j$), and there must be enough product units to cover possible demand from higher priority retailers.

$D(i,j)=0$, 1≤j≤n, if at state i, on replenishment order arrival at the wholesaler, no product is forwarded towards retailer j. This may be either because retailer j does not ask for a replenishment order ($I_j^t > s_j$), or because no inventory is left after meeting demand from higher priority retailers.

$D(i,n+1)=0$, when at state i total demand from the retailers is equal to or more than Q_w. In this case, inventory on hand at the wholesaler remains zero ($I_w^{t+dt}=0$).

$D(i,n+1)=T_w^t - td$, when total demand from the retailers (*td*) is less than Q_w. In this case, inventory on hand at the wholesaler increases ($I_w^{t+dt}=T_w^t - td$).

Matrix D can be computed recursively knowing that in every B_i block there is demand for a replenishment order from retailer i in the first s_i+1 states, and taking into account the priority rule for the retailers.

We denote as *state* the sequence number of the state in the B_n block under consideration. First we examine the case where the incoming order is less than or equal to the total retailers' demand $T_w^t \le td$. In such cases inventory at the wholesaler remains zero. The step to the right for a particular state depends on the number of the retailers whose outstanding demand is met from the incoming replenishment order of T_w^t units. If (x,y) the position of μ_w in the infinitesimal generator for a particular state:

$$x = Bl_n + NLw \cdot Cl_n + (\frac{T_w^t}{bsd} - 1) \cdot b + state$$

$$y = y_0 + \sum_{i=2}^{n}\left[\left(s_i + Q_i + 1 + \min\left(\frac{T_w^t}{bsw} - sumQ_i, nQ_i - 1\right) \cdot (s_i + 1)\right) \cdot Bl_{i-1} \cdot D(state, i)\right]$$

$$y_0 = \left[(s_1 + Q_1 + 1) + \min\left(\frac{T_w^t}{bsw} - sumQ_1, nQ_1 - 1\right) \cdot (s_1 + 1)\right] \cdot D(state, 1)$$

$$sumQ_i = 1 + \sum_{j=i+1}^{n} nQ_j \cdot D(state, j)\,,\ 1 \le i \le n-1$$

$$sumQ_n = 1$$

y_0 is the step to the right due to the initiation of a replenishment order towards retailer 1.

$sumQ_i$ corresponds to the product units that are needed to meet the demand for retailers i+1 to n.

When $T_w^t > td$ the inventory on hand at the wholesaler increases. The jump to the right because of the change in I_w^t will be:

$$y_0 = Bl_n + \left(\frac{D(state, n+1)}{bsw} - 1\right) \cdot Cl_n$$

Here we have transition from B-blocks to C-blocks. The exact position of μ_w in the C type block depends on the hierarchy of the initial state in the B-type block. To calculate the states jumped to the right at a particular state, first we must determine the position of each B block (ps_i) that corresponds to the state under consideration. This can be done recursively:

$$ps_i = ceil\left(\frac{nstate_i}{Bl_{n-i}}\right)$$

$$nstate_{i+1} = \begin{cases} nstate_i - (ps_i - 1) \cdot Bl_{n-i},\ 1 \le i \le n-1 \\ state,\ i = 0 \end{cases}$$

The jump to the right because of the position of a particular state in the hierarchy of states in B-type blocks will be:

$$y_1 = nstate_n \cdot D(state,1) + \sum_{i=1}^{n-1}\left((ps_i - 1) \cdot Cl_{n-i} \cdot D(state, n-i+1)\right)$$

The total jump to the right for the cases where there is no initiation of a replenishment order toward the retailer will be:

$$y_2 = \left(nstate_n - s_1 - 1\right) \cdot \left(1 - D(state,1)\right) + \sum_{i=1}^{n-1}\left(\left(ps_i - 1 - s_{n-i+1} - 1\right) \cdot Cl_{n-i} \cdot \left(1 - D(state, n-i+1)\right)\right)$$

The total jump to the right for the cases where there is initiation of a replenishment order toward the retailer will be:

$$y_3 = \left(Q_1 + (nQ_1 - 1) \cdot (s_1 + 1)\right) \cdot D(state,1) + \sum_{i=2}^{n}\left(\left(Q_i + (nQ_i - 1) \cdot (s_i + 1)\right) \cdot Cl_{i-1} \cdot D(state, i)\right)$$

If (x,y) the position of μ_w in the infinitesimal generator matrix:

$$x = Bl_n + NLw \cdot Cl_n + \left(\frac{T_w^t}{bsd} - 1\right) \cdot b + state$$

$$y = y_0 + y_1 + y_2 + y_3$$

I_d=0, I_w>0

Here the incoming order T_w^t increases the inventory on hand at the wholesaler $(I_w^{t+dt} = I_w^t + T_w^t)$.

These transitions correspond to a diagonal matrix of μ_w of $nsw \cdot Cl_n$ dimensions. If (x,y) the position of the upper left element:

$$x = Bl_n + NLw \cdot Cl_n + \left(\frac{T_w^t}{bsd} - 1\right) \cdot b + Bl_n + 1$$

$$y = Bl_n + \left(\frac{T_w^t}{bsw}\right) \cdot Cl_n + 1$$

I_d>0, I_w=0

Here, on the arrival of the replenishment order at the wholesaler, if $I_w^{t+dt} \le s_w$, a new replenishment order will be initiated and the inventory at the DC will decrease correspondingly. We work state-wise and denote as *state* the sequence number of the state under consideration in the B_n block.

First we examine the case where the incoming order is less than or equal to the total retailers' demand $T_w^t \leq td$. In such cases inventory at the wholesaler remains zero and obviously a new replenishment order is initiated towards the wholesaler.

If $I_d^t > Q_w$ then there is a jump to the right (counting from column 1) corresponding to the new value of I_d^{t+dt} and the new value of T_d^{t+dt}

$$y_d = L_0 + \left(\frac{I_d^t - Q_w}{bsd} - 1 \right) \cdot L_1 + (NLw - nsw) \cdot Cl_n + \left(\frac{Q_w}{bsd} - 1 \right) \cdot b$$

If $I_d^t \leq Q_w$ there is a jump to the right because of the new value of T_d^{t+dt}:

$$y_d = Bl_n + NLw \cdot Cl_n + \left(\frac{I_d^t}{bsd} - 1 \right) \cdot b$$

With regard to transitions inside the retailers' sub-matrix, the step to the right depends on the number of the retailers whose outstanding demand is met from the incoming replenishment order of T_w^t units. If (x,y) the position of μ_w in the infinitesimal generator for a particular state:

$$x = L_0 + \left(\frac{I_d^t}{bsd} - 1 \right) \cdot L_1 + (NLw - nsw) \cdot Cl_n + \left(\frac{T_w^t}{bsd} - 1 \right) \cdot b + state$$

$$y = y_d + y_0 + \sum_{i=2}^{n} \left[\left(s_i + Q_i + 1 + \min\left(\frac{T_w^t}{bsw} - sumQ_i, nQ_i - 1 \right) \cdot (s_i + 1) \right) \cdot Bl_{i-1} \cdot D(state, i) \right]$$

$$y_0 = \left[(s_1 + Q_1 + 1) + \min\left(\frac{T_w^t}{bsw} - sumQ_1, nQ_1 - 1 \right) \cdot (s_1 + 1) \right] \cdot D(state, 1)$$

$$sumQ_i = 1 + \sum_{j=i+1}^{n} nQ_j \cdot D(state, j), \quad 1 \leq i \leq n-1$$

$$sumQ_n = 1$$

y_0 is the step to the right due to the initiation of a replenishment order towards retailer 1.

$sumQ_i$ corresponds to the product units that are needed to address the demand for retailers i+1 to n.

When $T_w^t > td$, the inventory on hand at the wholesaler increases. If $I_w^{t+dt} > s_w$, there is no new replenishment order. The jump to the right (from column 1) because of the change of inventory at the wholesaler will be:

$$y_d = L_0 + \left(\frac{I_d^t}{bsd} - 1 \right) \cdot L_1 + \left(\frac{I_d^{t+dt}}{bsw} - nsw - 1 \right) \cdot Cl_n$$

If $I_w^{t+dt} \leq s_w$, a new replenishment order is initiated from the DC towards the Wholesaler and the inventory at the DC decreases. If $I_d^t > Q_w$, the jump to the right corresponding to I_d^{t+dt} and T_w^{t+dt} will be :

$$y_d = L_0 + \left(\frac{I_d^t - Q_w}{bsd} - 1 \right) \cdot L_1 + (NLw - nsw) \cdot Cl_n + \left(\frac{Q_w}{bsd} - 1 \right) \cdot b$$

If $I_d^t \leq Q_w$:

$$y_d = Bl_n + NLw \cdot Cl_n + \left(\frac{I_d^t}{bsd} - 1 \right) \cdot b$$

The jump to the right because of the new I_w^{t+dt} value:

$$y_w = Bl_n + \left(\frac{I_w^{t+dt}}{bsw} - 1 \right) \cdot Cl_n$$

With regard to transitions in the state of the retailers, we have transition from B-blocks to C-blocks. The exact position of μ_w in the C type block depends on the hierarchy of the initial state in the B-type block. To calculate the states jumped to the right at a particular state, first we must determine the position of each B block (ps_i) that corresponds to the state under consideration. This can be done recursively:

$$ps_i = ceil\left(\frac{nstate_i}{Bl_{n-i}} \right)$$

$$nstate_{i+1} = \begin{cases} nstate_i - (ps_i - 1) \cdot Bl_{n-i}, \ 1 \leq i \leq n-1 \\ state, \ i = 0 \end{cases}$$

The jump to the right because of the position of a particular state in the hierarchy of states in B-type blocks will be:

$$y_1 = nstate_n \cdot D(state,1) + \sum_{i=1}^{n-1} \left((ps_i - 1) \cdot Cl_{n-i} \cdot D(state, n-i+1) \right)$$

The total jump to the right corresponding to the cases where there is no initiation of a replenishment order toward the retailers will be:

$$y_2 = (nstate_n - s_1 - 1) \cdot (1 - D(state,1)) + \sum_{i=1}^{n-1} \left((ps_i - 1 - s_{n-i+1} - 1) \cdot Cl_{n-i} \cdot (1 - D(state, n-i+1)) \right)$$

The total jump to the right corresponding to the cases where there is initiation of a replenishment order toward the retailers will be:

$$y_3 = (Q_1 + (nQ_1 - 1) \cdot (s_1 + 1)) \cdot D(state,1) + \sum_{i=2}^{n} \left((Q_i + (nQ_i - 1) \cdot (s_i + 1)) \cdot Cl_{i-1} \cdot D(state,i) \right)$$

If (x,y) the position of μ_w in the infinitesimal generator matrix:

$$x = L_0 + \left(\frac{I_d^t}{bsd} - 1 \right) \cdot L_1 + (NLw - nsw) \cdot Cl_n + \left(\frac{T_w^t}{bsd} - 1 \right) \cdot b + state$$

$$y = y_d + y_w + y_0 + y_1 + y_2 + y_3$$

I_d>0, I_w>0

Here the incoming order T_w^t increases the inventory on hand at the wholesaler $(I_w^{t+dt} = I_w^t + T_w^t)$.

These transitions correspond to a diagonal matrix of μ_w of Cl_n dimensions. If (x,y) the position of the upper left element of this block:

$$x = L_0 + \left(\frac{I_d^t}{bsd} - 1 \right) \cdot L_1 + (NLw - nsw) \cdot Cl_n + \left(\frac{T_w^t}{bsd} - 1 \right) \cdot b + Bl_n + \left(\frac{I_w^t}{bsw} - 1 \right) \cdot Cl_n + 1$$

If $I_w^{t+dt} > s_w$, there is no re-ordering from the wholesaler:

$$y = L_0 + \left(\frac{I_d^t}{bsd} - 1 \right) \cdot L_1 + \left(\frac{I_w^{t+dt}}{bsw} - nsw - 1 \right) \cdot Cl_n + 1$$

If $I_w^{t+dt} \le s_w$, there will be new order from the DC towards the wholesaler:

If $I_d^t > Q_w$:

$$y = L_0 + \left(\frac{I_d^t - Q_w}{bsd} - 1 \right) \cdot L_1 + (NLw - nsw) \cdot Cl_n + \left(\frac{Q_w}{bsd} - 1 \right) \cdot b + Bl_n + \left(\frac{I_w^{t+dt}}{bsw} - 1 \right) \cdot Cl_n + 1$$

If $I_d^t \le Q_w$:

$$y = Bl_n + NLw \cdot Cl_n + \left(\frac{I_d^t}{bsd} - 1 \right) \cdot b + Bl_n + \left(\frac{I_w^{t+dt}}{bsw} - 1 \right) \cdot Cl_n + 1$$

7.5.3.2 Triggering of a replenishment order towards the retailers

Triggering of a replenishment order towards a retailer can occur when:

a) $I_i^t = s_i + 1$ and external demand occurs (instantaneous transition rate λ_i)

b) A replenishment order arrives at the retailer *i* and the updated I_i^{t+dt} is less than or equal to s_i.

With the dispatched replenishment order the wholesaler sends part, or whole of the inventory he held on hand. If $I_w^{t+dt} \le s_w$, $T_w^t = 0$, and $I_d^t > 0$ there will also be triggering of a replenishment order from the DC towards the wholesaler.

Transitions to state where I_w>0 $\left(I_w^{t+dt} > 0\right)$

In these cases there is more than enough inventory to meet the retailer's demand.

<u>*Retailer 1*</u>

Triggering of a replenishment order because of external demand can occur in the first state of each C_1 block, as this state corresponds to $I_1^t = s_1 + 1$.

The sub-diagonal block S_1^M describing triggering of a replenishment order after the arrival of a previous replenishment order is a $(s_1+1)\cdot k_1 \times s_1$ matrix. $k_1 \cdot bsw$ is the maximum T_1^t value for which there can be reordering: $k_1 = \min\left(nQ_1, floor(\frac{s_1}{bsw}) \right)$

S_1^M can be divided into k_1 $(s_1+1)\times s_1$ sub-blocks, each one corresponding to a different value of the incoming replenishment order T_1^t.

The sub-diagonal block for triggering of a replenishment order towards retailer 1 (S_1) will be a $\left(Q_1 + (s_1+1)\cdot k_1\right)\times s_1$ matrix:

$$
S_1 = \overbrace{\left[\begin{array}{cccc}
 & & & \lambda_1 \\
 & & & \\
\hline
\underbrace{\quad}_{bsw-1} & \mu_1 & & \\
 & \mu_1 & & \\
 & & \cdots & \\
 & & \cdots & \mu_1 \\
 & & & \\
\hline
\underbrace{\qquad}_{2\cdot bsw-1} & \mu_1 & & \\
 & & \mu_1 & \\
 & & \cdots & \\
 & & \cdots & \mu_1 \\
 & & & \\
\hline
\vdots & \vdots & \vdots & \\
\hline
\underbrace{\qquad\qquad}_{k_1\cdot bsw-1} & \mu_1 & & \\
 & & \cdots & \\
 & & & \mu_1 \\
 & & & \\
\end{array}\right]}^{s_1}
\begin{array}{l}
\left.\begin{array}{c} \\ \\ \end{array}\right\} Q_1 \\
\left.\begin{array}{c} \\ \\ \\ \\ \end{array}\right\} s_1+1, \quad \text{last rows: } \}bsw \\
\left.\begin{array}{c} \\ \end{array}\right\} 2\cdot bsw \\
\left.\begin{array}{c} \\ \end{array}\right\} k_1\cdot bsw
\end{array}
\left.\begin{array}{c} \\ \\ \\ \\ \\ \\ \\ \\ \end{array}\right\} k_1\cdot(s_1+1)
$$

The position of S_1 in the infinitesimal generator matrix can be defined iteratively. We define (x,y) the position of λ_1, j the sequence number of C_1 block counting from the first occurrence of C_1 at given level I_w^t , $Level_W = \frac{I_w^t}{bsw}$, $Level_T = \frac{T_w^t}{bsd}$, and

$Level_D = \frac{I_d^t}{bsd}$

$\boldsymbol{I_d^t = 0, T_w^t = 0}$

$$x = Bl_n + (Level_w - 1)\cdot Cl_n + (j-1)\cdot Cl_1 + 1$$
$$y = Bl_n + (Level_w - nQ_1 - 1)\cdot Cl_n + (j-1)\cdot Cl_1 + Q_1 + (nQ_1 - 1)\cdot(s_1+1) + s_1 + 1$$

$\boldsymbol{I_d^t = 0, T_w^t} > 0$

$$x = Bl_n + NLw\cdot Cl_n + (Level_T - 1)\cdot b + Bl_n + (Level_W - 1)\cdot Cl_n + (j-1)\cdot Cl_1 + 1$$
$$y = Bl_n + NLw\cdot Cl_n + (Level_T - 1)\cdot b + Bl_n + (Level_W - nQ_1 - 1)\cdot Cl_n + (j-1)\cdot Cl_1 + Q_1 + (nQ_1 - 1)\cdot(s_1+1) + s_1 + 1$$

$\boldsymbol{I_d^t} > 0, \boldsymbol{T_w^t = 0}$

Here there may be initiated a replenishment order from the DC towards the Wholesaler.

$$x = L_0 + (Level_D - 1)\cdot L_1 + (Level_W - nsw - 1)\cdot Cl_n + (j-1)\cdot Cl_1 + 1$$

If $I_w^{t+dt} > s_w$ (no replenishment order from DC):

$$step = L_0 + (Level_D - 1)\cdot L_1 + (Level_w - nQ_1 - nsw - 1)\cdot Cl_n$$

If $I_w^{t+dt} \le s_w$, there will be a replenishment order from the DC.

If $I_d^t > Q_w$

$$step = L_0 + \left(\frac{I_d^{t+dt}}{bsd} - 1\right)\cdot L_1 + (NLw - nsw)\cdot Cl_n + \left(\frac{T_w^{t+dt}}{bsd} - 1\right)\cdot b + Bl_n + (Level_W - nQ_1 - 1)\cdot Cl_n$$

If $I_d^t \le Q_w$

$$step = Bl_n + NLw\cdot Cl_n + \left(\frac{I_d^t}{bsd} - 1\right)\cdot b + Bl_n + (Level_W - nQ_1 - 1)\cdot Cl_n$$

$$y = step + (j-1)\cdot Cl_1 + (Q_1 + (nQ_1 - 1)\cdot(s_1+1) + s_1 + 1$$

$\boldsymbol{I_d^t} > 0, \boldsymbol{T_w^t} > 0$

$$x = L_0 + (Level_D - 1)\cdot L_1 + (NLw - nsw)\cdot Cl_n + (Level_T - 1)\cdot b + Bl_n + (Level_W - 1)\cdot Cl_n + (j-1)\cdot Cl_1 + 1$$

$$y = L_0 + (Level_D - 1)\cdot L_1 + (NLw - nsw)\cdot Cl_n + (Level_T - 1)\cdot b + Bl_n + \\ + (Level_W - nQ_1 - 1)\cdot Cl_n + (j-1)\cdot Cl_1 + Q_1 + (nQ_1 - 1)\cdot(s_1+1) + s_1 + 1$$

Retailers 2 - n

For a higher priority retailer i (i>1), the below-the-diagonal block is a $\left(Q_i + (s_i+1)\cdot k_i\right)\cdot Cl_{i-1} \times s_i \cdot Cl_{i-1}$ matrix, where $k_i = \min\left(nQ_i, floor(\frac{s_i}{bsw})\right)$

Triggering of a replenishment order because of external demand can occur in the first Cl_{i-1} states of each C_i block, as these states correspond to $I_i^t = s_i + 1$.

The sub-diagonal block S_i^M describing triggering of a replenishment order towards retailer i after the arrival of a previous replenishment order is a $(s_i+1)\cdot k_i \cdot Cl_{i-1} \times s_i \cdot Cl_{i-1}$ matrix. S_i^M can be divided into k_i $(s_i+1)\cdot Cl_{i-1} \times s_i \cdot Cl_{i-1}$ sub-blocks, each one corresponding to a different value of the incoming replenishment order T_i^t.

If I_{i-1} the identity matrix of Cl_{i-1} dimension, the sub-diagonal block describing replenishment order triggering towards retailer i (S_i) will be a $\left(Q_i + (s_i+1)\cdot k_i\right)\cdot Cl_{i-1} \times s_i \cdot Cl_{i-1}$ block:

$$
S_i = \overbrace{\left[\begin{array}{cccc}
 & & & \lambda_i \cdot I_{i-1} \\
 & & & \\
\hline
\underbrace{\quad}_{(bsw-1)\cdot Cl_{i-1}}\mu_i \cdot I_{i-1} & & & \\
 & \mu_i \cdot I_{i-1} & & \\
 & \cdots & & \\
 & & \cdots & \mu_i \cdot I_{i-1} \\
 & & & \\
\hline
\underbrace{\qquad\quad}_{(2\cdot bsw-1)\cdot Cl_{i-1}}\mu_i \cdot I_{i-1} & & & \\
 & \mu_i \cdot I_{i-1} & & \\
 & & \cdots & \\
 & & \cdots & \mu_i \cdot I_{i-1} \\
 & & & \\
\hline
\vdots & \vdots & \vdots & \\
\hline
\underbrace{\qquad\qquad\qquad}_{(k_i\cdot bsw-1)\cdot Cl_{i-1}}\mu_i \cdot I_{i-1} & & & \\
 & & \cdots & \\
 & & & \mu_i \cdot I_{i-1} \\
 & & & \\
\end{array}\right]}^{s_i \cdot Cl_{i-1}}
\begin{array}{l}
\left.\right\} Q_i \cdot Cl_{i-1} \\
\left.\begin{array}{l} \\ \left.\right\} bsw \cdot Cl_{i-1} \end{array}\right\} (s_i+1)\cdot Cl_{i-1} \\
\left.\right\} 2\cdot bsw \cdot Cl_{i-1} \\
\left.\right\} k_i \cdot bsw \cdot Cl_{i-1}
\end{array}
\quad \Bigg\} k_i \cdot (s_i+1)\cdot Cl_{i-1}
$$

The position of each sub-diagonal block S_r in the infinitesimal generator matrix can be defined recursively. We define (x,y) the position of the top left λ_r element, j the sequence number of C_r block counting from the first occurrence of C_r at given level I_w^t, $Level_W = \frac{I_w^t}{bsw}$, $Level_T = \frac{T_w^t}{bsd}$, and $Level_D = \frac{I_d^t}{bsd}$.

$\boldsymbol{I_d^t = 0, T_w^t = 0}$

$$x = Bl_n + (Level_w - 1)\cdot Cl_n + (j-1)\cdot Cl_r + 1$$

$$y = Bl_n + (Level_w - nQ_r - 1)\cdot Cl_n + (j-1)\cdot Cl_r + \left(Q_r + (nQ_r - 1)\cdot(s_r + 1) + s_r\right)\cdot Cl_{r-1} + 1$$

$\boldsymbol{I_d^t = 0, T_w^t} > 0$

$$x = Bl_n + NLw\cdot Cl_n + (Level_T - 1)\cdot b + Bl_n + (Level_W - 1)\cdot Cl_n + (j-1)\cdot Cl_r + 1$$

$$y = Bl_n + NLw\cdot Cl_n + (Level_T - 1)\cdot b + Bl_n + (Level_W - nQ_r - 1)\cdot Cl_n + \\ + (j-1)\cdot Cl_r + \left(Q_r + (nQ_r - 1)\cdot(s_r + 1) + s_r\right)\cdot Cl_{r-1} + 1$$

$\boldsymbol{I_d^t} > 0, \boldsymbol{T_w^t = 0}$

Here there may be initiated a replenishment order from the DC towards the Wholesaler.

$$x = L_0 + (Level_D - 1)\cdot L_1 + (Level_W - nsw - 1)\cdot Cl_n + (j-1)\cdot Cl_r + 1$$

If $I_w^{t+dt} > s_w$ (no replenishment order from the DC):

$$step = L_0 + (Level_D - 1)\cdot L_1 + (Level_w - nQ_r - nsw - 1)\cdot Cl_n$$

If $I_w^{t+dt} \le s_w$, there will be a replenishment order from the DC.

If $I_d^t > Q_w$

$$step = L_0 + \left(\frac{I_d^{t+dt}}{bsd} - 1\right)\cdot L_1 + (NLw - nsw)\cdot Cl_n + \left(\frac{T_w^{t+dt}}{bsd} - 1\right)\cdot b + Bl_n + (Level_W - nQ_r - 1)\cdot Cl_n$$

If $I_d^t \le Q_w$

$$step = Bl_n + NLw\cdot Cl_n + \left(\frac{I_d^t}{bsd} - 1\right)\cdot b + Bl_n + (Level_W - nQ_r - 1)\cdot Cl_n$$

$$y = step + (j-1)\cdot Cl_r + \left(Q_r + (nQ_r - 1)\cdot(s_r + 1) + s_r\right)\cdot Cl_{r-1} + 1$$

$\boldsymbol{I_d^t} > 0, \boldsymbol{T_w^t} > 0$

$$x = L_0 + (Level_D - 1)\cdot L_1 + (NLw - nsw)\cdot Cl_n + (Level_T - 1)\cdot b + Bl_n + (Level_W - 1)\cdot Cl_n + (j-1)\cdot Cl_r + 1$$

$$y = L_0 + (Level_D - 1)\cdot L_1 + (NLw - nsw)\cdot Cl_n + (Level_T - 1)\cdot b + Bl_n + \\ + (Level_W - nQ_r - 1)\cdot Cl_n + (j-1)\cdot Cl_r + \left(Q_r + (nQ_r - 1)\cdot(s_r + 1) + s_r\right)\cdot Cl_{r-1} + 1$$

Transitions to state where I_w=0 $\left(I_w^{t+dt} = 0\right)$

In these cases there is just enough or not enough inventory to meet the retailer's demand. The replenishment order towards the retailer may be partial, while I_w^{t+dt} becomes zero. Here we have transitions from states that belong to C-type blocks to states belonging to B-type blocks. We define $sumS_i$ such that:

$$sumS_i = (s_n+1)\cdot Bl_{n-1} + (s_{n-1}+1)\cdot Bl_{n-2} + ... + (s_{i+1}+1)\cdot Bl_i \quad , 1 \le \text{i} \le \text{n-1}$$

$$sumS_n = 0$$

Retailer 1

Triggering of a replenishment order towards retailer 1 because of external demand can occur in the first state of each C_1 block, as this state corresponds to $I_1^t = s_1 + 1$.

The sub-diagonal block S_1^M describing triggering of a replenishment order after the arrival of a previous replenishment order is a $(s_1+1)\cdot k_1 \times s_1$ matrix, where $k_1 = \min\left(nQ_1, floor(\frac{s_1}{bsw})\right)$. The structure of the sub-diagonal block (S_1) which describes triggering of a replenishment order towards retailer 1 is the same as that described for transition to state where $I_w^{t+dt} > 0$:

$$S_1 = \overbrace{\begin{bmatrix} & & & \lambda_1 \\ & & & \\ \hline \underbrace{\quad}_{bsw-1}\ \mu_1 & & & \\ & \mu_1 & & \\ & & \cdots & \\ & & \cdots & \mu_1 \\ & & & \\ \hline \underbrace{\qquad}_{2\cdot bsw-1}\ \mu_1 & & & \\ & \mu_1 & & \\ & & \cdots & \\ & & \cdots & \mu_1 \\ & & & \\ \hline \vdots & \vdots & \vdots & \\ \hline \underbrace{\qquad\qquad}_{k_1\cdot bsw-1}\ \mu_1 & & & \\ & & \cdots & \\ & & & \mu_1 \\ & & & \end{bmatrix}}^{s_1} \begin{matrix} \}\, Q_1 \\ \}\, bsw \;\; \}\, s_1+1 \\ \}\, 2\cdot bsw \\ \}\, k_1\cdot bsw \\ \}\, k_1\cdot(s_1+1) \end{matrix}$$

The position of S_1 in the infinitesimal generator matrix can be defined iteratively. We define (x,y) the position of λ_1 element in the infinitesimal generator, i the sequence number of C_1 block counting from the first occurrence of C_1 at a given level I_w^t, $Level_W = \frac{I_w^t}{bsw}$, $Level_T = \frac{T_w^t}{bsd}$, and $Level_D = \frac{I_d^t}{bsd}$

B_i blocks are $(s_i + 1) \cdot Bl_{i-1}$ states longer than C_i blocks. Every time we change C_i (i≥2) block in the rows of the Infinitesimal Generator Matrix, there is a further step of $(s_i + 1) \cdot Bl_{i-1}$ in the columns. The sum of these jumps will be:

$$sum_1 = \sum_{j=2}^{n-1} (ps_j - 1) \cdot (s_{n-j+1} + 1) \cdot Bl_{n-j}$$

$$ps_j = ceil\left(\frac{i \cdot Cl_1}{Cl_{n-j+1}}\right)$$

ps_j has to do with the position of the C_1 block under consideration and depends on the block sequence number i counting from the first occurrence of C_1 in each I_w^t Level.

If (x,y) the position of λ_1 element in the infinitesimal generator:

$\boldsymbol{I_d^t = 0, T_w^t = 0}$

$$x = Bl_n + (Level_w - 1) \cdot Cl_n + (i-1) \cdot Cl_1 + 1$$

$$y = sumS_1 + s_1 + Q_1 + 1 + (Level_w - 1) \cdot (s_1 + 1) + s_1 + 1 + (i-1) \cdot Bl_1 + sum_1$$

$\boldsymbol{I_d^t = 0, T_w^t} > 0$

$$x = Bl_n + NLw \cdot Cl_n + (Level_T - 1) \cdot b + Bl_n + (Level_W - 1) \cdot Cl_n + (i-1) \cdot Cl_1 + 1$$

$$\begin{aligned} y = {} & Bl_n + NLw \cdot Cl_n + (Level_T - 1) \cdot b + sumS_1 + s_1 + Q_1 + 1 + \\ & + (Level_W - 1) \cdot (s_1 + 1) + s_1 + 1 + (i-1) \cdot Bl_1 + sum_1 \end{aligned}$$

$\boldsymbol{I_d^t} > 0, \boldsymbol{T_w^t = 0}$

A replenishment order from the DC towards the Wholesaler will be initiated.

$$x = L_0 + (Level_D - 1) \cdot L_1 + (Level_W - nsw - 1) \cdot Cl_n + (i-1) \cdot Cl_1 + 1$$

If $I_d^t > Q_w$

$$step = L_0 + \left(\frac{I_d^{t+dt}}{bsd} - 1\right) \cdot L_1 + (NLw - nsw) \cdot Cl_n + \left(\frac{T_w^{t+dt}}{bsd} - 1\right) \cdot b$$

If $I_d^t \le Q_w$

$$step = Bl_n + NLw \cdot Cl_n + \left(\frac{I_d^t}{bsd} - 1\right) \cdot b$$

$$y = step + sumS_1 + s_1 + Q_1 + 1 + (Level_W - 1)\cdot(s_1 + 1) + s_1 + 1 + (i-1)\cdot Bl_1 + sum_1$$

$\boldsymbol{I_d^t} > 0, \boldsymbol{T_w^t} > 0$

$$x = L_0 + (Level_D - 1)\cdot L_1 + (NLw - nsw)\cdot Cl_n + (Level_T - 1)\cdot b + Bl_n + \\ + (Level_W - 1)\cdot Cl_n + (i-1)\cdot Cl_1 + 1$$

$$y = L_0 + (Level_D - 1)\cdot L_1 + (NLw - nsw)\cdot Cl_n + (Level_T - 1)\cdot b + sumS_1 + s_1 + Q_1 + 1 + \\ + (Level_W - 1)\cdot(s_1 + 1) + s_1 + 1 + (i-1)\cdot Bl_1 + sum_1$$

<u>*Retailers 2 to n*</u>

We define $S_r^L(r)$ the sub-diagonal block describing triggering of a replenishment order because of external demand at retailer *r*. Such transitions occur in the first Cl_{r-1} states of each C_r block, as these states correspond to $I_r^t = s_r + 1$. $S_i^L(r)$ is a $Cl_{i-1} \times Bl_{i-1}$ matrix. The first $(s_i + 1)\cdot Bl_{i-2}$ columns correspond to states where $T_i^t = 0$ and $I_i^t \le s_i$ and as a result they have only zero elements. The structure of the block can be defined recursively. If O_{i-2} a zero matrix of $Cl_{i-2} \times Bl_{i-2}$ dimensions, we define $S_i^L(r)$ such that for i=2:

$$S_2^L(r) = \overbrace{\begin{bmatrix} \overbrace{0 \;\cdots\; 0}^{s_1+1} & \lambda_r & & & & \\ 0 \;\cdots\; 0 & & \lambda_r & & & \\ \vdots \quad \vdots \quad \vdots & & & \cdots & & \\ 0 \;\cdots\; 0 & & & & \lambda_r & \\ 0 \;\cdots\; 0 & & & & & \lambda_r \end{bmatrix}}^{Bl_1} \left.\vphantom{\begin{bmatrix}0\\0\\0\\0\\0\end{bmatrix}}\right\} Cl_1$$

,and for i>2:

$$S_i^L(r) = \overbrace{\begin{bmatrix} \overbrace{O_{i-2} \;\cdots\; O_{i-2}}^{(s_{i-1}+1)\cdot Bl_{i-2}} & S_{i-1}^L(r) & & & & \\ O_{i-2} \;\cdots\; O_{i-2} & & S_{i-1}^L(r) & & & \\ \vdots \quad \vdots \quad \vdots & & & \cdots & & \\ O_{i-2} \;\cdots\; O_{i-2} & & & & S_{i-1}^L(r) & \\ O_{i-2} \;\cdots\; O_{i-2} & & & & & S_{i-1}^L(r) \end{bmatrix}}^{Bl_{i-1}} \left.\vphantom{\begin{bmatrix}0\\0\\0\\0\\0\end{bmatrix}}\right\} Cl_{i-1}$$

The position of $S_r^L(r)$ blocks in the infinitesimal generator matrix is defined iteratively. We define *z* the sequence number of C_r block counting from the first occurrence of C_r at a given level I_w^t , $Level_W = \frac{I_w^t}{bsw}$, $Level_T = \frac{T_w^t}{bsd}$, and $Level_D = \frac{I_d^t}{bsd}$

Since B_i blocks are $(s_i + 1) \cdot Bl_{i-1}$ states longer than C_i blocks, every time we change C_i block in the rows of the Infinitesimal Generator Matrix, there is a further step of $(s_i + 1) \cdot Bl_{i-1}$ states in the columns (i>r). The sum of these jumps will be

$$sum_r = \sum_{i=r+1}^{n-1} (ps_i - 1) \cdot (s_{n-i+r} + 1) \cdot Bl_{n-i+r-1}$$

ps_i is related to the position of the C_r block under consideration and depends on block sequence number z:

$$ps_i = ceil\left(\frac{z \cdot Cl_r}{Cl_{n-i+r}} \right)$$

If (x_L, y_L) the position of the upper left element of $S_r^L(r)$:

$\boldsymbol{I_d^t = 0, T_w^t = 0}$

$$x_L = Bl_n + (Level_w - 1) \cdot Cl_n + (z - 1) \cdot Cl_r + 1$$

$$y_L = sumS_r + (s_r + Q_r + 1) \cdot Bl_{r-1} + (Level_w - 1) \cdot (s_r + 1) \cdot Bl_{r-1} + s_r \cdot Bl_{r-1} + (z - 1) \cdot Bl_r + sum_r + 1$$

$\boldsymbol{I_d^t = 0, T_w^t} > 0$

$$x_L = Bl_n + NLw \cdot Cl_n + (Level_T - 1) \cdot b + Bl_n + (Level_W - 1) \cdot Cl_n + (z - 1) \cdot Cl_r + 1$$

$$\begin{aligned} y_L = {} & Bl_n + NLw \cdot Cl_n + (Level_T - 1) \cdot b + sumS_r + (s_r + Q_r + 1) \cdot Bl_{r-1} + \\ & + (Level_W - 1) \cdot (s_r + 1) \cdot Bl_{r-1} + s_r \cdot Bl_{r-1} + (z - 1) \cdot Bl_r + sum_r + 1 \end{aligned}$$

$\boldsymbol{I_d^t} > 0, \boldsymbol{T_w^t = 0}$

A replenishment order from the DC towards the Wholesaler will be initiated.

$$x_L = L_0 + (Level_D - 1) \cdot L_1 + (Level_W - nsw - 1) \cdot Cl_n + (z - 1) \cdot Cl_r + 1$$

If $I_d^t > Q_w$

$$step = L_0 + \left(\frac{I_d^{t+dt}}{bsd} - 1 \right) \cdot L_1 + (NLw - nsw) \cdot Cl_n + \left(\frac{T_w^{t+dt}}{bsd} - 1 \right) \cdot b$$

If $I_d^t \le Q_w$

$$step = Bl_n + NLw \cdot Cl_n + \left(\frac{I_d^t}{bsd} - 1 \right) \cdot b$$

$$\begin{aligned} y_L = {} & step + sumS_r + (s_r + Q_r + 1) \cdot Bl_{r-1} + (Level_W - 1) \cdot (s_r + 1) \cdot Bl_{r-1} + \\ & + s_r \cdot Bl_{r-1} + (z - 1) \cdot Bl_r + sum_r + 1 \end{aligned}$$

$\boldsymbol{I_d^t} > 0, \boldsymbol{T_w^t} > 0$

$$x_L = L_0 + (Level_D - 1)\cdot L_1 + (NLw - nsw)\cdot Cl_n + (Level_T - 1)\cdot b + Bl_n + \\ + (Level_W - 1)\cdot Cl_n + (z-1)\cdot Cl_r + 1$$

$$y_L = L_0 + (Level_D - 1)\cdot L_1 + (NLw - nsw)\cdot Cl_n + (Level_T - 1)\cdot b + sumS_r + \\ + \left(s_r + Q_r + 1 + (Level_W - 1)\cdot(s_r + 1) + s_r\right)\cdot Bl_{r-1} + (z-1)\cdot Bl_r + sum_r + 1$$

The sub-diagonal block S_r^M describing triggering of a replenishment order towards retailer r at the arrival of a previous replenishment order can also be defined recursively. We define a $Cl_1 \times Bl_1$ matrix $M_2(r)$ such that:

$$M_2(r) = \overbrace{\left[\begin{array}{ccc|ccccc} \multicolumn{3}{c}{\overbrace{\hphantom{0 \cdots 0}}^{s_1+1}} \\ 0 & \cdots & 0 & \mu_r & & & & \\ 0 & \cdots & 0 & & \mu_r & & & \\ \vdots & \vdots & \vdots & & & \cdots & & \\ 0 & \cdots & 0 & & & & \mu_r & \\ 0 & \cdots & 0 & & & & & \mu_r \end{array}\right]}^{Bl_1} \left.\vphantom{\begin{array}{c}0\\0\\0\\0\\0\end{array}}\right\} Cl_1$$

The first $s_1 + 1$ columns of $M_2(r)$ correspond to states where $T_1^t = 0$ and $I_1^t \le s_1$ and so they have only elements of zero.

For i>2, if O_{i-2} a zero matrix of $Cl_{i-2} \times Bl_{i-2}$ dimensions, $M_i(r)$ is a $Cl_{i-1} \times Bl_{i-1}$ matrix such that:

$$M_i(r) = \overbrace{\left[\begin{array}{ccc|ccccc} \multicolumn{3}{c}{\overbrace{\hphantom{O_{i-2} \cdots O_{i-2}}}^{(s_{i-1}+1)\cdot Bl_{i-2}}} \\ O_{i-2} & \cdots & O_{i-2} & M_{i-1}(r) & & & & \\ O_{i-2} & \cdots & O_{i-2} & & M_{i-1}(r) & & & \\ \vdots & \vdots & \vdots & & & \cdots & & \\ O_{i-2} & \cdots & O_{i-2} & & & & M_{i-1}(r) & \\ O_{i-2} & \cdots & O_{i-2} & & & & & M_{i-1}(r) \end{array}\right]}^{Bl_{i-1}} \left.\vphantom{\begin{array}{c}0\\0\\0\\0\\0\end{array}}\right\} Cl_{i-1}$$

The first $(s_{i-1}+1)\cdot Bl_{i-2}$ columns correspond to states where $T_{i-1}^t = 0$ and $I_{i-1}^t < s_{i-1}$ and include only zeros.

Block S_r^M describing the triggering of a replenishment order towards retailer r after the arrival of a previous replenishment order is a $\left((s_r+1)\cdot k_r \cdot Cl_{r-1}\right)\times s_r \cdot Bl_{r-1}$ matrix, where $k_r = \min\left(nQ_r, floor(\frac{s_r}{bsw})\right)$

$$S_r^M = \overbrace{\left[\begin{array}{cccccc} \underbrace{\quad}_{(bsw-1)\cdot Bl_{r-1}} M_r(r) & & & & & \\ & M_r(r) & & & & \\ & & .. & .. & & \\ & & & & & M_r(r) \\ \hline \underbrace{\qquad\qquad}_{(2\cdot bsw-1)\cdot Bl_{r-1}} M_r(r) & & & & & \\ & & M_r(r) & & & \\ & & & .. & & \\ & & & & & M_r(r) \\ \hline \hline \underbrace{\qquad\qquad\qquad}_{(k_r\cdot bsw-1)\cdot Bl_{r-1}} M_r(r) & & & & & \\ & & & M_r(r) & & \\ & & & & .. & M_r(r) \\ & & & & & \end{array}\right]}^{s_r \cdot Bl_{r-1}} \begin{array}{l} \left.\begin{array}{l} \\ \\ \\ \\ \end{array}\right\} (s_r+1)\cdot Cl_{r-1} \\ \left.\right\} bsw \cdot Cl_{r-1} \\ \\ \left.\right\} 2\cdot bsw \cdot Cl_{r-1} \\ \\ \left.\right\} k_r \cdot bsw \cdot Cl_{r-1} \end{array} \left.\begin{array}{l} \\ \\ \\ \\ \\ \\ \end{array}\right\} k_r \cdot (s_r+1)\cdot Cl_{r-1}$$

The position of S_r^M blocks in the infinitesimal generator Matrix can be defined relatively to that of $S_r^L(r)$ blocks. If (x_M, y_M) the position of the upper left element of S_r^M, then:

$$x_M = x_L + Q_r \cdot Cl_{r-1}$$
$$y_M = y_{L:} - s_r \cdot Bl_{r-1} + Bl_{r-1}$$

7.6 Performance Measures

Our analysis is based on the steady state solution of the system. If we denote **P:** the infinitesimal generator matrix, and $\vec{p}$: the vector of stationary probabilities with the i^{th} element of the vector $p(i)$ corresponding to the i^{th} state in the hierarchy of states defined according to the rules of paragraph 7.4.1, then in the steady state:

$$\vec{p}\cdot P = 0$$

$$\sum_{1}^{ns} p_i = 1$$

From the above a system of linear equations is extracted and the vector $\vec{p}$ can be computed numerically. Performance measures about the system can be computed algorithmically using the stationary probabilities and taking advantage of the infinitesimal generator matrix structure.

To facilitate the analysis, the performance measures concerning the retailers are computed for each B_n and C_n block separately. We denote L_0: the dimension of Basic Level for I_d=0; and L_1: the dimension of Basic Levels for I_d>0. If *lp+1* the number of the first state of the B_n or C_n block under consideration:

B_n blocks

$\boldsymbol{I_d^t} = \mathbf{0}, \boldsymbol{T_w^t} = \mathbf{0}$

$lp = 0$

$\boldsymbol{I_d^t} = \mathbf{0}, \boldsymbol{T_w^t} > 0$

$lp = Bl_n + NLw \cdot Cl_n + (Level_T - 1) \cdot b$

$1 \leq Level_T \leq nQ_w$

$\boldsymbol{I_d^t} > 0, \boldsymbol{T_w^t} > 0$

$lp = L_0 + (Level_D - 1) \cdot L_1 + (NLw - nsw) \cdot Cl_n + (Level_T - 1) \cdot b$

$1 \leq Level_D \leq NLd$

$1 \leq Level_T \leq nQ_w$

C_n blocks

$\boldsymbol{I_d^t} = \mathbf{0}, \boldsymbol{T_w^t} = \mathbf{0}$

$lp = Bl_n + (Level_W - 1) \cdot Cl_n$

$1 \leq Level_W \leq NLw$

$\boldsymbol{I_d^t} = \mathbf{0}, \boldsymbol{T_w^t} > 0$

$lp = Bl_n + NLw \cdot Cl_n + (Level_T - 1) \cdot b + Bl_n + (Level_W - 1) \cdot Cl_n$

$1 \leq Level_T \leq nQ_w$

$1 \leq Level_W \leq nsw$

$\boldsymbol{I_d^t} > 0, \boldsymbol{T_w^t} = \mathbf{0}$

$lp = L_0 + (Level_D - 1) \cdot L_1 + (Level - 1) \cdot Cl_n$

$1 \leq Level_D \leq NLd$

$1 \leq Level_W \leq NLw - nsw$

$\boldsymbol{I_d^t} > 0, \boldsymbol{T_w^t} > 0$

$$lp = L_0 + (Level_D - 1)\cdot L_1 + (NLw - nsw)\cdot Cl_n + (Level_T - 1)\cdot b + Bl_n + (Level_w - 1)\cdot Cl_n$$

$$1 \le Level_T \le nQ_w$$

$$1 \le Level_W \le nsw$$

7.6.1 Average inventory at the Distribution Centre – $\mathbf{WIP_d}$

WIP_d is the average inventory on hand at the Distribution Centre. Inventory at the DC (I_d^t) is designated as the basic level during modeling and WIP_d can be computed as the sum:

$$WIP_d = \sum_{Level=1}^{NLd}\left(Level \cdot bsd \cdot \sum_{j=1}^{L_1} p\big(L_0 + (Level-1)\cdot L_1 + j\big)\right)$$

7.6.2 Utilization of resource for transportation towards the Wholesaler – $\mathbf{u_w}$

Utilization of the resource for transportation towards the wholesaler is the percentage of time that there is a replenishment order in transit towards the Wholesaler. To calculate u_w we sum the stationary probabilities of states corresponding to $T_w^t > 0$.

$$u_w = \sum_{i=i_0}^{L_0} p(i) + \sum_{i=1}^{NLd}\sum_{j=j_0}^{L_1} p(L_0 + (i-1)\cdot L_1 + j)$$

$$i_0 = Bl_n + NLw \cdot Cl_n + 1$$

$$j_0 = (NLw - nsw)\cdot Cl_n + 1$$

7.6.3 Average inventory at the wholesaler - $\mathbf{WIP_w}$

WIP_w is the average inventory on hand at the wholesaler. We define a vector W such that the i[th] element W(i) is the probability of $I_w^t = i \cdot bsw$. We define $Level$, $Level_T$, and $Level_D$ positive integers.

$\boldsymbol{I_d^t = 0, T_w^t = 0}$

$$W(Level) = \sum_{i=1}^{Cl_n} p(Bl_n + (Level-1)\cdot Cl_n + i)$$

$$1 \le Level \le NLw$$

$\boldsymbol{I_d^t = 0, T_w^t} > 0$

$$W(Level) = W(Level) + \sum_{i=1}^{Cl_n} p(Bl_n + NLw \cdot Cl_n + (Level_T - 1)\cdot b + Bl_n + (Level-1)\cdot Cl_n + i)$$

$$1 \le Level_T \le nQ_w$$

$$1 \le Level \le nsw$$

$\boldsymbol{I_d^t > 0, T_w^t = 0}$

$$W(Level + nsw) = W(Level + nsw) + \sum_{i=1}^{Cl_n} p\big(L_0 + (Level_D - 1)\cdot L_1 + (Level - 1)\cdot Cl_n + i\big)$$

$1 \le Level_D \le NLd$

$1 \le Level \le NLw - nsw$

$\boldsymbol{I_d^t > 0, T_w^t > 0}$

$$W(Level) = W(Level) + \sum_{i=1}^{Cl_n} p\big(L_0 + (Level_D - 1)\cdot L_1 + (NLw - nsw)\cdot Cl_n + (Level_T - 1)\cdot b + Bl_n + (Level - 1)\cdot Cl_n + i\big)$$

$1 \le Level_D \le NLd$

$1 \le Level_T \le nQ_w$

$1 \le Level \le nsw$

Having constructed vector W, WIP_w can be easily calculated as the sum:

$$WIP_w = \sum_{i=1}^{NLw} \big(i \cdot bsw \cdot W(i)\big)$$

7.6.4 Average inventory at the retailers – $\mathbf{WIP_r}$

WIP_r is the average inventory on hand at the retailer r. We calculate WIP_r for each B_n and C_n block separately.

7.6.4.1 B-type blocks $(I_w^t = 0)$

<u>*Retailer 1*</u>

In B_1 blocks, for the states where $T_1^t = 0$, positive inventory at retailer 1 corresponds to s_1+Q_1 states. For $T_1^t > 0$, there are nQ_1 different levels of T_1, while in each level s_1 states correspond to $I_1^t > 0$. If $b_1 = Bl_n / Bl_1$ the number of B_1 blocks in B_n, and *lp+1* the number of the first state of the B_n block under consideration

$$WIP_1^{B,lp} = \sum_{i=1}^{b_1}\left(\sum_{j=1}^{s_1+Q_1} j\cdot p\big(lp + (i-1)\cdot Bl_1 + j + 1\big) + \sum_{j=1}^{nQ_1}\sum_{z=1}^{s_1} z\cdot p\big(lp + (i-1)\cdot Bl_1 + s_1 + Q_1 + 1 + (j-1)\cdot(s_1+1) + z + 1\big)\right)$$

<u>*Retailers 2 to n*</u>

With the same iterative approach we can calculate the average inventory on hand for higher priority retailers. In B_r blocks, for each state of retailer *r* correspond Bl_{r-1} states of the lower priority retailers. In B-type blocks, for the states where $T_r^t = 0$, inventory at retailer *r* can take s_r+Q_r values. For $T_r^t > 0$, there are nQ_r different levels of T_r^t, while in each level s_r values correspond to $I_r^t > 0$. If $b_r = Bl_n / Bl_r$ and *lp+1* the number of the first state of the B_n block under consideration:

$$WIP_r^{B1} = \sum_{i=1}^{b_r}\left(\sum_{j=1}^{s_r+Q_r}\sum_{z=1}^{Bl_{r-1}} j\cdot p\big(lp+(i-1)\cdot Bl_r + Bl_{r-1} + (j-1)\cdot Bl_{r-1} + z\big)\right)$$

$$WIP_r^{B2} = \sum_{i=1}^{b_r}\sum_{j=1}^{nQ_r}\sum_{z=1}^{s_r}\sum_{y=1}^{Bl_{r-1}} z\cdot p\big(lp+(i-1)\cdot Bl_r + (s_r+Q_r+1)\cdot Bl_{r-1} + (j-1)\cdot(s_r+1)\cdot Bl_{r-1} + Bl_{r-1} + (z-1)\cdot Bl_{r-1} + y\big)$$

$$WIP_r^{B,lp} = WIP_r^{B1} + WIP_r^{B2}$$

7.6.4.2 C-type blocks $(I_w^t > 0)$

<u>*Retailer 1*</u>

When $T_1^t = 0$, there are Q_1 different states where $I_1^t > 0$. For $T_1^t > 0$, there are nQ_1 different levels of T_1^t, and in each level s_1 states correspond to $I_1^t > 0$. If $c_1 = Cl_n / Cl_1$ the number of C_1 blocks in C_n, counting from the first occurrence of C_1 in each I_w^t level and *lp+1* the number of the first state of the C_n block under consideration

$$WIP_1^{C1} = \sum_{j=1}^{c_1}\sum_{z=1}^{Q_1}(z+s_1)\cdot p(lp+(j-1)\cdot Cl_1 + z$$

$$WIP_1^{C2} = \sum_{j=1}^{c_1}\sum_{z=1}^{nQ_1}\sum_{y=1}^{s_1} y\cdot p\big(lp+(j-1)\cdot Cl_1 + Q_1 + (z-1)\cdot(s_1+1) + y + 1\big)$$

$$WIP_1^{C,lp} = WIP_1^{C1} + WIP_1^{C2}$$

<u>*Retailers 2 to n*</u>

In C_r blocks each state of retailer *r* corresponds to Cl_{r-1} states of the lower priority retailers. When $T_r^t = 0$, there are Q_r different values for $I_r^t > 0$. For $T_r^t > 0$, there are nQ_r different levels of T_r^t, while in each level correspond s_r different levels of positive I_r^t. If $c_r = Cl_n / Cl_r$ the number of C_r blocks in C_n:

$$WIP_1^{C1} = \sum_{j=1}^{c_r}\sum_{z=1}^{Q_r}\sum_{y=1}^{Cl_{r-1}}(s_r+z)\cdot p\big(lp+(j-1)\cdot Cl_r + (z-1)\cdot Cl_{r-1} + y\big)$$

$$WIP_r^{C2} = \sum_{j=1}^{c_r}\sum_{z=1}^{nQ_r}\sum_{y=1}^{s_r}\sum_{x=1}^{Cl_{r-1}} y\cdot p\big(lp+(j-1)\cdot Cl_r + Q_r\cdot Cl_{r-1} + (z-1)\cdot(s_r+1)\cdot Cl_{r-1} + (y-1)\cdot Cl_{r-1} + Cl_{r-1} + x\big)$$

$$WIP_r^{C,lp} = WIP_r^{C1} + WIP_r^{C2}$$

7.6.4.3 WIP$_r$

The average inventory at retailer *r* ($1 \le r \le n$) will be the sum $WIP_r^{B,lp}$ for B_n blocks and $WIP_r^{C,lp}$ for C_n blocks.

$$WIP_r = \sum_{lp, I_w^t = 0} WIP_r^{B,lp} + \sum_{lp, I_w^t > 0} WIP_r^{C,lp}$$

7.6.5 Stock-out probability for the retailers – SO_r

SO_r is the probability that the external demand at retailer *r* will become lost sales. Since external demand at the retailers is independent and uniformly distributed in time, SO_r will be same as the probability of inventory on hand at retailer *r* being zero. The calculation of SO_r is done for each B_n and C_n block separately. We denote *lp+1* the number of the first state of each B_n/C_n block.

7.6.5.1 B-type blocks $(I_w^t = 0)$

Retailer 1

I_1^t is zero in the first state of each B_1 block of states, where also $T_1^t = 0$. For $T_1^t > 0$, I_1^t is zero in the first state of each (s_1+1)-dimension sub-blocks corresponding to different T_1^t values. If $b_1 = Bl_n / Bl_1$

$$SO_1^{B,lp} = \sum_{i=1}^{b_1}\left(p\big(lp+(i-1)\cdot Bl_1+1\big) + \sum_{j=1}^{nQ_1} p\big(lp+(i-1)\cdot Bl_1+s_1+Q_1+1+(j-1)\cdot(s_1+1)+1\big)\right)$$

Retailers 2 to n

In B_r blocks, each retailer *r* state (*r>1*) corresponds to Bl_{r-1} states of the lower priority retailers. I_r^t is zero in the first Bl_{r-1} states of each B_r block of states, where T_r^t is also zero. For $T_r^t > 0$, $I_r^t = 0$ in the first Bl_{r-1} states of each $(s_r+1)\cdot Bl_{r-1}$ dimension sub-block corresponding to a particular T_r^t value. If $b_r = Bl_n / Bl_r$:

$$SO_r^{B,lp} =$$

$$\sum_{i=1}^{b_r}\left(\sum_{j=1}^{Bl_{r-1}}\big(p\big(lp+(i-1)\cdot Bl_r+j\big)\big) + \sum_{j=1}^{nQ_r}\sum_{z=1}^{Bl_{r-1}} p\big(lp+(i-1)\cdot Bl_r+(s_r+Q_r+1)\cdot Bl_{r-1}+(j-1)\cdot(s_r+1)\cdot Bl_{r-1}+z\big)\right)$$

7.6.5.2 C-type blocks $(I_w^t > 0)$

Retailer 1

In C_1 blocks $I_1^t = 0$ only when $T_1^t > 0$ and in the first state of each of (s_1+1)-dimension sub-blocks corresponding to different T_1^t values. If $c_1 = Cl_n / Cl_1$

$$SO_1^{C,lp} = \sum_{j=1}^{c_1}\sum_{z=1}^{nQ_1} p\big(lp+(j-1)\cdot Cl_1+Q_1+(z-1)\cdot(s_1+1)+1\big)$$

Retailers 2 to n

In C_r blocks each retailer *r* state corresponds to Cl_{r-1} states of the lower priority retailers. $I_r^t = 0$ only when $T_r^t > 0$ and in the first state of each $(s_r+1)\cdot Bl_{r-1}$ dimension sub-block corresponding to a particular T_r^t value. If $c_r = Cl_n / Cl_r$

$$SO_r^{C,lp} = \sum_{j=1}^{c_r}\sum_{z=1}^{nQ_r}\sum_{y=1}^{Cl_{r-1}} p\big(lp+(j-1)\cdot Cl_r+Q_r\cdot Cl_{r-1}+(z-1)\cdot(s_r+1)\cdot Cl_{r-1}+y\big)$$

7.6.5.3 SO_r

Stock out probability for retailer *r* ($1 \le r \le n$) will be the sum $SO_r^{B,lp}$ for B_n blocks and $SO_r^{C,lp}$ for C_n blocks.

$$SO_r = \sum_{lp, I_w^t = 0} SO_r^{B,lp} + \sum_{lp, I_w^t > 0} SO_r^{C,lp}$$

7.6.6 Stock-out probability for the Wholesaler – SOw_r

Stock-out at the wholesaler with regard to retailer *r* is the probability that the wholesaler will have no inventory to respond to retailer's *r* demand for a replenishment order. It is the probability that $I_w^t = 0$, given that $I_r^t \le s_r$ and $T_r^t = 0$. Only B_n blocks are of concern. SOw_r is calculated iteratively for each Bl_n block. We denote $b_r = Bl_n / Bl_r$ and *lp+1* the number of the first state of each B_n block.

For retailer 1, $T_1^t = 0$ while $I_1^t \le s_1$ occurs only in the first (s_1+1) states of each B_1 block. For higher priority retailers, $T_r^t = 0$ while $I_r^t \le s_r$ occurs in the first $(s_r+1)\cdot Bl_{r-1}$ states of each B_r block of states.

$$SOw_1^{lp} = \sum_{i=1}^{b_1} \sum_{j=1}^{s_1+1} p\big(lp + (i-1) \cdot Bl_1 + j\big)$$

$$SOw_r^{lp} = \sum_{i=1}^{b_r} \sum_{j=1}^{s_r+1} \sum_{z=1}^{Bl_{r-1}} p\big(lp + (i-1) \cdot Bl_r + (j-1) \cdot Bl_{r-1} + z\big)$$

The Stock-out probability for the wholesaler with regard to retailer *r* ($1 \le r \le n$) will be the sum SOw_r^{lp} for all B_n blocks.

$$SOw_r = \sum_{lp, I_w^t = 0} SOw_r^{lp}$$

7.6.7 Stock-out probability for the DC – SO_d

Stock-out probability at the DC is the probability that the DC will have no inventory to respond to wholesaler's demand for a replenishment order. It is the probability that $I_d^t = 0$, given that $I_w^t \le s_w$ and $T_w^t = 0$.

$$SO_d = \sum_{i=1}^{Bl_n + nsw \cdot Cl_n} p(i)$$

7.6.8 Utilization of transportation resource towards the retailers – u_r

Utilization of the transportation resource towards retailer *r* is the percentage of time that there is a replenishment order in transit towards retailer *r*. To calculate u_r we sum the stationary probabilities of states corresponding to $T_r^t > 0$. We calculate u_r in each B_n and C_n block iteratively. We denote $b_r = Bl_n / Bl_r$, $c_r = Cl_n / Cl_r$ and *lp+1* the number of the first state of each B_n block.

B_n blocks

For Retailer 1:

$$u_1^{B,lp} = \sum_{i=1}^{b_1} \sum_{j=1}^{nQ_1} \sum_{z=1}^{s_1+1} p\big(lp + (i-1) \cdot Bl_1 + s_1 + Q_1 + 1 + (j-1) \cdot (s_1 + 1) + z\big)$$

For higher priority retailers 2 to n:

$$u_r^{B,lp} = \sum_{i=1}^{b_r} \sum_{j=1}^{nQ_r} \sum_{z=1}^{s_r+1} \sum_{y=1}^{Bl_{r-1}} p\big(lp + (i-1) \cdot Bl_r + (s_r + Q_r + 1) \cdot Bl_{r-1} + (j-1) \cdot (s_r + 1) \cdot Bl_{r-1} + (z-1) \cdot Bl_{r-1} + y\big)$$

C_n blocks

For Retailer 1:

$$u_1^{C,lp} = \sum_{j=1}^{c_1} \sum_{z=1}^{nQ_1} \sum_{y=1}^{s_1+1} p\big(lp + (j-1) \cdot Cl_1 + Q_1 + (z-1) \cdot (s_1 + 1) + y\big)$$

For higher priority retailers 2 to n:

$$u_r^{C,lp} = \sum_{j=1}^{c_r} \sum_{z=1}^{nQ_r} \sum_{y=1}^{s_r+1} \sum_{x=1}^{Cl_{r-1}} p\big(lp + (j-1) \cdot Cl_r + Q_r \cdot Cl_{r-1} + (z-1) \cdot (s_r + 1) \cdot Cl_{r-1} + (y-1) \cdot Cl_{r-1} + x\big)$$

The Utilization u_r for retailer r ($1 \le r \le n$) will be the sum of $u_r^{B,lp}$ of B_n blocks and $u_r^{C,lp}$ of C_n blocks.

$$u_r = \sum_{lp, I_w^t = 0} u_r^{B,lp} + \sum_{lp, I_w^t > 0} u_r^{C,lp}$$

7.6.9 Fill rate at the retailers – FR_r

Fill rate is the percentage of external customers arriving at retailer r whose demand is met from inventory on hand at the retailer.

$$FR_r = 1 - SO_r$$

7.6.10 Throughput at the retailers – Thr_r

Throughput is the number of product-units per time-unit that flow through retailer r. Alternatively, Thr_r could be defined as the rate of sales at retailer r.

$$Thr_r = \lambda_r \cdot FR_r$$

7.6.11 Average inventory in transit - $WIPtr_r$

$WIPtr_r$ is the average (in time) inventory in transit from the wholesaler towards retailer r. It can be calculate through Little's Law:

$$WIPtr_r = \frac{Thr_r}{\mu_r}$$

7.6.12 Average replenishment order - ARO_r

ARO_r is the average number of product units per replenishment order sent to retailer *r*. Along with SOw_r it allows us to evaluate the performance of the wholesaler.

$$ARO_r = \frac{WIPtr_r}{u_r}$$

7.6.13 Total Throughput – Thr_{total}

Total Throughput is the number of product-units per time-unit that flow through the system. It can be easily calculated using the retailers' Throughput.

$$Thr_{total} = \sum_{r=1}^{n} Thr_r$$

7.6.14 Average inventory in transit towards the wholesaler - $WIPtr_w$

$WIPtr_w$ is the average (in time) inventory in transit from the DC towards the wholesaler. It can be calculate through Little's Law:

$$WIPtr_w = \frac{Thr_{total}}{\mu_w}$$

7.6.15 Average replenishment order towards the Wholesaler – ARO_w

ARO_w is the average number of product units per replenishment order sent from the DC to the Wholesaler. Along with SO_d it allows us to evaluate the performance of the DC.

$$ARO_w = \frac{WIPtr_w}{u_w}$$

7.7 Illustrative example

To illustrate the algorithmic steps described above, we present the analysis for a simple example with two retailers and parameters s_d=0, Q_d=2 ,s_w=0, Q_w=2, s_1=2, Q_1=2, s_2=0, Q_2=1.

7.7.1 States definition and states transitions

7.7.1.1 States definition

The system is a 7-dimensional continuous time Markov chain $\{I_d^t, T_w^t, I_w^t, T_2^t, I_2^t, T_1^t, I_1^t, t \geq 0\}$ and at any given time the state of the system can be defined by the 7-dimensional vector $\bar{S}_t = (I_d^t, T_w^t, I_w^t, T_2^t, I_2^t, T_1^t, I_1^t)$, where:

I_d^t: The Inventory on hand at the Distribution Centre (DC) at time *t*. $0 \leq I_d^t \leq 2$
T_w^t: The Inventory in transit towards the Wholesaler at time t. $0 \leq T_w^t \leq 2$
I_w^t: The Inventory on hand at the Wholesaler at time *t*. $0 \leq I_w^t \leq 2$
T_2^t: The Inventory in transit towards the Retailer 2 at time t. $0 \leq T_2^t \leq 1$
I_2^t: The Inventory on hand at retailer 2 at time *t*. $0 \leq I_2^t \leq 1$

T_1^t: The Inventory in transit towards retailer 1 at time t. $0 \leq T_1^t \leq 2$
I_1^t: The Inventory on hand at retailer 1 at time t. $0 \leq I_1^t \leq 4$

Since we assume the Plant to be saturated, the inventory at the Distribution Centre always increases by Q_d=2 units. At the same time, the wholesaler always asks for replenishment orders of Q_w=2 units. Even if the system started with $I_d^t = 1$, this state would be transient and in the long term the possible values for the inventory on hand at the DC will be 0 and 2 (*bsd*=2). The possible values for inventory in transit towards the wholesaler will also be 0 and 2.

The inventory at the wholesaler always increases by Q_w=2, while the retailers ask for 1 (Q_2) or 2 (Q_1) units per order. The possible values for the inventory on hand at the wholesaler will be 0, 1 and 2 (*bsw*=1). Inventories in transit towards the retailers will also be multiples of *bsw*. Summing up the possible values for each system variable:

$I_d^t = \{0,2\}$

$T_w^t = \{0,2\}$

$I_w^t = \{0,1,2\}$

$T_2^t = \{0,1\}$

$I_2^t = \{0,1\}$

$T_1^t = \{0,1,2\}$

$I_1^t = \{0,1,2,3,4\}$

The state space Ω of the Markov process is comprised of all permissible $\bar{S}_t$ vectors. In the example under consideration there are 163 possible states. These states are ordered linearly, using a lexicographical ordering. We use as basic level the inventory at the DC I_d^t, and the basic levels are ordered from lower to higher values. Within a basic level, the states are ordered according to T_w^t and then I_w^t. Again the ordering is from lower to higher values. For fixed values of I_d^t, T_w^t, and I_w^t, the states are grouped in increasing order of T_2^t, I_2^t, T_1^t, and I_1^t. The possible states for our example and their respective ordering are given in figure 7.2.

Figure 7.2: States definition and hierarchy for s_d=0, Q_d=2, s_w=0, Q_w=2, s_2=1, Q_2=1, s_1=2, Q_1=2

S/N	State	I_d^t	T_w^t	I_w^t	T_2^t	I_2^t	T_1^t	I_1^t		S/N	State	I_d^t	T_w^t	I_w^t	T_2^t	I_2^t	T_1^t	I_1^t
1	000-00.00	0	0	0	0	0	0	0		83	020-01.11	0	2	0	0	1	1	1
2	000-00.01	0	0	0	0	0	0	1		84	020-01.12	0	2	0	0	1	1	2
3	000-00.02	0	0	0	0	0	0	2		85	020-01.20	0	2	0	0	1	2	0
4	000-00.03	0	0	0	0	0	0	3		86	020-01.21	0	2	0	0	1	2	1
5	000-00.04	0	0	0	0	0	0	4		87	020-01.22	0	2	0	0	1	2	2
6	000-00.10	0	0	0	0	0	1	0		88	020-10.00	0	2	0	1	0	0	0
7	000-00.11	0	0	0	0	0	1	1		89	020-10.01	0	2	0	1	0	0	1
8	000-00.12	0	0	0	0	0	1	2		90	020-10.02	0	2	0	1	0	0	2
9	000-00.20	0	0	0	0	0	2	0		91	020-10.03	0	2	0	1	0	0	3
10	000-00.21	0	0	0	0	0	2	1		92	020-10.04	0	2	0	1	0	0	4
11	000-00.22	0	0	0	0	0	2	2		93	020-10.10	0	2	0	1	0	1	0
12	000-01.00	0	0	0	0	1	0	0		94	020-10.11	0	2	0	1	0	1	1
13	000-01.01	0	0	0	0	1	0	1		95	020-10.12	0	2	0	1	0	1	2
14	000-01.02	0	0	0	0	1	0	2		96	020-10.20	0	2	0	1	0	2	0
15	000-01.03	0	0	0	0	1	0	3		97	020-10.21	0	2	0	1	0	2	1
16	000-01.04	0	0	0	0	1	0	4		98	020-10.22	0	2	0	1	0	2	2
17	000-01.10	0	0	0	0	1	1	0		99	201-01.03	2	0	1	0	1	0	3
18	000-01.11	0	0	0	0	1	1	1		100	201-01.04	2	0	1	0	1	0	4
19	000-01.12	0	0	0	0	1	1	2		101	201-01.10	2	0	1	0	1	1	0
20	000-01.20	0	0	0	0	1	2	0		102	201-01.11	2	0	1	0	1	1	1
21	000-01.21	0	0	0	0	1	2	1		103	201-01.12	2	0	1	0	1	1	2
22	000-01.22	0	0	0	0	1	2	2		104	201-01.20	2	0	1	0	1	2	0
23	000-10.00	0	0	0	1	0	0	0		105	201-01.21	2	0	1	0	1	2	1
24	000-10.01	0	0	0	1	0	0	1		106	201-01.22	2	0	1	0	1	2	2
25	000-10.02	0	0	0	1	0	0	2		107	201-10.03	2	0	1	1	0	0	3
26	000-10.03	0	0	0	1	0	0	3		108	201-10.04	2	0	1	1	0	0	4
27	000-10.04	0	0	0	1	0	0	4		109	201-10.10	2	0	1	1	0	1	0
28	000-10.10	0	0	0	1	0	1	0		110	201-10.11	2	0	1	1	0	1	1
29	000-10.11	0	0	0	1	0	1	1		111	201-10.12	2	0	1	1	0	1	2
30	000-10.12	0	0	0	1	0	1	2		112	201-10.20	2	0	1	1	0	2	0
31	000-10.20	0	0	0	1	0	2	0		113	201-10.21	2	0	1	1	0	2	1
32	000-10.21	0	0	0	1	0	2	1		114	201-10.22	2	0	1	1	0	2	2
33	000-10.22	0	0	0	1	0	2	2		115	202-01.03	2	0	2	0	1	0	3
34	001-01.03	0	0	1	0	1	0	3		116	202-01.04	2	0	2	0	1	0	4
35	001-01.04	0	0	1	0	1	0	4		117	202-01.10	2	0	2	0	1	1	0
36	001-01.10	0	0	1	0	1	1	0		118	202-01.11	2	0	2	0	1	1	1
37	001-01.11	0	0	1	0	1	1	1		119	202-01.12	2	0	2	0	1	1	2
38	001-01.12	0	0	1	0	1	1	2		120	202-01.20	2	0	2	0	1	2	0
39	001-01.20	0	0	1	0	1	2	0		121	202-01.21	2	0	2	0	1	2	1
40	001-01.21	0	0	1	0	1	2	1		122	202-01.22	2	0	2	0	1	2	2
41	001-01.22	0	0	1	0	1	2	2		123	202-10.03	2	0	2	1	0	0	3
42	001-10.03	0	0	1	1	0	0	3		124	202-10.04	2	0	2	1	0	0	4
43	001-10.04	0	0	1	1	0	0	4		125	202-10.10	2	0	2	1	0	1	0
44	001-10.10	0	0	1	1	0	1	0		126	202-10.11	2	0	2	1	0	1	1
45	001-10.11	0	0	1	1	0	1	1		127	202-10.12	2	0	2	1	0	1	2
46	001-10.12	0	0	1	1	0	1	2		128	202-10.20	2	0	2	1	0	2	0
47	001-10.20	0	0	1	1	0	2	0		129	202-10.21	2	0	2	1	0	2	1
48	001-10.21	0	0	1	1	0	2	1		130	202-10.22	2	0	2	1	0	2	2
49	001-10.22	0	0	1	1	0	2	2		131	220-00.00	2	2	0	0	0	0	0
50	002-01.03	0	0	2	0	1	0	3		132	220-00.01	2	2	0	0	0	0	1
51	002-01.04	0	0	2	0	1	0	4		133	220-00.02	2	2	0	0	0	0	2
52	002-01.10	0	0	2	0	1	1	0		134	220-00.03	2	2	0	0	0	0	3
53	002-01.11	0	0	2	0	1	1	1		135	220-00.04	2	2	0	0	0	0	4
54	002-01.12	0	0	2	0	1	1	2		136	220-00.10	2	2	0	0	0	1	0

55	002-01.20	0	0	2	0	1	2	0		137	220-00.11	2	2	0	0	0	1	1
56	002-01.21	0	0	2	0	1	2	1		138	220-00.12	2	2	0	0	0	1	2
57	002-01.22	0	0	2	0	1	2	2		139	220-00.20	2	2	0	0	0	2	0
58	002-10.03	0	0	2	1	0	0	3		140	220-00.21	2	2	0	0	0	2	1
59	002-10.04	0	0	2	1	0	0	4		141	220-00.22	2	2	0	0	0	2	2
60	002-10.10	0	0	2	1	0	1	0		142	220-01.00	2	2	0	0	1	0	0
61	002-10.11	0	0	2	1	0	1	1		143	220-01.01	2	2	0	0	1	0	1
62	002-10.12	0	0	2	1	0	1	2		144	220-01.02	2	2	0	0	1	0	2
63	002-10.20	0	0	2	1	0	2	0		145	220-01.03	2	2	0	0	1	0	3
64	002-10.21	0	0	2	1	0	2	1		146	220-01.04	2	2	0	0	1	0	4
65	002-10.22	0	0	2	1	0	2	2		147	220-01.10	2	2	0	0	1	1	0
66	020-00.00	0	2	0	0	0	0	0		148	220-01.11	2	2	0	0	1	1	1
67	020-00.01	0	2	0	0	0	0	1		149	220-01.12	2	2	0	0	1	1	2
68	020-00.02	0	2	0	0	0	0	2		150	220-01.20	2	2	0	0	1	2	0
69	020-00.03	0	2	0	0	0	0	3		151	220-01.21	2	2	0	0	1	2	1
70	020-00.04	0	2	0	0	0	0	4		152	220-01.22	2	2	0	0	1	2	2
71	020-00.10	0	2	0	0	0	1	0		153	220-10.00	2	2	0	1	0	0	0
72	020-00.11	0	2	0	0	0	1	1		154	220-10.01	2	2	0	1	0	0	1
73	020-00.12	0	2	0	0	0	1	2		155	220-10.02	2	2	0	1	0	0	2
74	020-00.20	0	2	0	0	0	2	0		156	220-10.03	2	2	0	1	0	0	3
75	020-00.21	0	2	0	0	0	2	1		157	220-10.04	2	2	0	1	0	0	4
76	020-00.22	0	2	0	0	0	2	2		158	220-10.10	2	2	0	1	0	1	0
77	020-01.00	0	2	0	0	1	0	0		159	220-10.11	2	2	0	1	0	1	1
78	020-01.01	0	2	0	0	1	0	1		160	220-10.12	2	2	0	1	0	1	2
79	020-01.02	0	2	0	0	1	0	2		161	220-10.20	2	2	0	1	0	2	0
80	020-01.03	0	2	0	0	1	0	3		162	220-10.21	2	2	0	1	0	2	1
81	020-01.04	0	2	0	0	1	0	4		163	220-10.22	2	2	0	1	0	2	2
82	020-01.10	0	2	0	0	1	1	0										

7.7.1.2 State transitions

The state of the system can be altered instantaneously by four kinds of events:

1. The arrival of an outstanding order from the Plant at the Distribution Centre (DC). Such orders occur when $I_d^t = 0$ and always Q_d=2 units are delivered. If there is no outstanding demand from the Wholesaler $(I_w^t = 2)$, then the inventory at the DC increases $(I_d^{t+dt} = 2)$. When there is demand from the Wholesaler $(I_w^t = 0)$, the incoming order is forwarded to the Wholesaler $(T_w^{t+dt} = 2)$. The instantaneous transition rate of the event is μ_d.
2. The arrival of an outstanding order from the DC at the Wholesaler. Such orders occur when $I_w^t = 0$ and always Q_w=2 units are delivered. If there is no outstanding demand from the retailers, inventory at the wholesaler increases $(I_w^{t+dt} = 2)$. When there is demand only from retailer 1, the incoming order is immediately forwarded $(T_1^{t+dt} = 2)$. If there is demand only from retailer 2, one unit is forwarded $(T_2^{t+dt} = 1)$ and one unit remains at the wholesaler $(I_w^{t+dt} = 1)$. If there is demand from both the retailers, first the demand from the higher priority retailer 2 is met $(T_2^{t+dt} = 1)$ and then a partial order is sent to retailer 1 $(T_1^{t+dt} = 1)$. The instantaneous transition rate of such events is μ_w.

3. The arrival of an outstanding order at one of the retailers. The inventory on hand at the respective retailer increases accordingly. In the case of retailer 2, since s_2=0, there can be no reordering on arrival. In the case of retailer 1, if $I_1^{t+dt} \leq 2$, a new replenishment order is initiated. T_1^{t+dt} depends on the available inventory at the wholesaler I_w^t. The instantaneous transition rate for these transitions is μ_1 and μ_2 for retailer 1 and 2 respectively.
4. The occurrence of external demand at one of the retailers. Since we assume unitary demand, inventory on hand of the retailer always decreases by one unit $(I_i^{t+dt} = I_i^t - 1)$. In the case of retailer 2, the occurrence of external demand will always trigger a replenishment order (base-stock policy). In the case of retailer 1, a new order is asked from the wholesaler when $I_1^{t+dt} \leq 2$. T_1^{t+dt} depends on the available inventory at the wholesaler I_w^t. The instantaneous rate for these transitions will be λ_1 and λ_2 for retailer 1 and 2 respectively.

7.7.2 The Infinitesimal Generator Matrix

7.7.2.1 Diagonal elements

$bsd = GCD(Q_d, Q_w) = 2$, $bsw = GCD(bsd, Q_1, Q_2) = 1$

$nQ_1 = \dfrac{Q_1}{bsw} = 2$, $nQ_2 = \dfrac{Q_2}{bsw} = 1$

We use as building blocks the sub-matrices describing transitions in the state of the retailers for a given state of the upstream nodes (B_2, C_2).

B-type Blocks (I_w=0)

In the case where $I_w^t = 0$ (B-blocks), Bl_1 = 11, Bl_2 = 33.

Retailer 1

$$B_1^z = \begin{bmatrix} -\mu_d-\mu_w & 0 & 0 & 0 & 0 \\ \lambda_1 & -\mu_d-\mu_w-\lambda_1 & 0 & 0 & 0 \\ 0 & \lambda_1 & -\mu_d-\mu_w-\lambda_1 & 0 & 0 \\ 0 & 0 & \lambda_1 & -\mu_d-\mu_w-\lambda_1 & 0 \\ 0 & 0 & 0 & \lambda_1 & -\mu_d-\mu_w-\lambda_1 \end{bmatrix}$$

$$B_1^p = \begin{bmatrix} -\mu_d-\mu_w-\mu_1 & 0 & 0 \\ \lambda_1 & -\mu_d-\mu_w-\mu_1-\lambda_1 & 0 \\ 0 & \lambda_1 & -\mu_d-\mu_w-\mu_1-\lambda_1 \end{bmatrix}$$

$$M_1 = \begin{bmatrix} 0 & \mu_1 & 0 & 0 & 0 \\ 0 & 0 & \mu_1 & 0 & 0 \\ 0 & 0 & 0 & \mu_1 & 0 \\ 0 & 0 & \mu_1 & 0 & 0 \\ 0 & 0 & 0 & \mu_1 & 0 \\ 0 & 0 & 0 & 0 & \mu_1 \end{bmatrix}$$

$$B_1 = \begin{bmatrix}
-\mu d-\mu w & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\lambda 1 & -\mu d-\mu w-\lambda 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \lambda 1 & -\mu d-\mu w-\lambda 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \lambda 1 & -\mu d-\mu w-\lambda 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \lambda 1 & -\mu d-\mu w-\lambda 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \mu 1 & 0 & 0 & 0 & -\mu d-\mu w-\mu 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \mu 1 & 0 & 0 & \lambda 1 & -\mu d-\mu w-\mu 1-\lambda 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \mu 1 & 0 & 0 & \lambda 1 & -\mu d-\mu w-\mu 1-\lambda 1 & 0 & 0 & 0 \\
0 & 0 & \mu 1 & 0 & 0 & 0 & 0 & 0 & -\mu d-\mu w-\mu 1 & 0 & 0 \\
0 & 0 & 0 & \mu 1 & 0 & 0 & 0 & 0 & \lambda 1 & -\mu d-\mu w-\mu 1-\lambda 1 & 0 \\
0 & 0 & 0 & 0 & \mu 1 & 0 & 0 & 0 & 0 & \lambda 1 & -\mu d-\mu w-\mu 1-\lambda 1
\end{bmatrix}$$

<u>Retailer 2</u>

B_2^z is a 22 x 22 block corresponding to $T_2^t = 0$. If I_{B1} the 11 x 11 identity matrix and O_1 an 11 x 11 zero matrix:

$$B_2^z = \begin{bmatrix} B_1 & O_1 \\ \lambda_2 \cdot I_{B1} & B_1 - \lambda_2 \cdot I_{B1} \end{bmatrix}$$

B_2^p is an 11 x 11 matrix corresponding to $T_2^t = 1$:

$$B_2^p = B_1 - \mu_2 \cdot I_{B1}$$

$$B_2^p = \begin{bmatrix}
-\mu d-\mu w-\mu 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\lambda 1 & -\mu d-\mu w-\mu 2-\lambda 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \lambda 1 & -\mu d-\mu w-\mu 2-\lambda 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \lambda 1 & -\mu d-\mu w-\mu 2-\lambda 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \lambda 1 & -\mu d-\mu w-\mu 2-\lambda 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \mu 1 & 0 & 0 & 0 & -\mu d-\mu w-\mu 2-\mu 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \mu 1 & 0 & 0 & \lambda 1 & -\mu d-\mu w-\mu 2-\mu 1-\lambda 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \mu 1 & 0 & 0 & \lambda 1 & -\mu d-\mu w-\mu 2-\mu 1-\lambda 1 & 0 & 0 & 0 \\
0 & 0 & \mu 1 & 0 & 0 & 0 & 0 & 0 & -\mu d-\mu w-\mu 2-\mu 1 & 0 & 0 \\
0 & 0 & 0 & \mu 1 & 0 & 0 & 0 & 0 & \lambda 1 & -\mu d-\mu w-\mu 2-\mu 1-\lambda 1 & 0 \\
0 & 0 & 0 & 0 & \mu 1 & 0 & 0 & 0 & 0 & \lambda 1 & -\mu d-\mu w-\mu 2-\mu 1-\lambda 1
\end{bmatrix}$$

M_2 is an 11 x 22 matrix.

$$M_2 = \begin{bmatrix} O_1 & \mu_2 \cdot I_{B1} \end{bmatrix}$$

If $O_{22\times 11}$ a zero matrix of 22 x 11 dimensions, the overall structure of block B_2 will be

$$B_2 = \begin{bmatrix} B_2^z & O_{22\times 11} \\ M_2 & B_2^p \end{bmatrix}$$

C-type Blocks ($I_w>0$)

In the case where $I_w^t > 0$ (C-blocks), $Cl_1 = 8$, $Cl_2 = 16$.

Retailer 1

Block C_1 is an 8x8 block

C_1^z is an 2x2 block :

$$C_1^z = \begin{bmatrix} -\mu_d-\mu_w-\lambda_2-\lambda_1 & 0 \\ \lambda_1 & -\mu_d-\mu_w-\lambda_2-\lambda_1 \end{bmatrix}$$

C_1^p is a 3x3 matrix that corresponds to states where $T_1^t > 0$. There are two C_1^p blocks in C_1, one corresponding to $T_1^t = 1$ and the other to $T_1^t = 2$.

$$C_1^p = \begin{bmatrix} -\mu_d-\mu_w-\mu_1-\lambda_2 & 0 & 0 \\ \lambda_1 & -\mu_d-\mu_w-\mu_1-\lambda_2-\lambda_1 & 0 \\ 0 & \lambda_1 & -\mu_d-\mu_w-\mu_1-\lambda_2-\lambda_1 \end{bmatrix}$$

M_1^C is an 6x2 matrix:

$$M_1^C = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ \mu_1 & 0 \\ 0 & 0 \\ \mu_1 & 0 \\ 0 & \mu_1 \end{bmatrix}$$

The overall structure of C_1 will be

$$C_1 = \begin{bmatrix} -\mu d-\mu w-\lambda 2-\lambda 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \lambda 1 & -\mu d-\mu w-\lambda 2-\lambda 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\mu d-\mu w-\lambda 2-\mu 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \lambda 1 & -\mu d-\mu w-\lambda 2-\mu 1-\lambda 1 & 0 & 0 & 0 & 0 \\ \mu 1 & 0 & 0 & \lambda 1 & -\mu d-\mu w-\lambda 2-\mu 1-\lambda 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -\mu d-\mu w-\lambda 2-\mu 1 & 0 & 0 \\ \mu 1 & 0 & 0 & 0 & 0 & \lambda 1 & -\mu d-\mu w-\lambda 2-\mu 1-\lambda 1 & 0 \\ 0 & \mu 1 & 0 & 0 & 0 & 0 & \lambda 1 & -\mu d-\mu w-\lambda 2-\mu 1-\lambda 1 \end{bmatrix}$$

Retailer 2

C_2 is a 16x16 matrix. We denote $\mathbf{I}_{1b}$ the identity matrix of 8x8 dimensions and $\mathbf{O}_1$ a zero square matrix of 8x8 dimensions. C_2^z is an 8x8 matrix corresponding to states where $T_2^t = 0$, $I_2^t = 1$. In the example under consideration, since Q_2=1, C_2^z is identical to C_1:

$$C_2^z = C_1$$

C_2^p is an 8x8 matrix corresponding to $T_2^t = 1$ and $I_2^t = 0$

$$C_2^p = C_1 + (\lambda_2 - \mu_2) \cdot I_{1b}$$

M_2^C is an 8x8 diagonal matrix of μ_2:

$M_2^C = \mu_2 \cdot I_{1b}$

Having constructed the constituent matrices:

$$C_2 = \begin{bmatrix} C_2^z & O_1 \\ M_2^C & C_2^p \end{bmatrix}$$

General structure of the diagonal tier

nQw=1

nsw=0

NLw=2

NLd=1

nsd=0

We also denote I_{2b} the identity matrix of 33 dimension, and I_{2c} the identity matrix 16 dimension.

The sub-matrix D_0 corresponding to basic level where $I_d^t = 0$ is a 98x98 block (L_0=98):

$$D_0 = \begin{bmatrix} D_0^z & \\ & D_0^p \end{bmatrix}$$

$$D_0^z = \begin{bmatrix} B_2 + \mu_w \cdot I_{2b} & & \\ & C_2 + \mu_w \cdot I_{2c} & \\ & & C_n + \mu_w \cdot I_{2c} \end{bmatrix}$$

$$D_0^p = B_2$$

For basic levels where $I_d^t > 0$, the corresponding block D_1 is a 65x65 block (L_1=65) :

$$D_1 = \begin{bmatrix} D_1^z & \\ & D_1^p \end{bmatrix}$$

,where

$$D_1^z = \begin{bmatrix} C_2 + \mu_w \cdot I_{2c} & \\ & C_2 + \mu_w \cdot I_{2c} \end{bmatrix}$$

$$D_1^p = B_2$$

If I_L the identity matrix of L_1=65 dimension, the infinitesimal generator matrix P will be:

$$P = \begin{bmatrix} D_0 & \\ & D_1 + \mu_d \cdot I_L \end{bmatrix}$$

7.7.2.2 Sub-Diagonal elements

Retailer 1

Given the parameters of the system, whenever there is triggering of a replenishment order the inventory on hand at the wholesaler will become zero $\left(I_w^{t+dt} = 0\right)$. k_1=2. The block S_1 which describes triggering of a replenishment order towards retailer 1 is a 8x2 matrix:

$$S_1 = \begin{bmatrix} 0 & \lambda_1 \\ 0 & 0 \\ \mu_1 & 0 \\ 0 & \mu_1 \\ 0 & 0 \\ 0 & \mu_1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Retailer 2

Since s_2=0, there cannot be reordering on the arrival of a previous replenishment order. In sub-diagonal blocks there are no μ_2 elements. k_2=0.

When there is transition to states where $I_w^{t+dt} > 0$, S_2 is 8x8 diagonal matrix of λ_2 .

In the case where $I_w^{t+dt} = 0$, the block $S_r^L(r)$ that describes triggering of a replenishment order towards retailer 2 , is a 8x11 matrix:

$$S_2^L(2) = \begin{bmatrix} 0 & 0 & 0 & \lambda_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \lambda_2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \lambda_2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \lambda_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \lambda_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \lambda_2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \lambda_2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \lambda_2 \end{bmatrix}$$

The general structure of the infinitesimal generator matrix is given in figure 7.3

7.7.3 Performance Measures

The performance measures are computed arithmetically based on the stationary probabilities. We denote **p** the vector of the stationary probabilities and p(i) its i[th] element according the hierarchy of states laid out in 7.7.1.1.

Average inventory at the Distribution Centre

$$WIP_d = 2 \cdot \sum_{i=99}^{163} p(i)$$

Utilization of resource for transportation towards the Wholesaler

$$u_w = \sum_{i=66}^{98} p(i) + \sum_{i=131}^{163} p(i)$$

Average inventory at the wholesaler

The probability of $I_w^t = 1$:

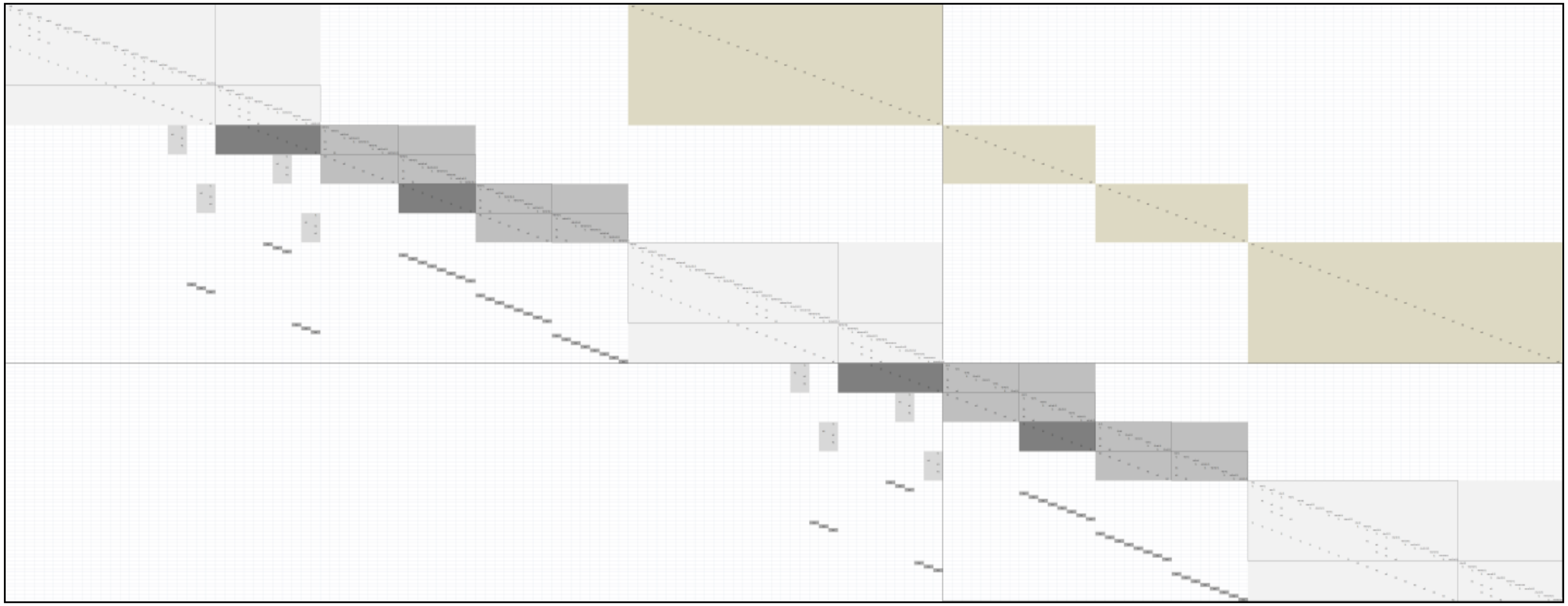
$$\Pr(I_w^t = 1) = \sum_{i=34}^{49} p(i) + \sum_{i=99}^{114} p(i)$$

The probability of $I_w^t = 2$:

$$\Pr(I_w^t = 2) = \sum_{i=50}^{65} p(i) + \sum_{i=115}^{130} p(i)$$

$$WIP_w = 1 \cdot \Pr(I_w^t = 1) + 2 \cdot \Pr(I_w^t = 2)$$

Figure 7.3: General structure of the infinitesimal generator matrix s_d=0, Q_d=2, s_w=0, Q_w=2, s_2=1, Q_2=1, s_1=2, Q_1=2. Different shades denote different blocks



Average inventory at the retailers

Retailer 1:

$$WIP_1^B = 1\cdot[\,p(2)+p(7)+p(10)+p(13)+p(18)+p(21)+p(24)+p(29)+p(32) + p(67)+p(72)+p(75)+p(78)+p(83)+p(86)+p(89)+p(94)+p(97) + p(132)+p(137)+p(140)+p(143)+p(148)+p(151)+p(154)+p(159)+p(162)] + 2\cdot[\,p(3)+p(8)+p(11)+p(14)+p(19)+p(22)+p(25)+p(30)+p(33) + p(68)+p(73)+p(76)+p(79)+p(84)+p(87)+p(90)+p(95)+p(98) + p(133)+p(138)+p(141)+p(144)+p(149)+p(152)+p(155)+p(160)+p(163)] + 3\cdot[\,p(4)+p(15)+p(26)+p(69)+p(80)+p(91)+p(134)+p(145)+p(156)] + 4\cdot[\,p(5)+p(16)+p(27)+p(70)+p(81)+p(92)+p(135)+p(146)+p(157)]$$

$$WIP_1^C = 1\cdot[\,p(37)+p(40)+p(45)+p(48)+p(53)+p(56)+p(61)+p(64) + p(102)+p(105)+p(110)+p(113)+p(118)+p(121)+p(126)+p(129)] + 2\cdot[\,p(38)+p(41)+p(46)+p(49)+p(54)+p(57)+p(62)+p(65) + p(103)+p(106)+p(111)+p(114)+p(119)+p(122)+p(127)+p(130)] + 3\cdot[\,p(34)+p(42)+p(50)+p(58)+p(99)+p(107)+p(115)+p(123)] + 4\cdot[\,p(35)+p(43)+p(51)+p(59)+p(100)+p(108)+p(116)+p(124)]$$

$$WIP_1 = WIP_1^B + WIP_1^C$$

Retailer 2:

$$WIP_2^B = \sum_{i=12}^{22} p(i) + \sum_{i=77}^{87} p(i) + \sum_{i=142}^{152} p(i)$$

$$WIP_2^C = \sum_{i=34}^{41} p(i) + \sum_{i=50}^{57} p(i) + \sum_{i=99}^{106} p(i) + \sum_{i=115}^{122} p(i)$$

$$WIP_2 = WIP_2^B + WIP_2^C$$

Stock-out probability for the retailers

Retailer 1:

$$SO_1^B = p(1)+p(6)+p(9)+p(12)+p(17)+p(20)+p(23)+p(28) +p(31) + p(66)+p(71)+p(74)+p(77)+p(82)+p(85)+p(88)+p(93) +p(96) + p(131)+p(136)+p(139)+p(142)+p(147)+p(150)+p(153)+p(158) +p(161)$$

$$SO_1^C = p(36)+p(39)+p(44)+p(47)+p(52)+p(55)+p(60)+p(63) + p(101)+p(104)+p(109)+p(112)+p(117)+p(120)+p(125)+p(128)$$

$$SO_1 = SO_1^B + SO_1^C$$

Retailer 2:

$$SO_2^B = \sum_{i=1}^{11} p(i) + \sum_{i=23}^{33} p(i) + \sum_{i=66}^{76} p(i) + \sum_{i=88}^{98} p(i) + \sum_{i=131}^{141} p(i) + \sum_{i=153}^{163} p(i)$$

$$SO_2^C = \sum_{i=42}^{49} p(i) + \sum_{i=58}^{65} p(i) + \sum_{i=107}^{114} p(i) + \sum_{i=123}^{130} p(i)$$

$$SO_2 = SO_2^B + SO_2^C$$

Stock-out probability for the Wholesaler

$$\begin{aligned} SOw_1 = {} & p(1)+p(2)+p(3)+p(12)+p(13)+p(14)+p(23)+p(24)+p(25) + \\ & p(66)+p(67)+p(68)+p(77)+p(78)+p(79)+p(88)+p(89)+p(90) + \\ & p(131)+p(132)+p(133)+p(142)+p(143)+p(144)+p(153)+p(154)+p(155) \end{aligned}$$

$$SOw_2 = \sum_{i=1}^{11} p(i) + \sum_{i=66}^{76} p(i) + \sum_{i=131}^{141} p(i)$$

Stock-out probability for the Distribution Centre

$$SO_d = \sum_{i=1}^{33} p(i)$$

Utilization of transportation resource towards the retailers

$$u_1 = \sum_{i=6}^{11} p(i) + \sum_{i=17}^{22} p(i) + \sum_{i=71}^{76} p(i) + \sum_{i=82}^{87} p(i) + \sum_{i=136}^{141} p(i) + \sum_{i=147}^{152} p(i) + \sum_{i=36}^{41} p(i) + \sum_{i=52}^{57} p(i) + \sum_{i=101}^{106} p(i) + \sum_{i=117}^{122} p(i)$$

$$u_2 = \sum_{i=23}^{33} p(i) + \sum_{i=88}^{98} p(i) + \sum_{i=153}^{163} p(i) + \sum_{i=42}^{49} p(i) + \sum_{i=58}^{65} p(i) + \sum_{i=107}^{114} p(i) + \sum_{i=124}^{130} p(i)$$

The other performance measures can be calculated from the above using simple relations.

7.7.4 Validation of algorithmic results

The infinitesimal generator matrix of the system was constructed manually and the linear system of equations was solved in Mathematica to get the vector of stationary probabilities. Then the performance measures of the system were calculated as described in the previous section. The results were identical to those produced algorithmically. The algorithmic results were also contrasted to simulation results (see section 7.8). Three replications of 2000000 time units each were used. Arithmetic values for parameters μ_d=0.9 μ_w=1.8, μ_1=1, μ_2=1.2, λ_1=0.9, λ_2=0.7 are given below.

Performance measure	Algorithm - Matlab	Manually - Mathematica	Simulation - Arena
FR_1	0.786822	0.786822	0.787 ± 0.001
FR_2	0.527915	0.527915	0.528 ± 0.001
$WIPr_1$	1.852801	1.852801	1.853 ± 0.001
$WIPr_2$	0.527915	0.527915	0.528 ± 0.001
$WIPtr_1$	0.708140	0.708140	0.708 ± 0.001
$WIPtr_2$	0.307950	0.307950	0.308 ± 0.001
WIP_w	0.732491	0.732491	0.732 ± 0.001
WIP_d	0.802578	0.802578	0.803 ± 0.001
SOw_1	0.166467	0.166467	0.167 ± 0.001
SOw_2	0.164135	0.164135	0.164 ± 0.001
SO_d	0.225061	0.225061	0.225 ± 0.001

7.8 Validation of the model

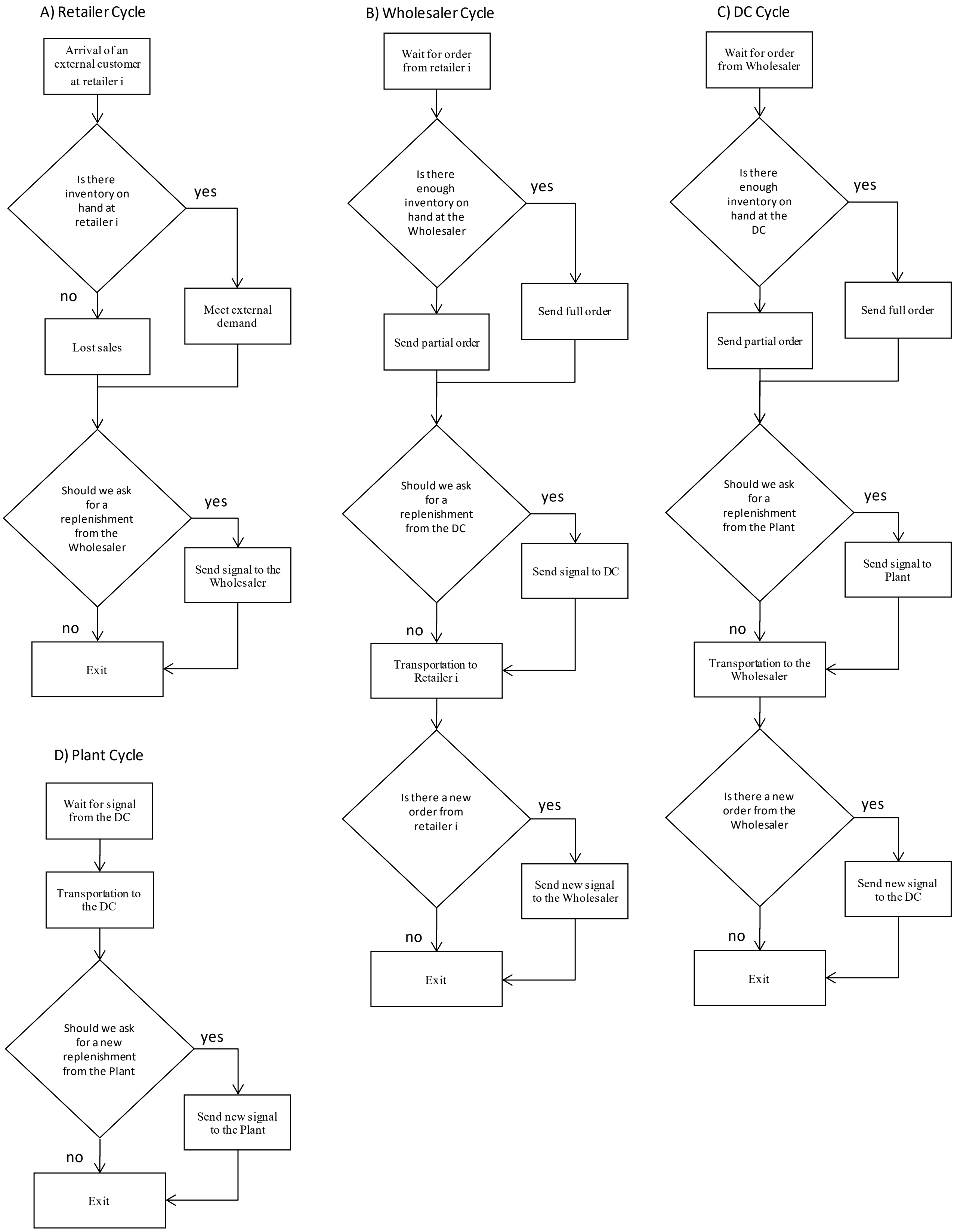
7.8.1 Simulation Model

Infinitesimal Generator Matrices for simple scenarios were constructed manually and the resulting systems of equations were solved in Mathematica. All the results were found to be identical to those of the algorithm.

For a more rigorous testing such an approach is not practical, as for bigger systems models of several thousand states may be involved. To check the validity of the developed algorithm, a simulation model of the system under consideration was developed. The system was modeled as a series of cycles, each cycle describing the interface between successive members of the supply network. The basic logic of the simulation model is given in figure 7.4.

The simulation model was constructed in Arena simulation package. Test runs were executed to determine the specific parameters of the simulation that would give statistically rigorous results within a reasonable computation time. A warm-up period of 10000 time units was deemed enough to ensure that any initial conditions effects are eliminated. A run time of 2000000 time units was selected as it was found to give results with an appropriate margin of error.

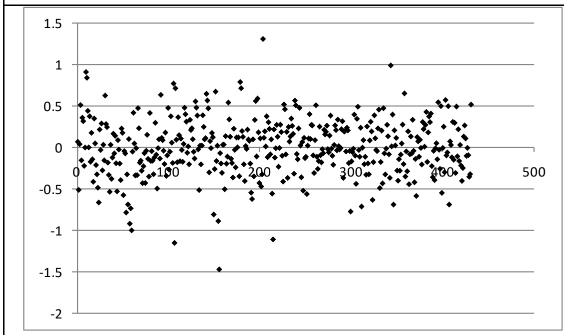
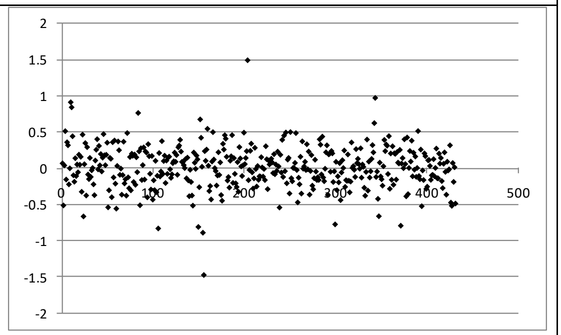
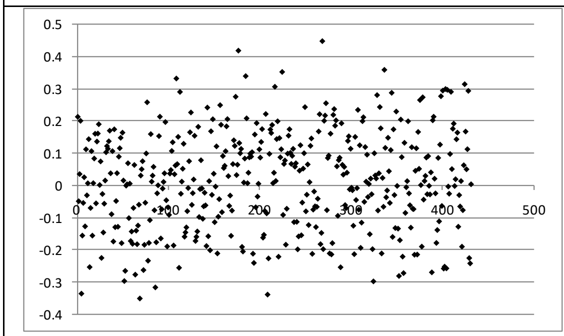
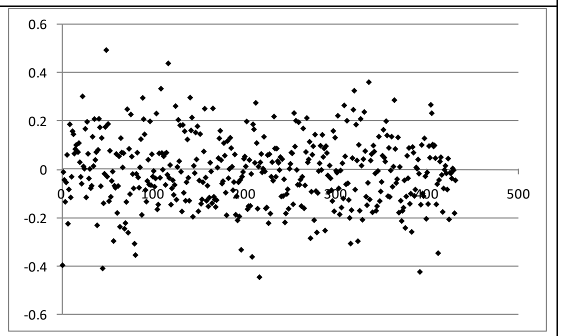
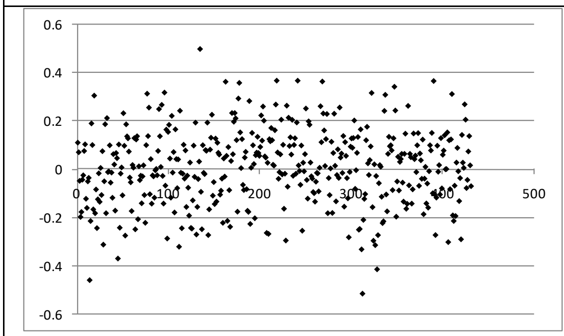
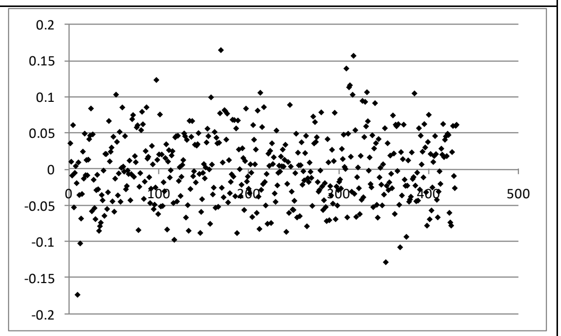
Figure 7.4: Simulation model logic

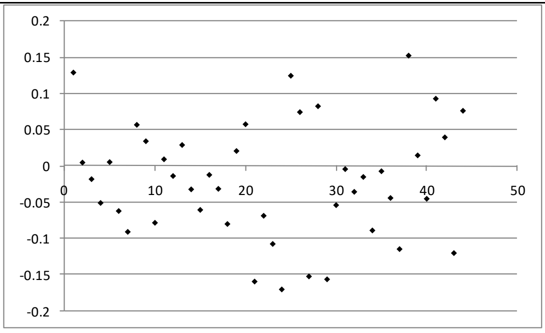
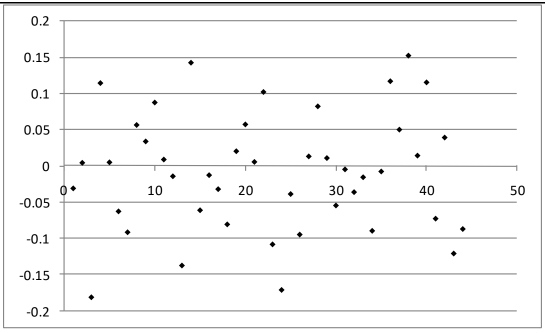
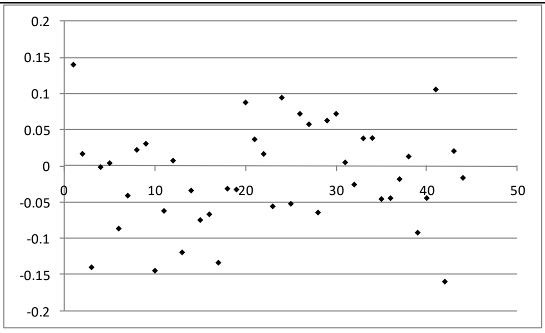
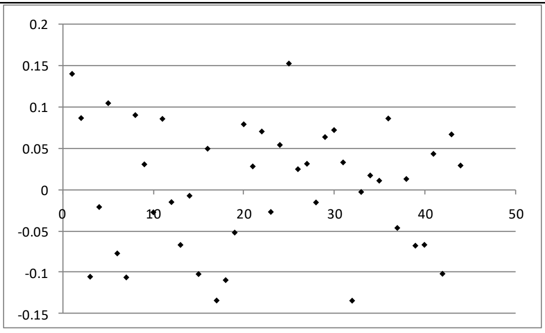
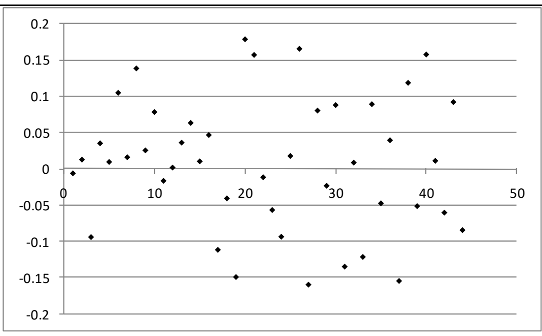
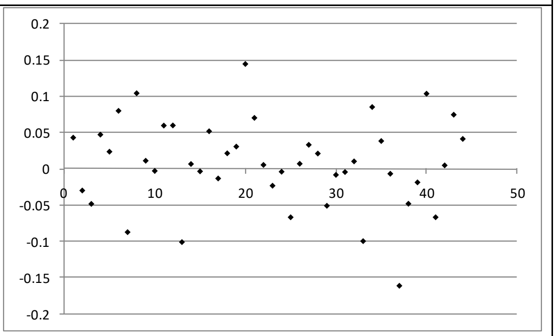


7.8.2 Arithmetic results

Scenarios with one to five retailers where tested. In all cases the difference between analytic results and simulation results was within the expected deviation attributed to the experimental nature of the simulation approach. In total, more than 900 different scenarios were tested for various parameter relations. The analysis included ten different performance measures (FR_r, $WIPr_r$, WIP_w, WIP_d, $WIPtr_r$, $WIPtr_w$, ARO_r , AROw, SOw_r and SO_d. Some results for two and five retailers are given in the figures below and a sample of validation data is given in the appendix. In the diagrams we give the deviation as a percentage of the analytical value:

$$\%\, deviation = 100 \times \frac{analytic - simulation}{analytic}$$

<table>
<tr><td colspan="2">% deviation between algorithmic solution and simulation results for a system with two retailers, s_d=2, Q_d=2, s_w=2, Q_w=\{2,3,4\}, s_1=\{0,1,2\}, Q_1=\{1,2,3,4\}, s_2=\{0,1,2\}, Q_2=\{1,2,3,4\} and parameters μ_d=0.4 μ_w=0.6, μ_1=1, μ_2=2, λ_1=1.5, λ_2=2.5. Simulation parameters: One replication of 2000000 time units with a warm up period of 10000 time units.</td></tr>
<tr><td>FR_1</td><td>WIP_1</td></tr>
<tr><td>
</td><td>
</td></tr>
<tr><td>FR_2</td><td>WIP_2</td></tr>
<tr><td>
</td><td>
</td></tr>
<tr><td>WIP_d</td><td>SOw_1</td></tr>
<tr><td>
</td><td>
</td></tr>
</table>

% deviation between algorithmic solution and simulation results for a system with five retailers, $s_d=\{2,4\}$, $Q_d=2$, $s_w=0$, $Q_w=2$, $s_1=0$, $Q_1=1$, $s_2=1$, $Q_2=1$, $s_3=0$, $Q_3=\{1,2\}$, $s_4=0$, $Q_4=\{1,2\}$, $0 \leq s_5 \leq 3$, $1 \leq Q_5 \leq 2$ and parameters $\mu_d=2.5$, $\mu_w=3.6$, $\mu_1=1$, $\mu_2=1.2$, $\mu_3=1.4$, $\mu_4=1.6$, $\mu_5=1.8$, $\lambda_1=0.5$, $\lambda_2=0.7$ $\lambda_3=0.9$, $\lambda_4=1.2$ $\lambda_5=1.5$. Simulation parameters: One replication of 2000000 time units with a warm up period of 10000 time units.	
FR_1	WIP_1
	
FR_5	WIP_5
	
WIP_w	WIP_d
	

7.9 Model Performance and limitations

The algorithm was programmed in Matlab, version 2018a, 9.4.0.813654. For the runs commented here a computer with Core i-3-4005U CPU at 1.70 GHz processor and 4GB installed RAM was used. Its operating system was Windows 7 – Ultimate, 64-bit.

The proposed algorithm is valid for any combination of input variables, but as the systems under consideration become more complex, the dimension of the infinitesimal generator increases and the solution of the model becomes computationally demanding. The increasing number of states for big systems is a common problem of Markovian models (Mehmood and Lu, 2011) and although it is alleviated with rising computational power, it still imposes limitations to the application of exact Markov models in real-life scale systems.

In the proposed model the exact size of the infinitesimal generator matrix depends on the number of the retailers and the specific values of the inventory policy parameters at each member of the network. The number of possible states depends on the relation between the decision variables, but as a general trend, the dimension of the infinitesimal generator increases with increasing number of retailers and increasing values of the inventory policy parameters. The exact number of states (*ns*) can be computed as:

$$ns = Bl_n + NL_w \cdot Cl_n + nQ_w \cdot (Bl_n + nsw \cdot Cl_n) + NLd \cdot \left((NLw - nsw) \cdot Cl_n + nQ_w \cdot (Bl_n + nsw \cdot Cl_n)\right)$$

$$Bl_n = \prod_{i=1}^{n} \left(s_i + Q_i + 1 + nQ_i \cdot (s_i + 1)\right)$$

$$Cl_n = \prod_{i=1}^{n} \left(Q_i + nQ_i \cdot (s_i + 1)\right)$$

$$NLw = floor\left(\frac{s_w + Q_w}{bsw}\right)$$

$$nsw = floor\left(\frac{s_w}{bsw}\right)$$

$$nQ_w = floor\left(\frac{Q_w}{bsd}\right)$$

$$NLd = floor\left(\frac{s_d + Q_d}{bsd}\right)$$

$$bsd = Greatest\ Common\ Divisor(Q_d, Q_w)$$

$$bsw = Greatest\ Common\ Divisor(bsd, Q_i), i = 1{:}n$$

In the tables below some examples are given of the number of states in relation to the decision variables:

n=4, s_d=0, Q_d=2, s_w=0, Q_w=2, s_1=0, Q_1=1, s_2=1, Q_2=2, s_3=1, Q_3=1		
s_4	**Q_4**	**Number of states**
0	1	1368
0	2	2376
1	1	2232
1	2	3744
2	1	3096
2	2	5112
3	1	3960
3	2	6480
4	1	4824
4	2	7848

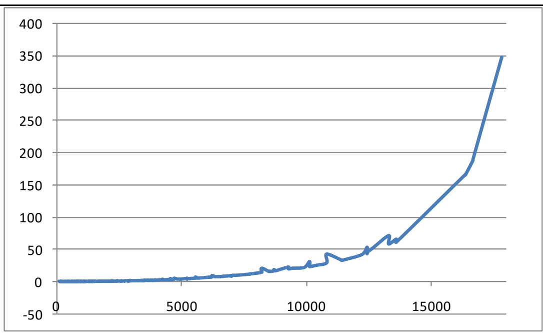
n=2, s_d=0, Q_d=4, s_w=0, Q_w=4, s_1=2, Q_1=3		
s_2	**Q_2**	**Number of states**
0	1	327
0	2	609
0	3	891
0	4	1173
1	1	513
1	2	936
1	3	1359
1	4	1782
2	1	699
2	2	1263
2	3	1827
2	4	2391
3	1	885
3	2	1590
3	3	2295
3	4	3000
4	1	1071
4	2	1917
4	3	2763
4	4	3609

s_d=0, Q_d=4, s_w=0, Q_w=4, all retailers (i) follow policy s_i=0,Q_i=1							
Number of retailers	1	2	3	4	5	6	7
Number of states	17	43	113	307	857	2443	7073

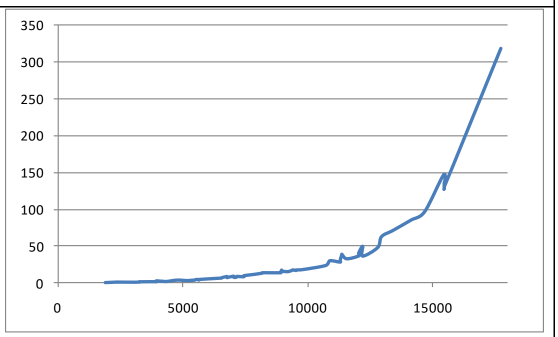
Computational time depends mainly on the number of possible states. Below, the computational time in relation to the infinitesimal generator dimension is given for systems of 2 and 5 retailers:

Computational time (sec) as a function of generator matrix dimension

n=2



n=5



The main limitation of the Matlab model is the required RAM memory. The demand for RAM depends on the size of the infinitesimal generator matrix. A problem with 18112 states was the biggest problem that was solved in the aforementioned computer. It must be noted that during program development our priority was clarity and transparency with regard to the translation of the theory into a computer program. Algorithmic efficiency issues were not concerned. Both algorithm efficiency (number of steps to the solution) and memory consumption can be improved by rephrasing the computer program and by exploiting embedded features of Matlab such as sparse matrices. However, in any case, the problem of increasing system states with increasing decision variable values would persist.

Despite size limitations, the proposed algorithm still offers certain advantages. The exact algorithm is significantly faster than simulation. In some cases the difference in computation time is several orders of magnitude, as for high precision results a typical simulation run of several minutes was required. Moreover, the exact solution poses no limits on precision in contrast to simulation where the results are always in the form of a confidence interval. This can be especially helpful in cases where low values of the performance measures are concerned, where the specific value is comparable to the last significant digit. Finally, the proposed algorithm can be easily integrated with other components in the framework of a more generic model, as for example in the context of an optimization model.

7.10 Numerical Results

7.10.1 Effect of the design variables – Balanced systems

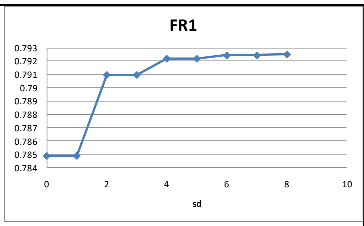
We investigate the performance of a system with two retailers (n=2). As we want to focus on the effect of the inventory policies we limit our investigation to a "balanced" system where $\mu_1 = \mu_2 = \lambda_1 = \lambda_2 = \frac{\mu_w}{2} = \frac{\mu_d}{2}$.

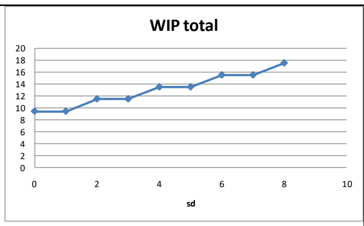
Some of the related data are given in the Appendix.

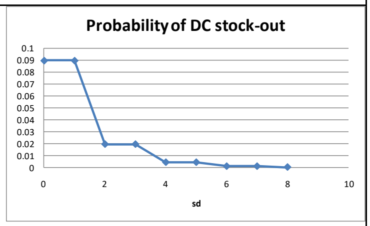
7.10.1.1 Distribution Centre's policy (s_d, Q_d)

The effect of s_d on system performance measures

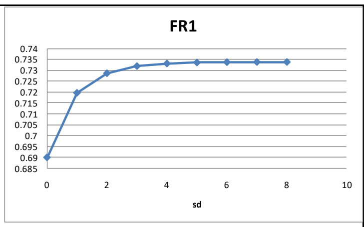
Q_d=2, s_w=2, Q_w=2, s_1=1, Q_1=2, s_2=1, Q_2=2

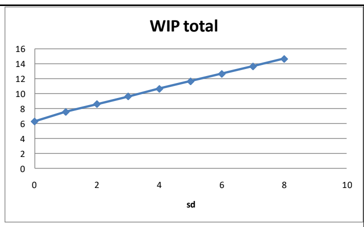


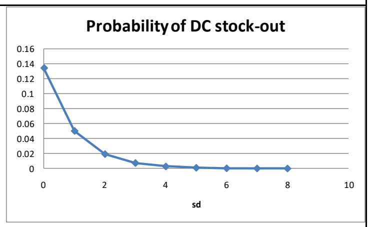




Q_d=2, s_w=2, Q_w=1, s_1=1, Q_1=2, s_2=1, Q_2=1

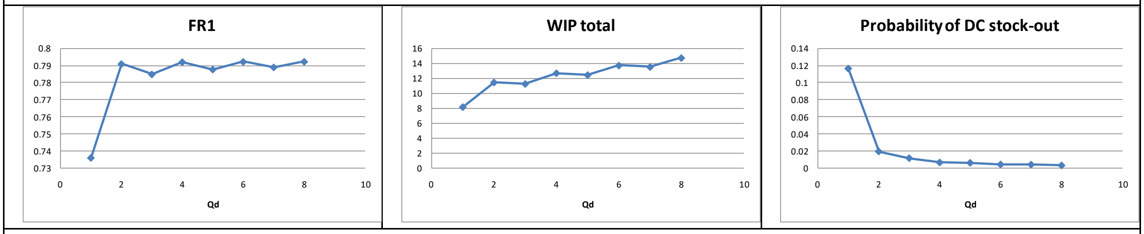




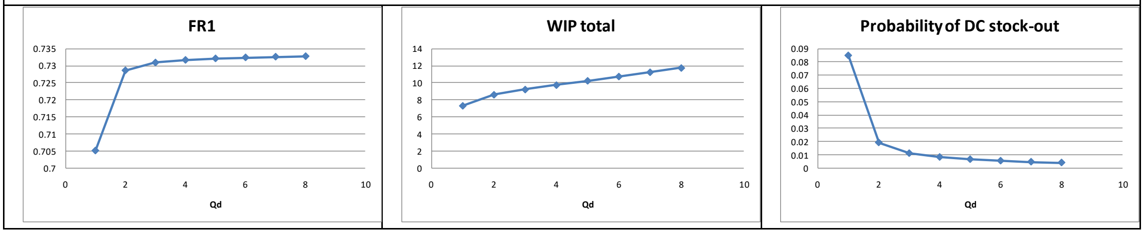


The effect of Q_d on system performance measures

s_d=2, s_w=2, Q_w=2, s_1=1, Q_1=2, s_2=1, Q_2=2



s_d=2, s_w=2, Q_w=1, s_1=1, Q_1=2, s_2=1, Q_2=1

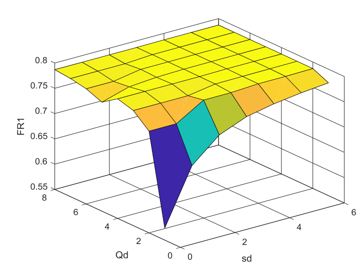


The performance of the retailers increases with increasing s_d, but at the cost of an almost linear increase in the average total inventory of the system. Depending on the values of Q_d and Q_w, small changes in s_d may have no impact on the performance of the system.

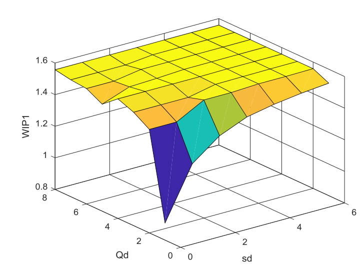
The effect of Q_d depends on the value of the other parameters. The Fill rate at the retailers in general tends to increase. Total inventory also increases with Q_d, but the effect is less pronounced compared to that of s_d. The behavior of the system is dynamic, and in some cases local minima were observed for Fill rate and WIP total. (WIP total is the average inventory from the DC and downstream)

Compound effect of DC policy: s_w=1, Q_w=4, s_1=1, Q_1=2, s_2=1, Q_2=2

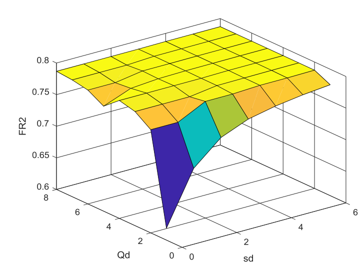
OFR_1



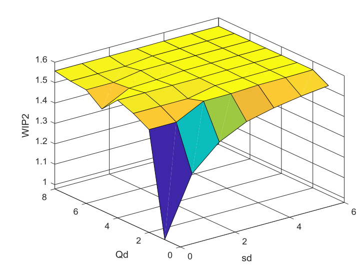
WIP_1



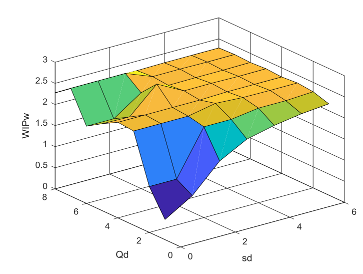
OFR_2



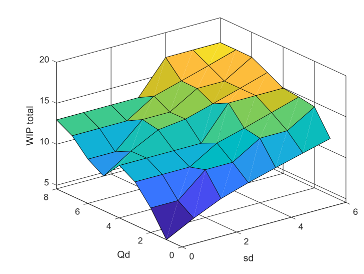
WIP_2

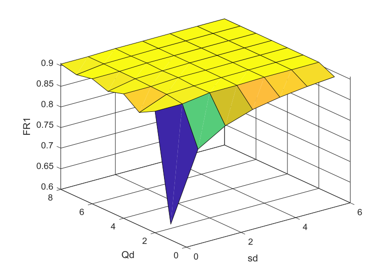
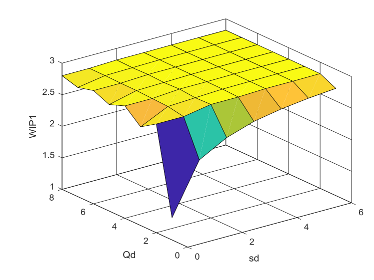
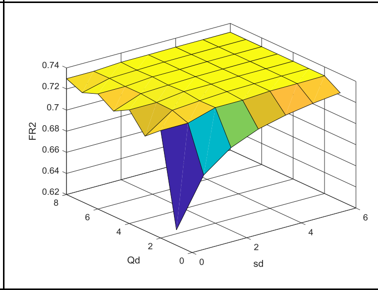
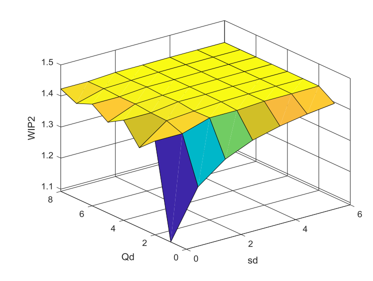
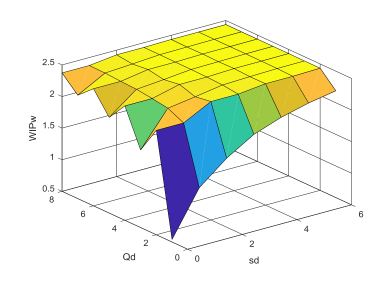
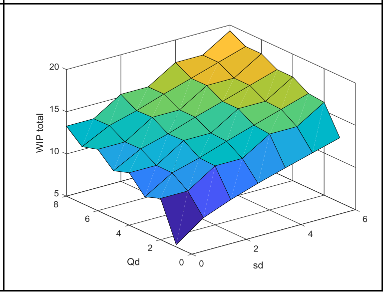


WIP_w



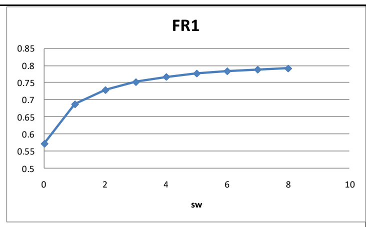
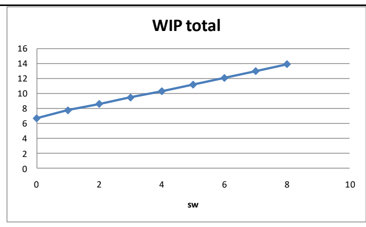
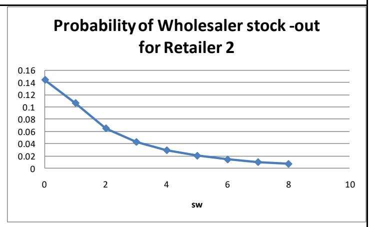
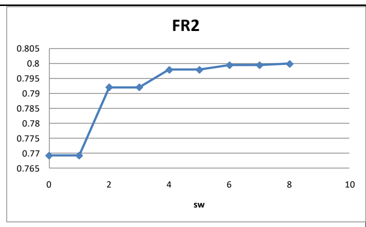
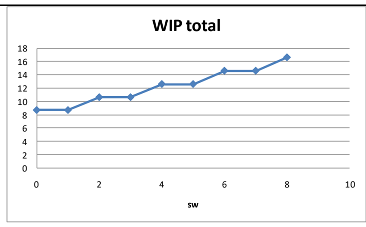
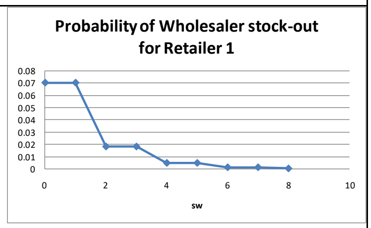
WIP total

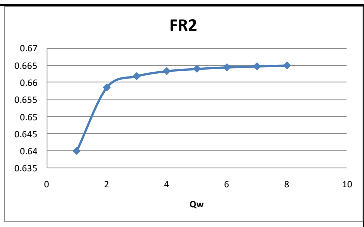
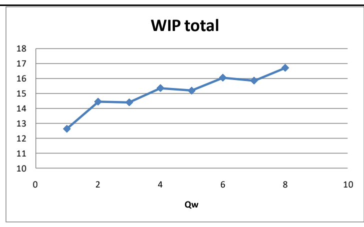
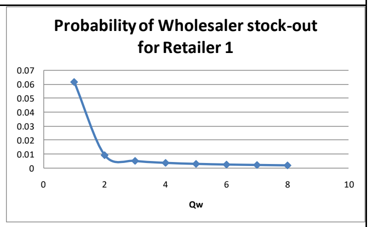
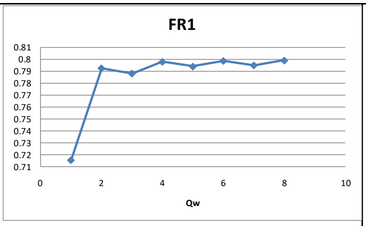
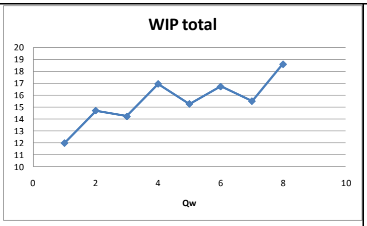
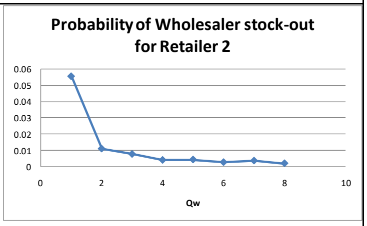


Compound effect of DC policy: s_w=2, Q_w=2, s_1=2, Q_1=3, s_2=2, Q_2=1		
OFR_1	WIP_1	OFR_2
		
WIP_2	WIP_w	WIP total
		

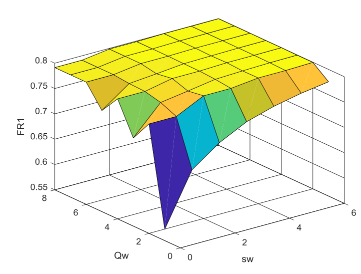
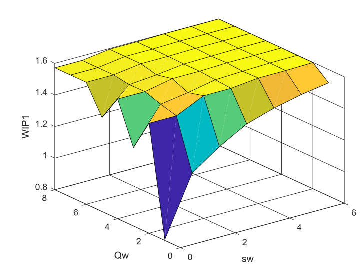
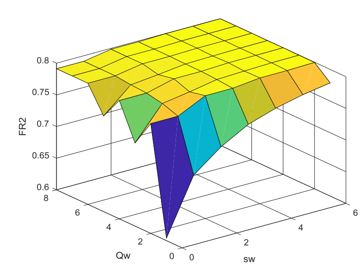
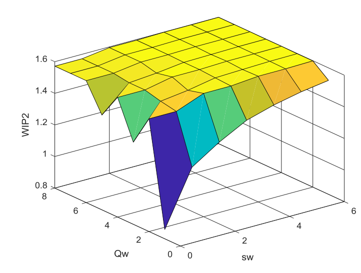
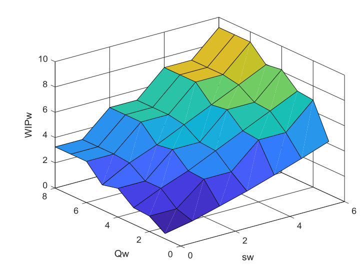
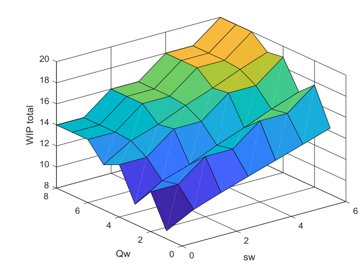
DC policy is important for the retailers' performance only for low s_d and Q_d values when the Distribution Center is the "bottleneck" of the system. The presence of some safety stock at the DC seems preferable, as for s_d=0 the system is less stable and more difficult to predict. WIP total tends to increase with increasing s_d and Q_d, but often local minima and maxima are observed. It is possible that with minor policy adjustments lower total inventory can be achieved without any serious negative effect in retailers' performance.

7.10.1.2 Wholesaler policy (s_w, Q_w)

The effect of s_w on system performance measures		
s_d=2, Q_d=2, Q_w=1, s_1=1, Q_1=2, s_2=1, Q_2=1		
		
s_d=0, Q_d=4, Q_w=2, s_1=1, Q_1=2, s_2=1, Q_2=2		
		

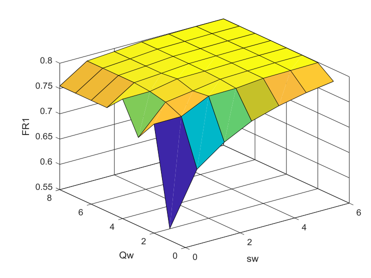
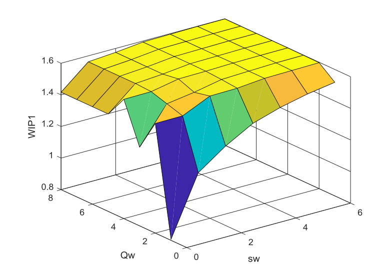
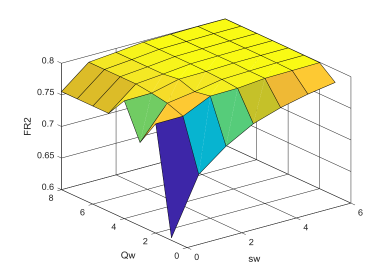
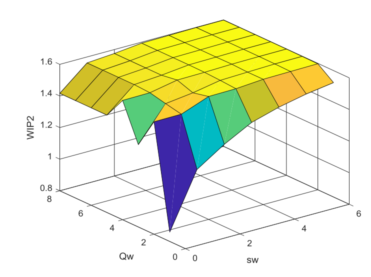
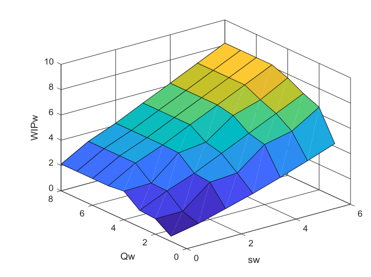
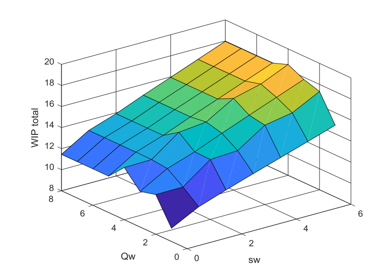
The effect of Q_w on system performance measures		
s_d=6, Q_d=2, s_w=2, s_1=1, Q_1=2, s_2=1, Q_2=1		
		
s_d=4, Q_d=4, s_w=2, s_1=1, Q_1=2, s_2=1, Q_2=2		
		

By increasing s_w we can achieve better service levels at the retailers, but at the cost of an almost linear increase in the average total inventory. There is interplay between the parameters and for certain Q_d, Q_1 and Q_2 values, small changes in s_w do not have any effect on the performance of the system.

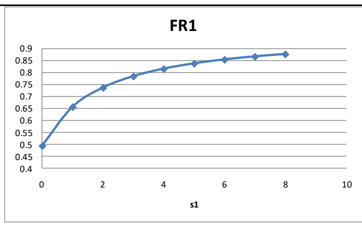
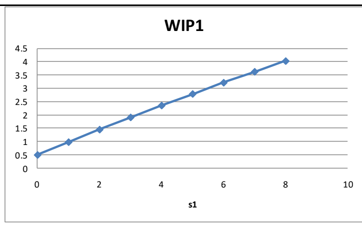
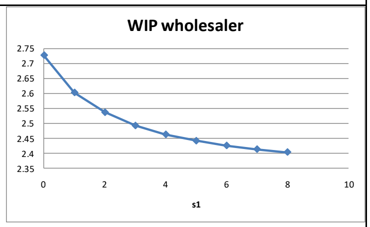
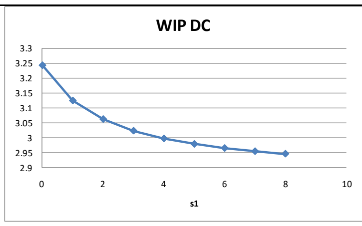
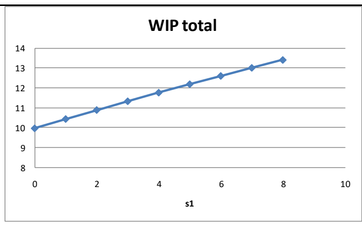
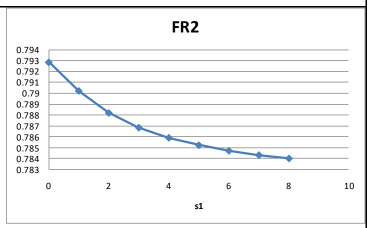
Compound effect of wholesaler's policy: s_d=2, Q_d=6, s_1=1, Q_1=2, s_2=1, Q_2=2		
OFR_1	WIP_1	OFR_2
		
WIP_2	WIP_w	WIP total
		

The effect of Q_w is less straightforward. In general, Fill rates and average inventories at the retailers tend to increase with increasing Q_w. In some cases small decreases were observed in total WIP, so that the performance of the retailers could be enhanced while actually decreasing the average inventory in the system. There is strong interplay between the parameters and for certain scenarios, incremental increases in Q_w caused small decreases in fill rates.

Both s_w and Q_w are more important when low values are concerned. The presence of some safety stock is advantageous. When $s_w=0$, the behavior of the system is more erratic and less predictable. In general the wholesaler's policy can be used to enhance the performance of the system in terms of customer satisfaction, but the upper limit for the Fill rate is still depended on the retailers' policy.

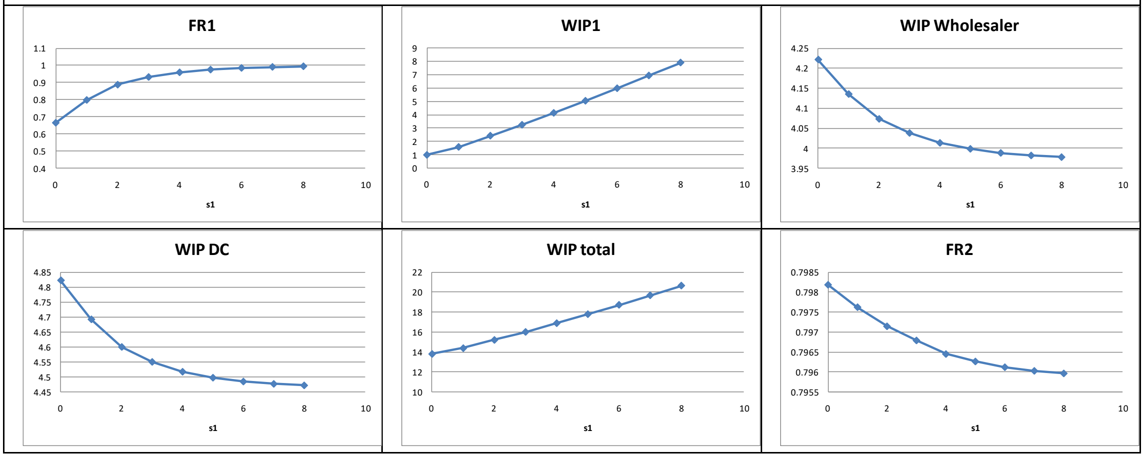
Compound effect of wholesaler's policy: $s_d=3$, $Q_d=5$, $s_1=1$, $Q_1=2$, $s_2=1$, $Q_2=2$		
OFR_1	WIP_1	OFR_2
		
WIP_2	WIP_w	WIP total
		

7.10.1.3 Retailer 1 policy (s_1, Q_1)

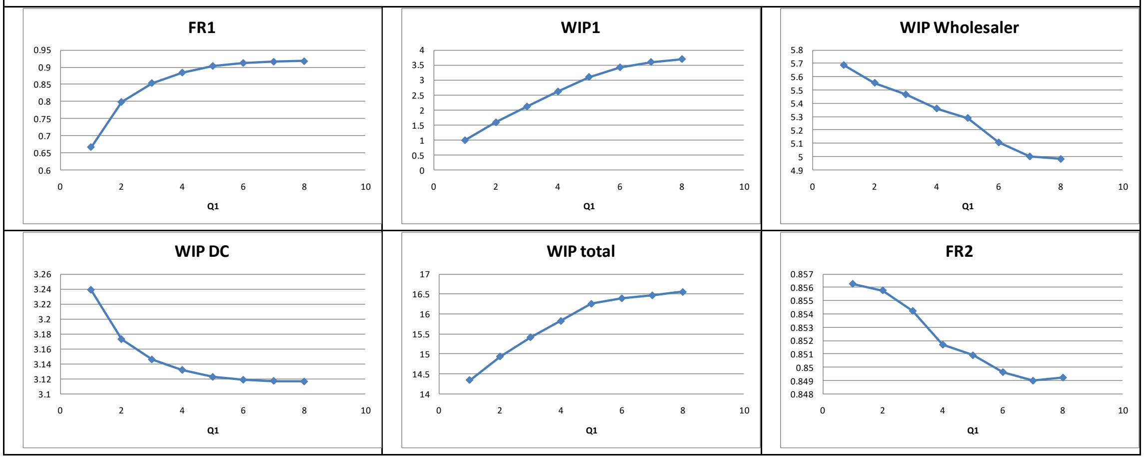
The effect of s_1 on system performance measures: $s_d=2$, $Q_d=2$, $s_w=2$, $Q_w=2$, $Q_1=1$, $s_2=1$, $Q_2=2$		
		
		

Predictably, Fill rate at retailer 1 increases with increasing s_1 but at the cost of a practically linear increase in the average inventory at Retailer 1. Both average inventory at the DC and average inventory at the wholesaler are negatively correlated with s_1. The overall effect on WIP total is an almost linear increase, but of a lower inclination compared to WIP_1. Up to a point, the increase of the reorder point at retailer 1 causes a transfer of available inventory downstream. This has a negative effect on the performance of retailer 2.

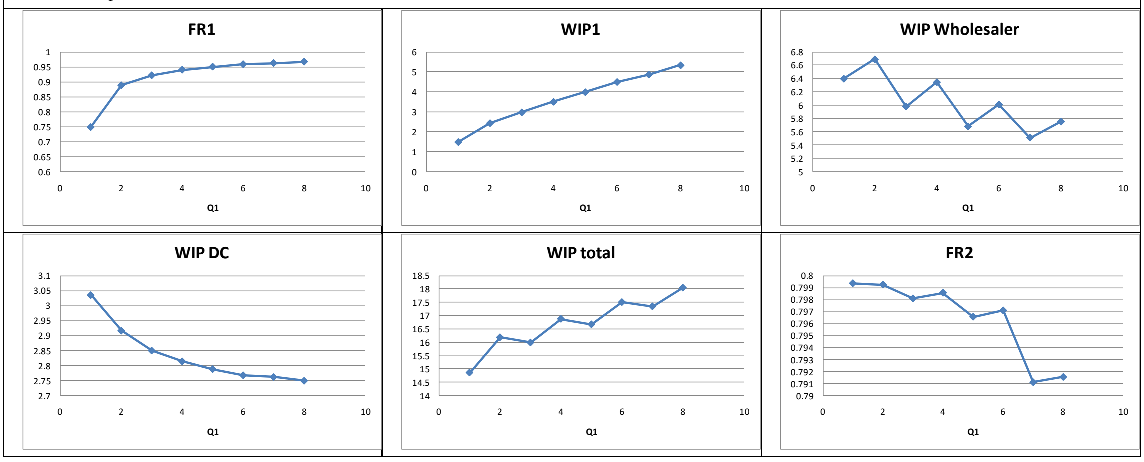
The effect of s_1 on system performance measures: $s_d=4$, $Q_d=2$, $s_w=2$, $Q_w=4$, $Q_1=2$, $s_2=1$, $Q_2=2$



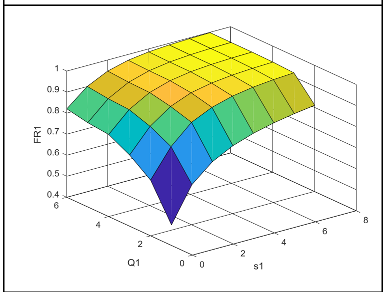
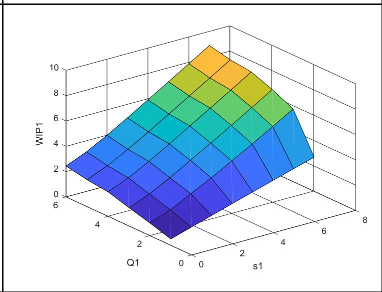
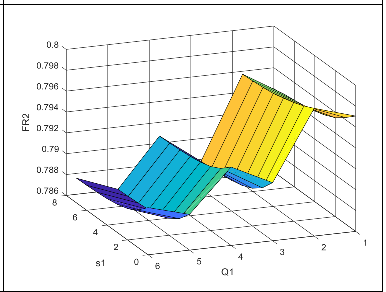
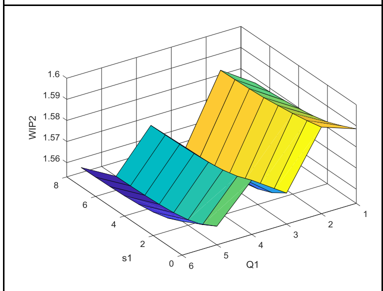
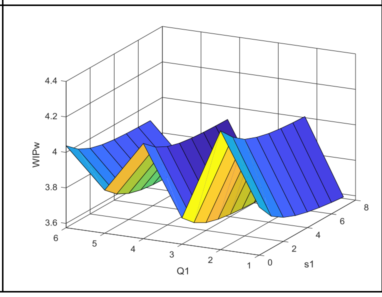
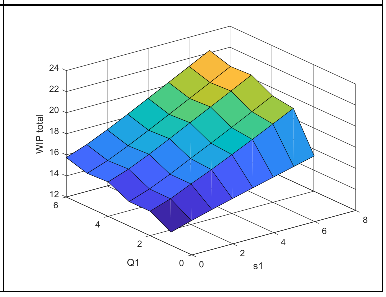
The effect of Q_1 on system performance measures: $s_d=0$, $Q_d=4$, $s_w=4$, $Q_w=4$, $s_1=1$, $s_2=1$, $Q_2=3$

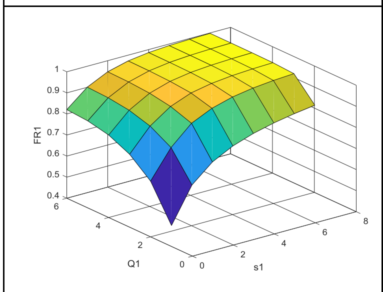
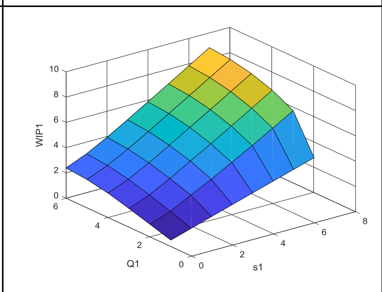
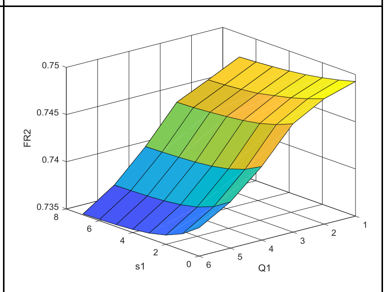
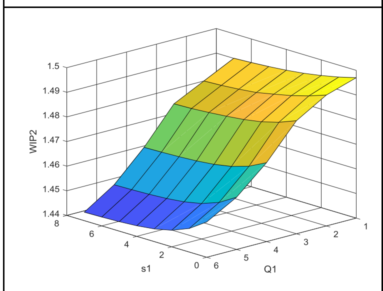
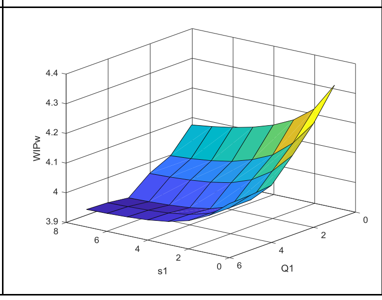
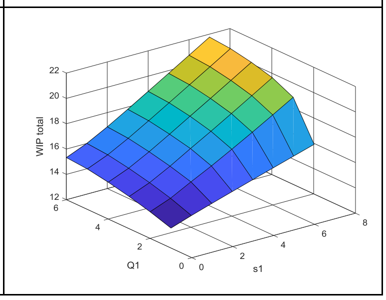


The effect of Q_1 on system performance measures: $s_d=2$, $Q_d=2$, $s_w=6$, $Q_w=2$, $s_1=2$, $s_2=1$, $Q_2=2$



Fill rate at retailer 1 increases with increasing Q_1, but at the cost of an increasing inventory at retailer 1. The effect on wholesaler's inventory is more dynamic. In general WIP wholesaler tends to decrease with increasing Q_1, but local maxima and minima may be observed. Average inventory at the distribution Centre decreases with Q_1. The average total inventory in the system is generally positively correlated with Q_1, but due to the WIP wholesaler contribution a jagged pattern may be observed. As was the case with s_1, the increase of Q_1 causes a downstream transfer of available inventory and this has a negative effect on the performance of the second retailer.

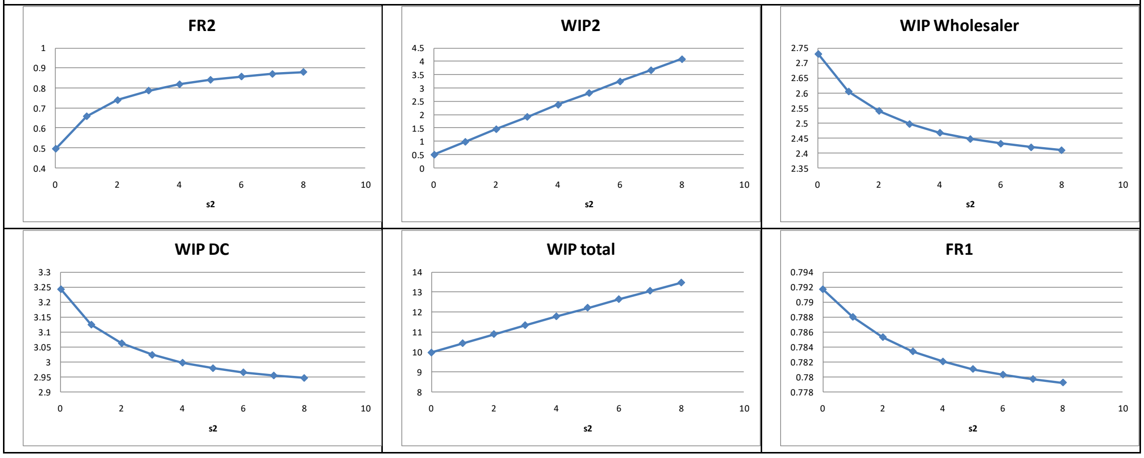
Compound effect of retailers' 1 policy: s_d=2, Q_d=6, s_w=2, Q_w=4, s_2=1, Q_2=2		
OFR_1	WIP_1	OFR_2
		
WIP_2	WIP_w	WIP total
		

Compound effect of retailer's 1 policy: s_d=3, Q_d=3, s_w=3, Q_w=3, s_2=2, Q_2=1		
OFR_1	WIP_1	OFR_2
		
WIP_2	WIP_w	WIP total
		

The managerial implication of the dynamic system behavior is that there are good reasons for the fine-tuning of the system. Small changes in inventory policies may achieve an enhanced performance in terms of both customer satisfaction and total system inventory.

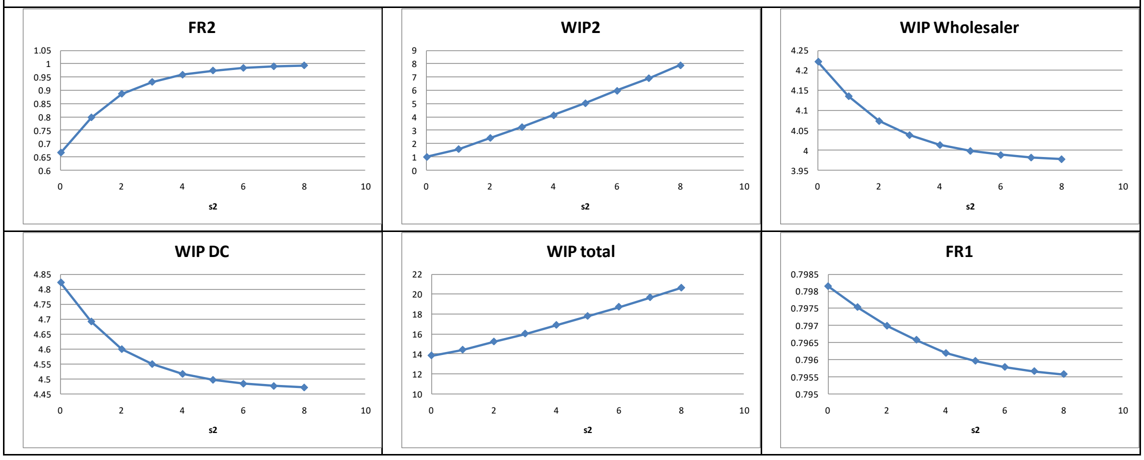
7.10.1.4 Retailer 2 policy (s_2,Q_2)

The effect of s_2 on system performance measures: s_d=2, Q_d=2, s_w=2, Q_w=2, s_1=1, Q_1=2, Q_2=1



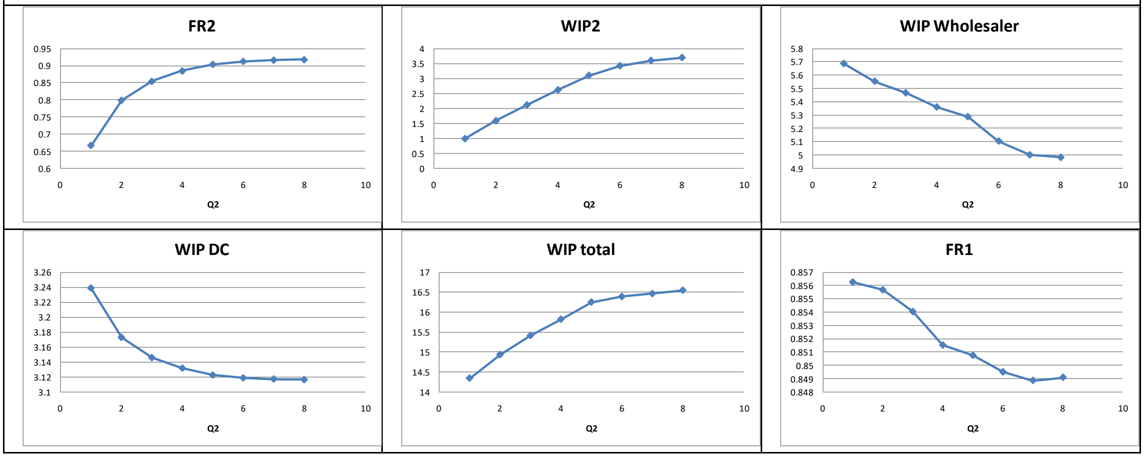
The behavior of retailer 2 with changing s_2 is similar to that of retailer 1 with changing s_1. For the higher priority retailer the elasticities of the performance measures with changing s_2 take slightly higher values (greater curve grades).

The effect of s_2 on system performance measures: s_d=4, Q_d=2, s_w=2, Q_w=4, s_1=1, Q_1=2, Q_2=2

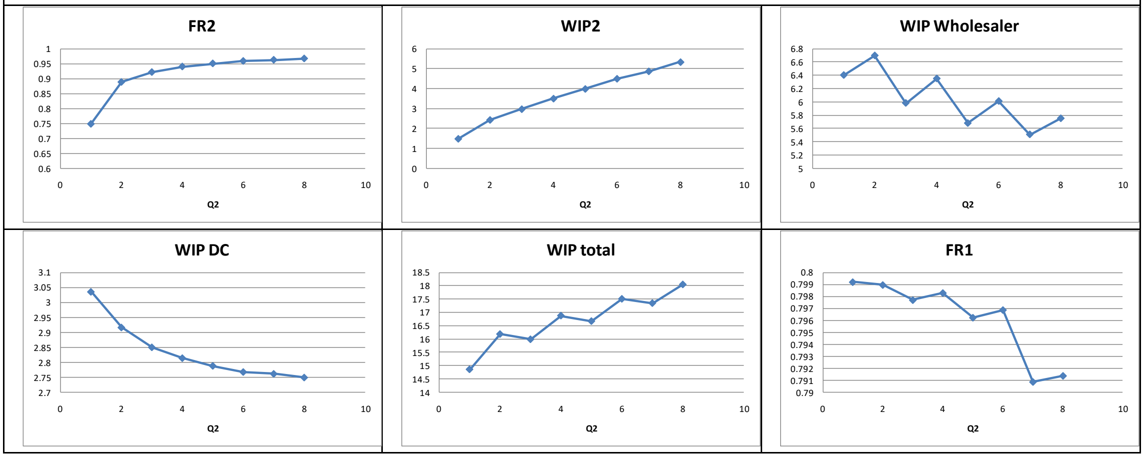


The effect of Q_2 on the performance measures is similar to that of Q_1. Numerically the results for Q_1 and Q_2 are close and the importance of each parameter depends on the specific scenario under investigation.

The effect of Q_2 on system performance measures: s_d=0, Q_d=4, s_w=4, Q_w=4, s_1=1, Q_1=3, s_2=1

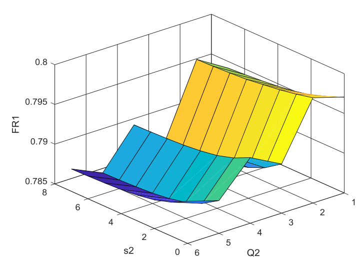


The effect of Q_2 on system performance measures: s_d=2, Q_d=2, s_w=6, Q_w=2, s_1=1, Q_1=2, s_2=2

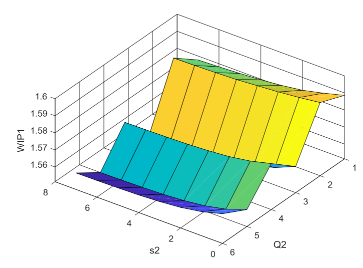


Compound effect of retailer's 2 policy: s_d=2, Q_d=6, s_w=2, Q_w=4, s_1=1, Q_1=2

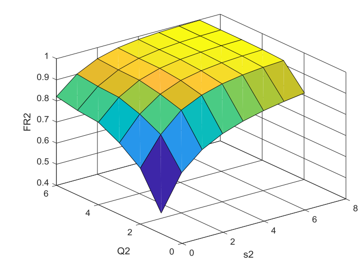
OFR_1



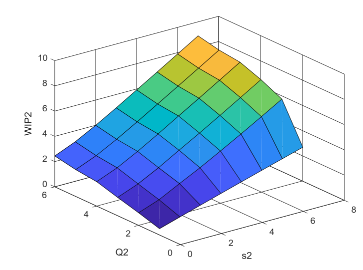
WIP_1



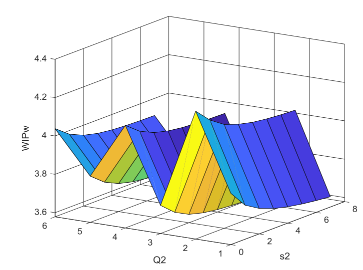
OFR_2



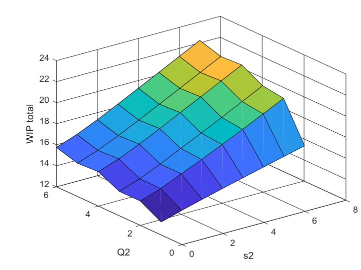
WIP_2

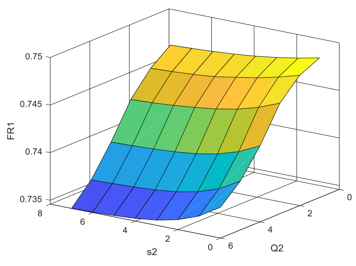
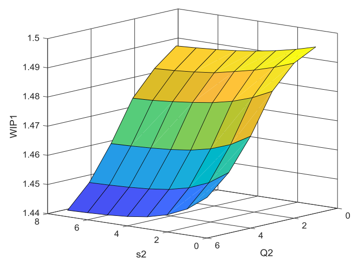
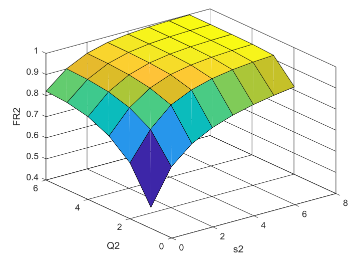
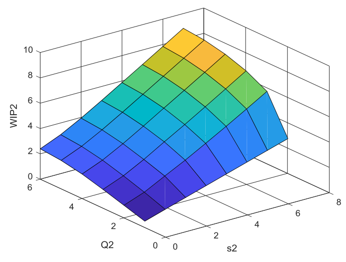
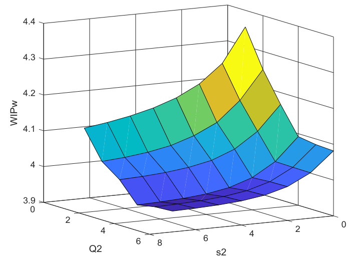
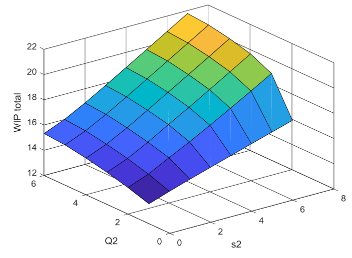


WIP_w



WIP total

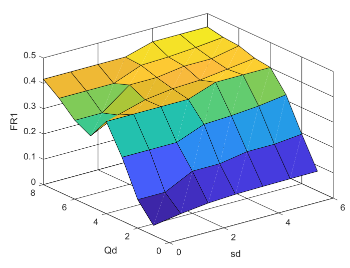
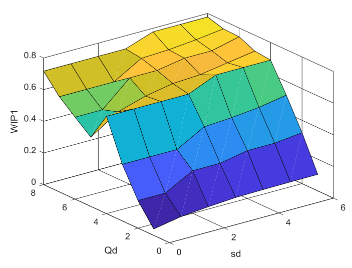
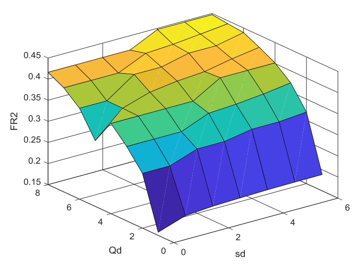
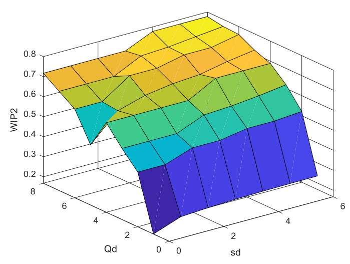
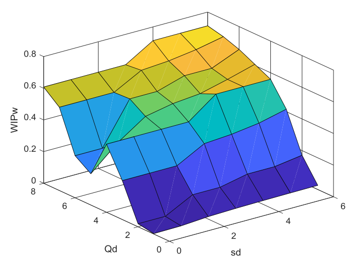
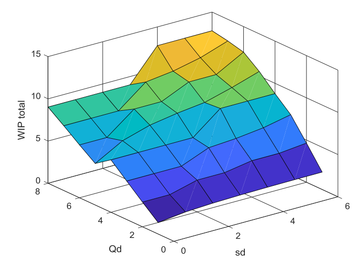


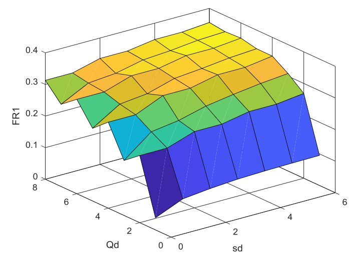
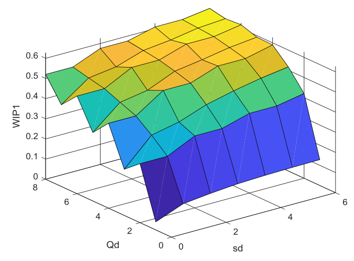
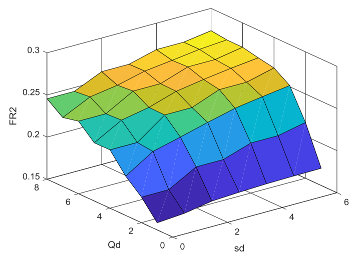
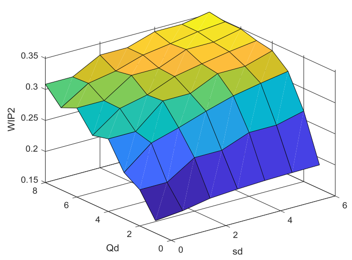
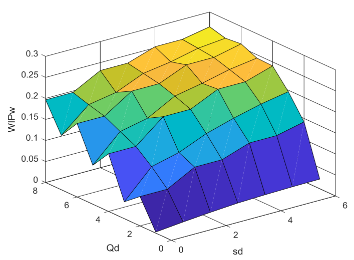
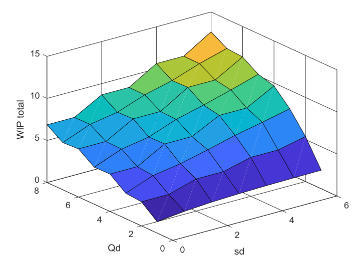
Compound effect of retailer's 2 policy: s_d=3, Q_d=3, s_w=3, Q_w=3, s_1=2, Q_1=1		
OFR_1	WIP_1	OFR_2
		
WIP_2	WIP_w	WIP total
		

7.10.2 Effect of the design variables – Supply constrained systems

We investigate a supply constrained system where $\lambda_1 = \lambda_2 > \mu_1 = \mu_2 > \mu_w > \mu_d$. For the numerical examples that follow the parameters were $\lambda_1=\lambda_2=3$, $\mu_1=\mu_2=2$, $\mu_w=1$, and $\mu_d=0.8$.

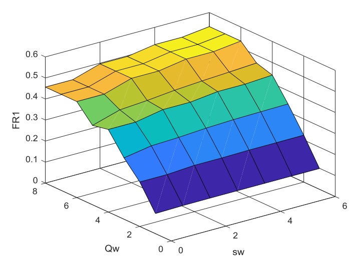
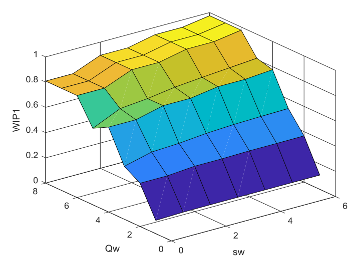
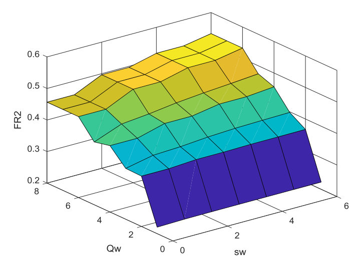
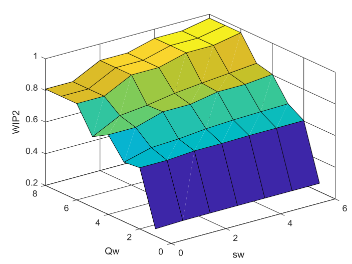
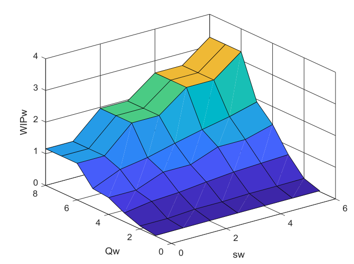
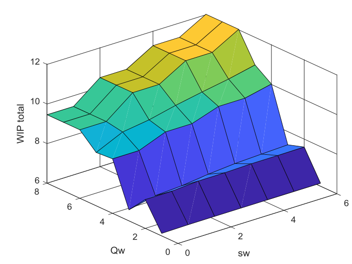
7.10.2.1 Distribution Centre's policy (s_d, Q_d)

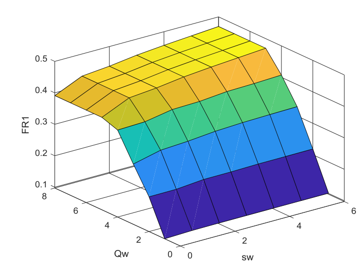
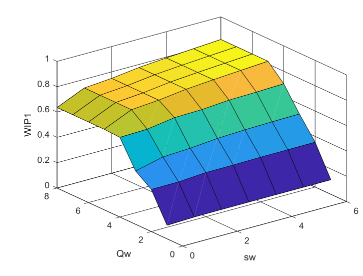
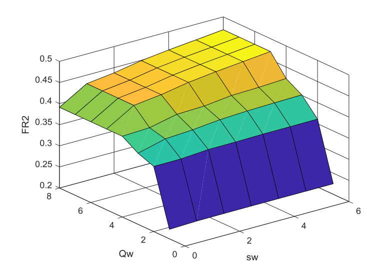
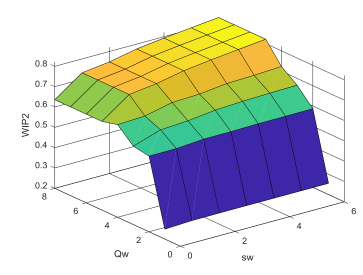
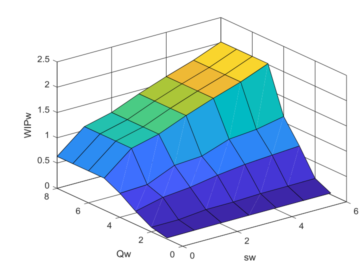
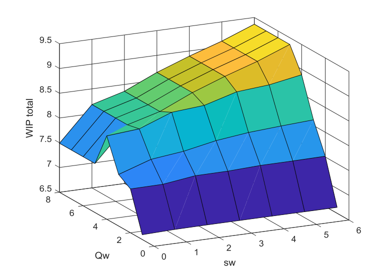
Compound effect of DC policy: s_w=1, Q_w=4, s_1=1, Q_1=2, s_2=1, Q_2=2		
OFR_1	WIP_1	OFR_2
		
WIP_2	WIP_w	WIP total
		

Compound effect of DC policy: s_w=2, Q_w=2, s_1=2, Q_1=3, s_2=2, Q_2=1		
OFR_1	WIP_1	OFR_2
		
WIP_2	WIP_w	WIP total
		

In general the re-order quantity Q_d has a greater effect on the performance measures. Depending on the specific scenario under consideration, local minima and maxima may occur, so attention should be given in fine-tuning the system. Compared to balanced systems, in supply constrained systems the retailers are more sensitive to DC policy changes. At the same time there is greater scope for performance improvement, as the system is harder to reach maximum fill rates.

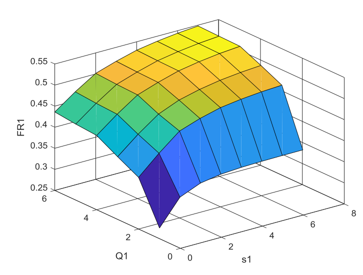
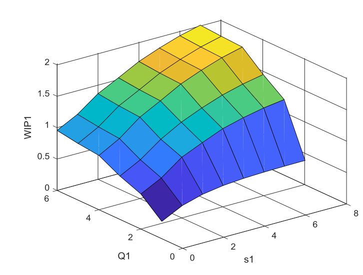
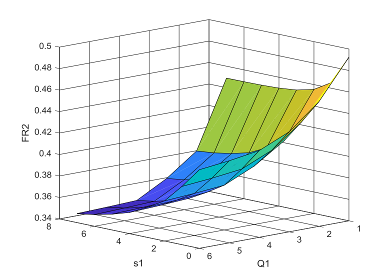
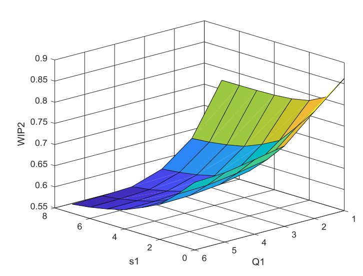
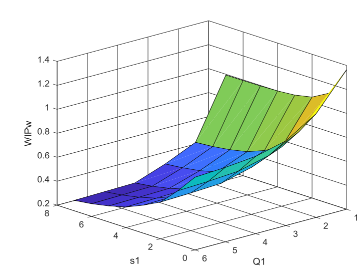
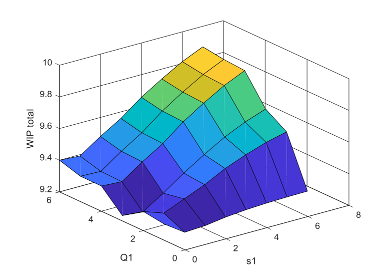
7.10.2.2 Wholesaler policy (s_w, Q_w)

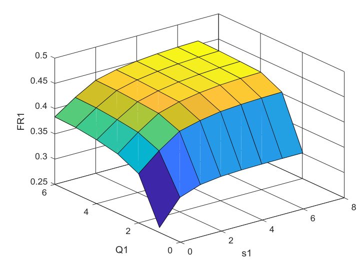
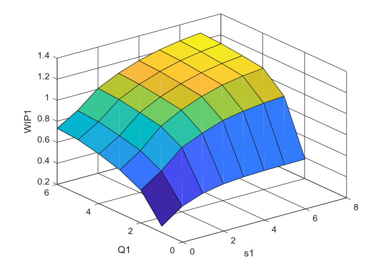
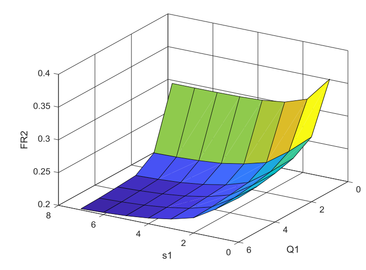
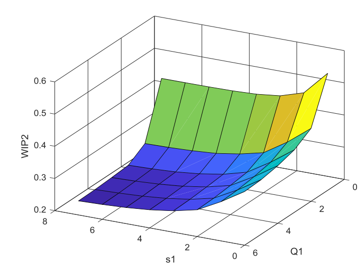
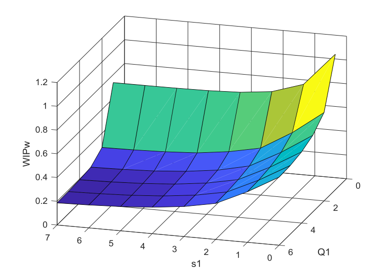
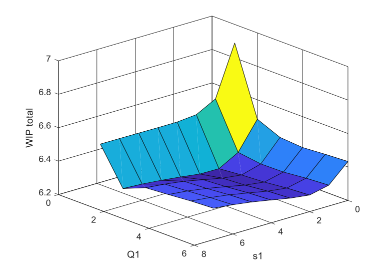
Compound effect of wholesaler's policy: s_d=2, Q_d=6, s_1=1, Q_1=2, s_2=1, Q_2=2		
OFR_1	WIP_1	OFR_2
		
WIP_2	WIP_w	WIP total
		

Compound effect of wholesaler's policy: $s_d=3$, $Q_d=5$, $s_1=1$, $Q_1=2$, $s_2=1$, $Q_2=2$		
OFR_1	WIP_1	OFR_2
		
WIP_2	WIP_w	WIP total
		

In general the effect of reorder quantity Q_w is more important than the reorder point s_w. Jagged patterns in the performance measures may occur, while compared to balanced systems, supply constrained systems are "sensitive" to a wider range of parameter values. Wholesaler's policy can be an effective way to enhance the retailers' performance. Higher s_w and Q_w values lead to higher availability of product at the wholesaler and a better service towards the retailers. This causes an increase in average inventories not only at the wholesaler, but also downstream at the retailers.

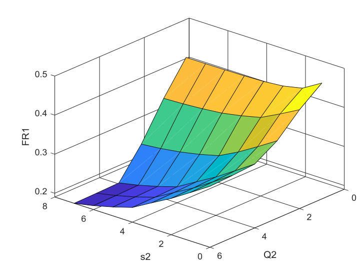
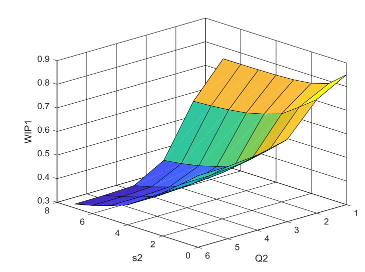
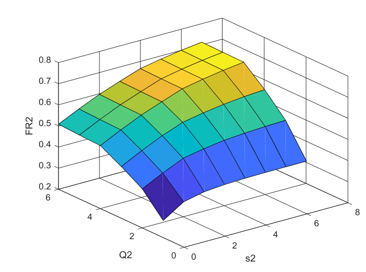
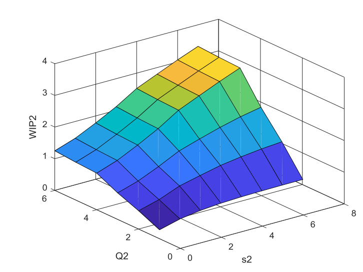
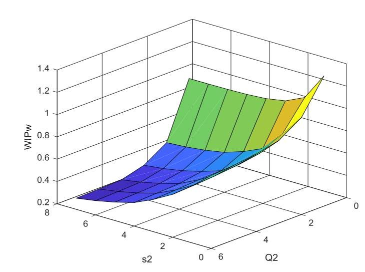
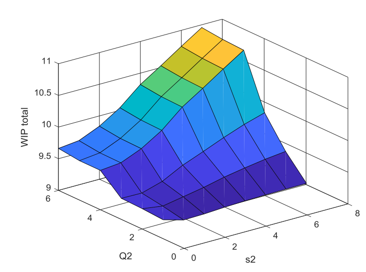
7.10.2.3 Retailer 1 policy (s_1, Q_1)

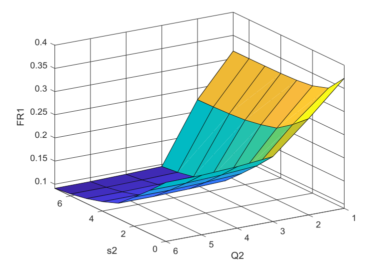
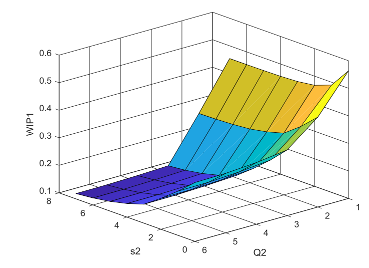
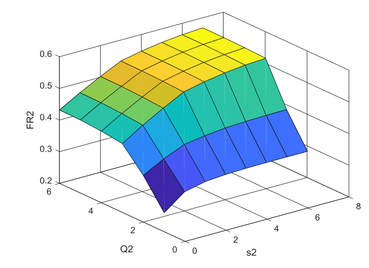
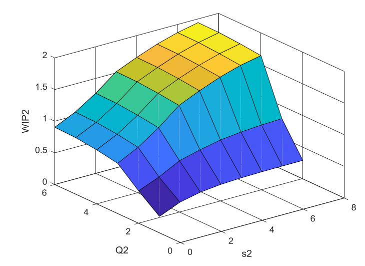
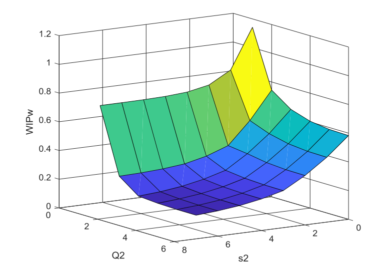
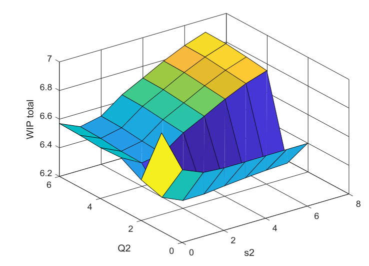
Compound effect of Retailer's 1 policy: $s_d=2$, $Q_d=6$, $s_w=2$, $Q_w=4$, $s_2=1$, $Q_2=2$		
OFR_1	WIP_1	OFR_2
		
WIP_2	WIP_w	WIP total
		

<table>
<tr><td colspan="3">Compound effect of Retailer's 1 policy : s_d=3, Q_d=3, s_w=3, Q_w=3, s_2=2, Q_2=1</td></tr>
<tr><td>OFR_1</td><td>WIP_1</td><td>OFR_2</td></tr>
<tr><td>
</td><td>
</td><td>
</td></tr>
<tr><td>WIP_2</td><td>WIP_w</td><td>WIP total</td></tr>
<tr><td>
</td><td>
</td><td>
</td></tr>
</table>

Both policy parameters have an impact on the performance measures. Increasing s_1 and Q_1 causes a transfer of available inventory downstream, but total average inventories also increase. High demand smoothes out the interplay between the parameters and for certain scenarios the jagged pattern observed in balanced systems does not occur. As the system is supply constrained, the increases in average inventories are not as dramatic as those observed for balanced systems.

7.10.2.4 Retailer 2 policy (s_2,Q_2)

<table>
<tr><td colspan="3">Compound effect of retailer's 2 policy: s_d=2, Q_d=6, s_w=2, Q_w=4, s_1=1, Q_1=2</td></tr>
<tr><td>OFR_1</td><td>WIP_1</td><td>OFR_2</td></tr>
<tr><td>
</td><td>
</td><td>
</td></tr>
<tr><td>WIP_2</td><td>WIP_w</td><td>WIP total</td></tr>
<tr><td>
</td><td>
</td><td>
</td></tr>
</table>

Compound effect of retailer's 2 policy: s_d=3, Q_d=3, s_w=3, Q_w=3, s_1=2, Q_1=1		
OFR_1	WIP_1	OFR_2
		
WIP_2	WIP_w	WIP total
		

The general trends for higher priority retailer 2 are the same to those in the case of lower priority retailer 1. In general greater elasticities are observed (for the same changes in parameter values there are greater changes in the performance measures). With regard to total inventory, local minima may be observed at intermediate values of s_2 and Q_2.

7.10.3 Interplay between the retailers

Each retailer acts independently from the others and no demand correlation is assumed. However, since all retailers are supplied by a finite capacity Wholesaler it is expected that some interplay amongst them occurs. We study a system with three retailers and we investigate how the inventory policy of one retailer affects the performance of the others.

7.10.3.1 Balanced systems

We assume

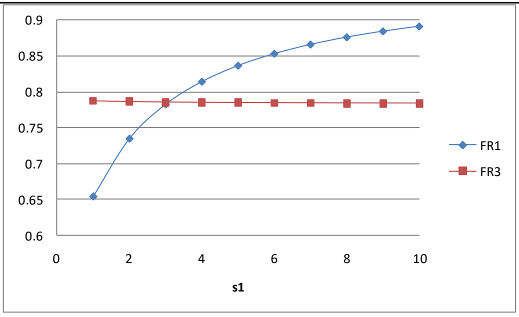
$$\mu_1 = \mu_2 = \mu_3 = \lambda_1 = \lambda_2 = \lambda_3 = \frac{\mu_w}{3} = \frac{\mu_d}{3}$$

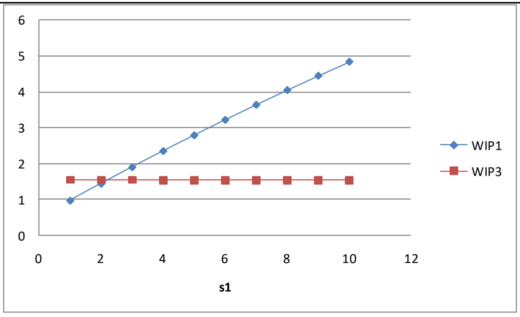
<u>Lowest priority retailer</u>

The policy of the lowest priority retailer 1 has a weak effect on the performance of the highest priority retailer 3. Increasing s_1 causes a small but consistent decrease in FR_3 and WIP_3. The effect can be described with good precision with a linear relation. In the investigated scenarios the unitary increase of s_1 caused a decreased in FR_3 that did not exceed 0.15%.

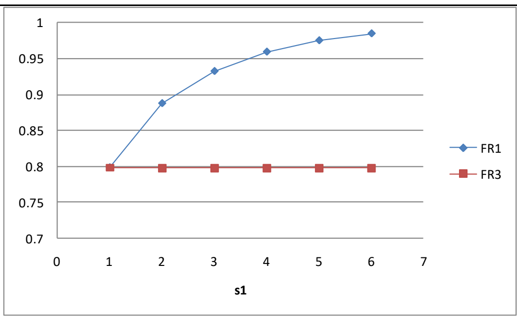
Effect of lowest priority retailer – s_1

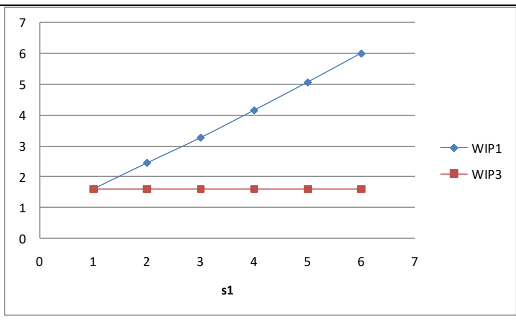
s_d=0, Q_d=4, s_w=2, Q_w=2, Q_1=1 s_2=1, Q_2=2, s_3=1, Q_3=2



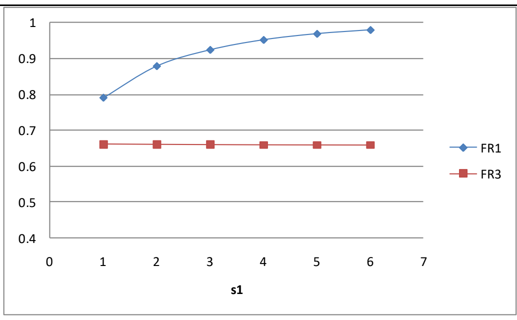


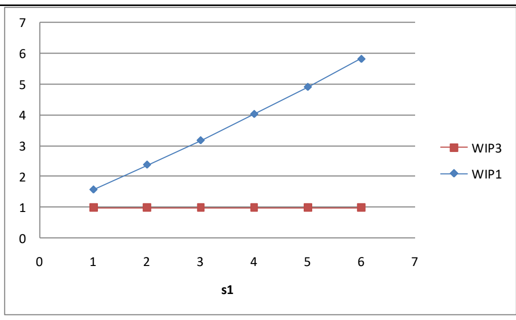
s_d=0, Q_d=4, s_w=2, Q_w=4, Q_1=2, s_2=1, Q_2=2, s_3=1, Q_3=2





s_d=2, Q_d=2, s_w=2, Q_w=2, Q_1=2, s_2=1, Q_2=1, s_3=1, Q_3=1

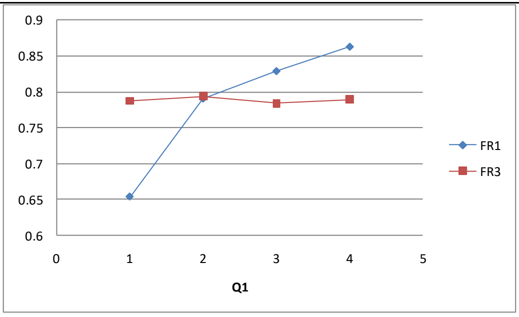


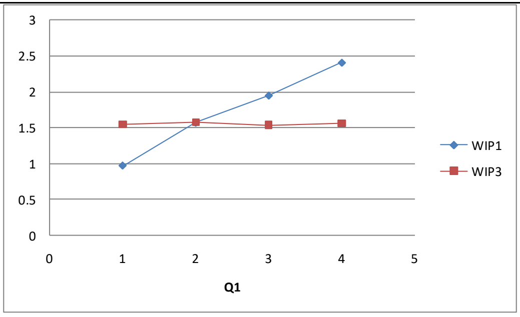


The effect of Q_1 on the performance of the highest priority retailer is more dynamic. Q_1 directly affects the availability of inventory at the wholesaler, and thus the ability of the wholesaler to respond to demand from the other retailers. In some cases FR_3 increased slightly with increasing Q_1, simultaneously with an increase in FR_1. Such cases indicate the coordination between the inventory policies of the various network members and are of interest from a managerial point of view. For a unitary change of Q_1 changes in FR_3 up to 1.25% were observed.

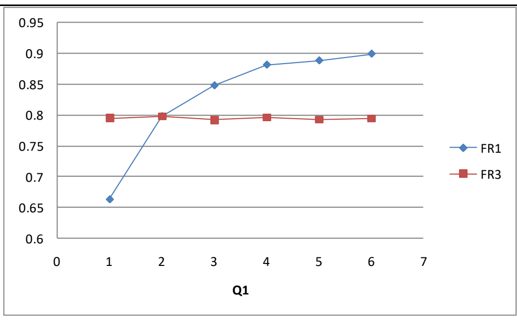
Effect of lowest priority retailer – Q_1

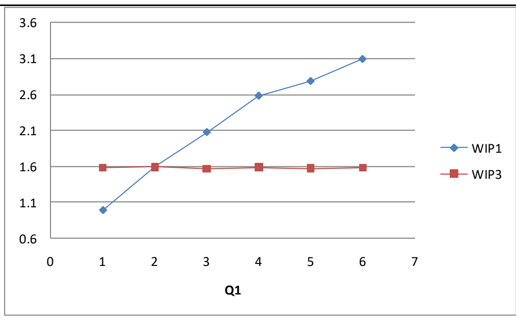
s_d=0, Q_d=4, s_w=2, Q_w=2, s_1=1 s_2=1, Q_2=2, s_3=1, Q_3=2



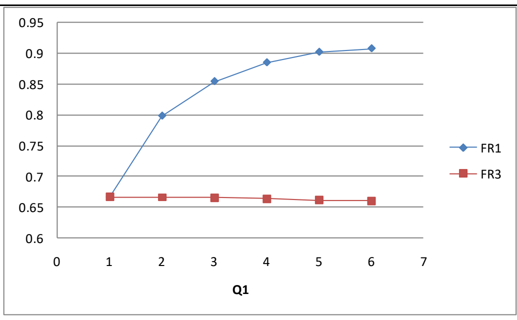


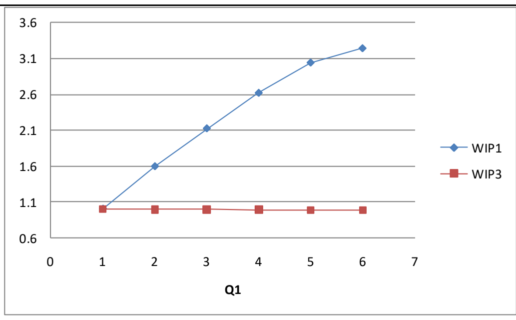
s_d=0, Q_d=4, s_w=2, Q_w=4, s_1=1, s_2=1, Q_2=2, s_3=1, Q_3=2





s_d=2, Q_d=2, s_w=4, Q_w=2, s_1=1, s_2=1, Q_2=1, s_3=1, Q_3=1



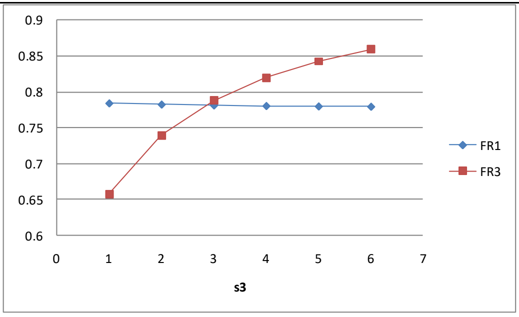


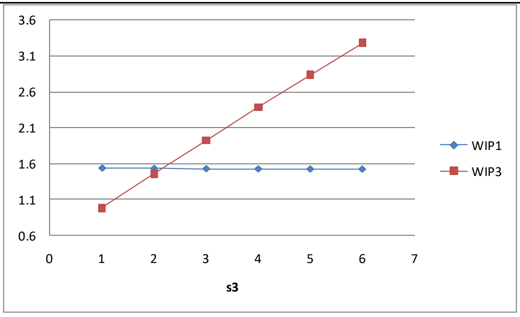
Highest priority retailer

Increasing s_3 causes FR_1 to decrease. In general the effect is weak, but stronger than the effect of s_1 on FR_3. The changes can be described with good accuracy by a logarithmic relation, while for a unitary change of s_3 changes in FR_1 up to 0.2 % where observed.

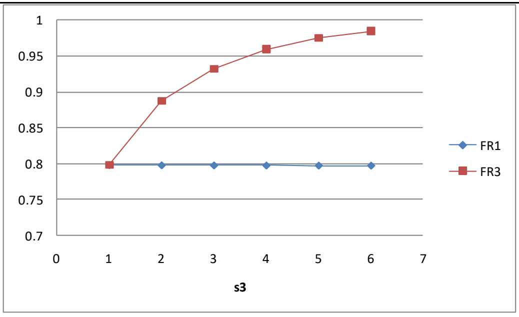
Effect of highest priority retailer – s_3

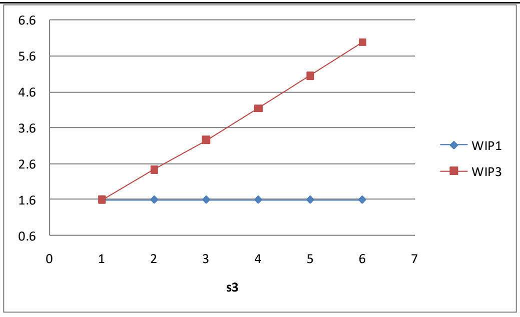
s_d=0, Q_d=4, s_w=2, Q_w=2, s_1=1, Q_1=2 s_2=1, Q_2=2, Q_3=1



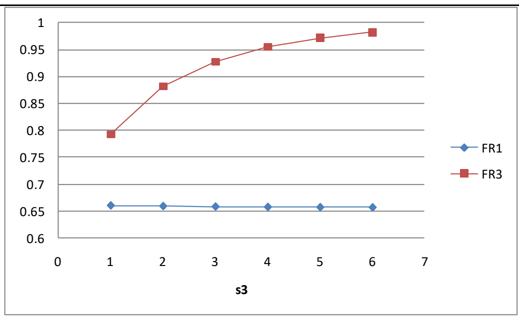


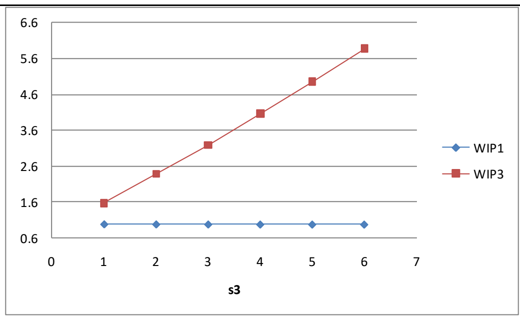
s_d=0, Q_d=4, s_w=2, Q_w=4, s_1=1, Q_1=2, s_2=1, Q_2=2, Q_3=2





s_d=2, Q_d=2, s_w=2, Q_w=2, s_1=1, Q_1=1, s_2=1, Q_2=1, Q_3=2

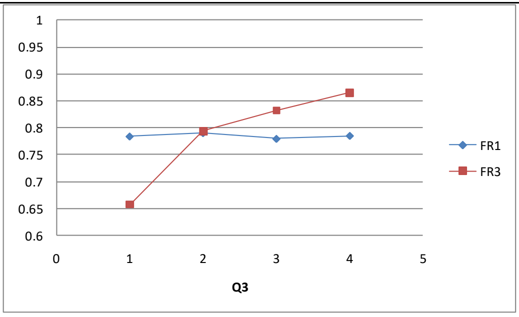


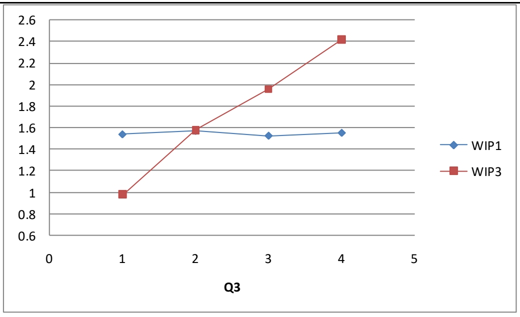


The effect of Q_3 on FR_1 is more dynamic. In some cases FR_1 decreases with increasing Q_3, but in some scenarios local maxima and minima were observed. The effect is similar to, but somewhat stronger than the effect of Q_1 on FR_3. In the investigated scenarios and for unitary changes in Q_3, changes of up to 1.36% in FR_1 were observed.

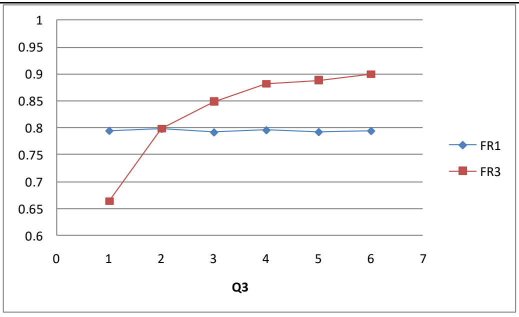
Effect of highest priority retailer – Q_3

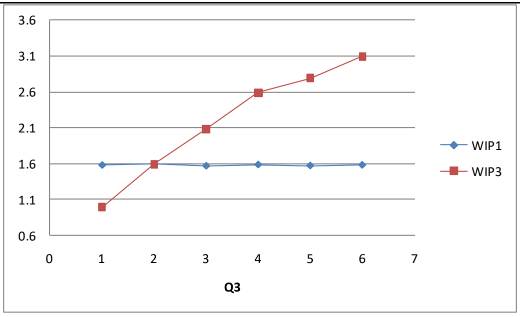
$s_d=0$, $Q_d=4$, $s_w=2$, $Q_w=2$, $s_1=1$, $Q_1=2$, $s_2=1$, $Q_2=2$, $s_3=1$



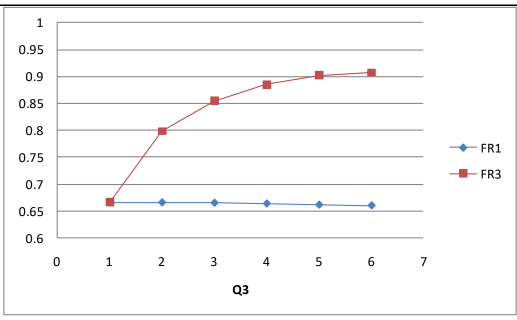


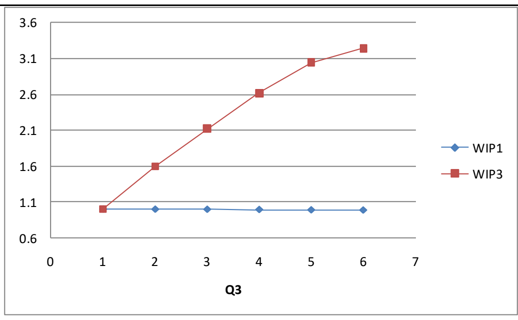
$s_d=0$, $Q_d=4$, $s_w=2$, $Q_w=4$, $s_1=1$, $Q_1=2$, $s_2=1$, $Q_2=2$, $s_3=1$





$s_d=2$, $Q_d=2$, $s_w=4$, $Q_w=2$, $s_1=1$, $Q_1=1$, $s_2=1$, $Q_2=1$, $s_3=1$





7.10.3.2 Unbalanced systems

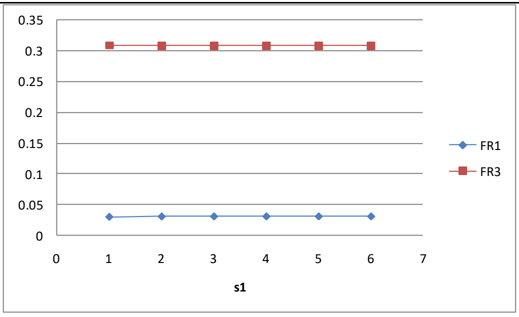
We investigate a system where $\lambda_1 = \lambda_2 = \lambda_3 > \mu_1 = \mu_2 = \mu_3 > \mu_w > \mu_d$. Such a system is considered supply constrained and a greater competition between the retailers for the resources of the wholesaler is to be expected. We use the values $\lambda_1=\lambda_2=\lambda_3=3$, $\mu_1=\mu_2=\mu_3=2$, $\mu_w=1$, and $\mu_d=0.8$.

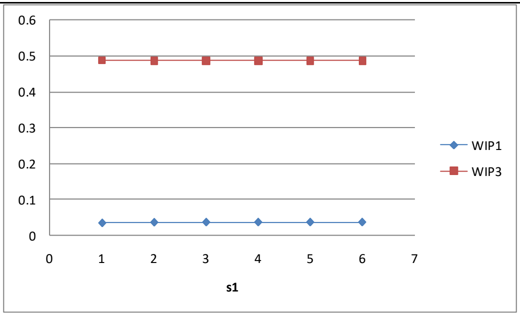
<u>Lowest priority retailer</u>

The increase of s_1 causes FR_3 to decrease. The effect becomes less important as s_1 increases. In the investigated scenarios and for unitary increases in s_1, the greatest change in FR_3 that was observed was 2.7%.

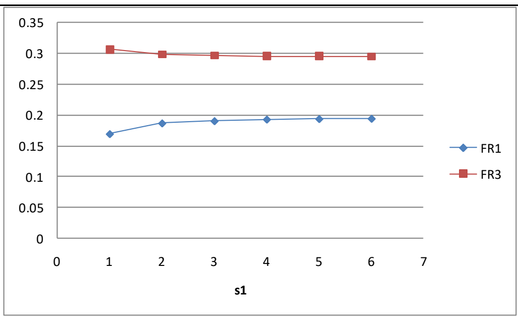
Effect of lowest priority retailer – s_1

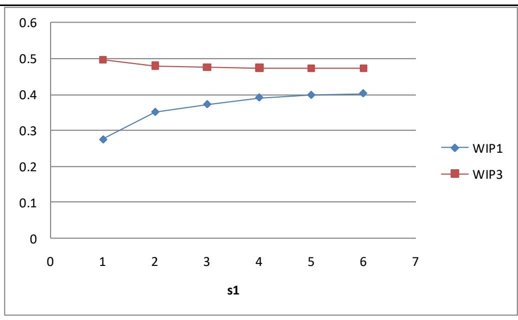
s_d=0, Q_d=4, s_w=2, Q_w=2, Q_1=1 s_2=1, Q_2=2, s_3=1, Q_3=2



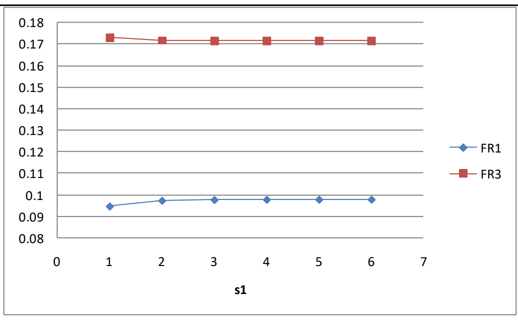


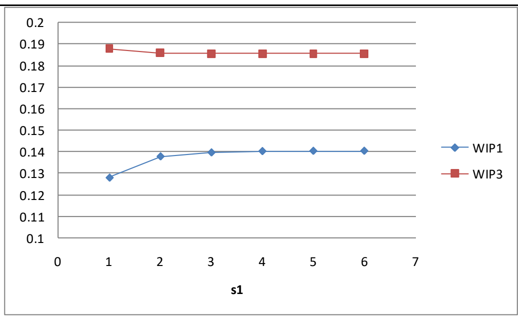
s_d=0, Q_d=4, s_w=2, Q_w=4, Q_1=2, s_2=1, Q_2=2, s_3=1, Q_3=2





s_d=2, Q_d=2, s_w=2, Q_w=2, Q_1=2, s_2=1, Q_2=1, s_3=1, Q_3=1

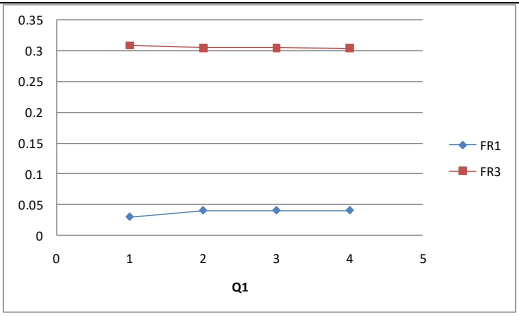


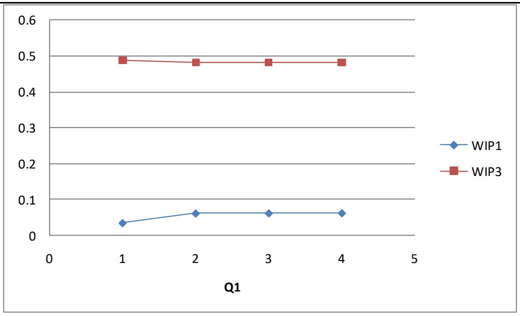


With regard to the Q_1 effect, increasing Q_1 causes a decrease in FR_3. In general, the effect is more pronounced for low Q_1 values. Q_1 is more important than s_1. For the investigated scenarios and for unitary increases in Q_1, the greatest decrease of FR_3 that was observed was 4.7%.

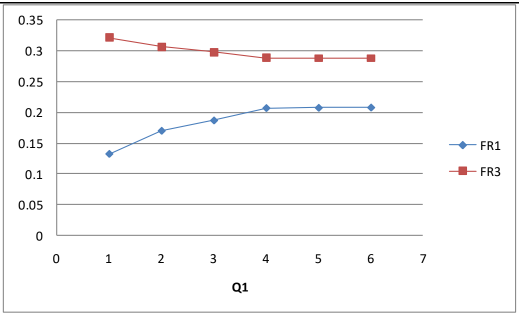
Effect of lowest priority retailer – Q_1

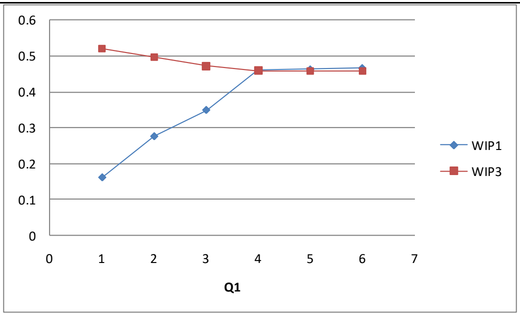
$s_d=0$, $Q_d=4$, $s_w=2$, $Q_w=2$, $s_1=1$ $s_2=1$, $Q_2=2$, $s_3=1$, $Q_3=2$



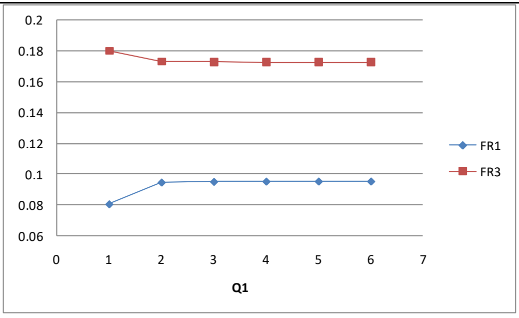


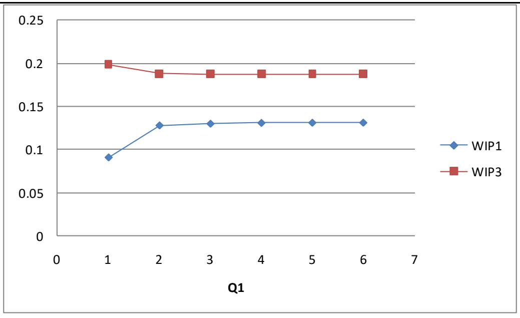
$s_d=0$, $Q_d=4$, $s_w=2$, $Q_w=4$, $s_1=1$, $s_2=1$, $Q_2=2$, $s_3=1$, $Q_3=2$





$s_d=2$, $Q_d=2$, $s_w=4$, $Q_w=2$, $s_1=1$, $s_2=1$, $Q_2=1$, $s_3=1$, $Q_3=1$



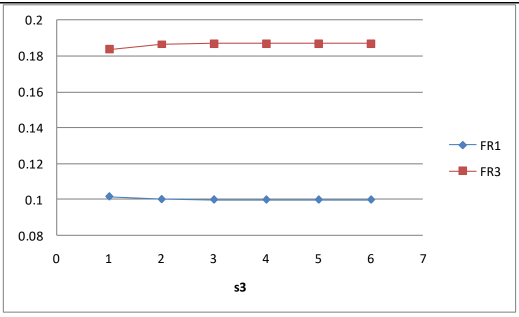


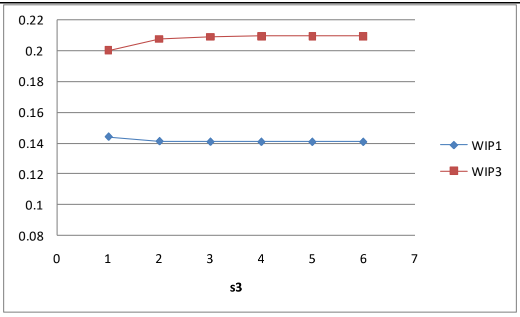
Highest priority retailer

The increase of s_3 causes FR_1 to decrease, with the effect being more important for low s_3 values. Compared to the effect of s_1 to FR_3, the lowest priority retailer is more sensitive to s_3 changes. In the investigated scenarios, for changes from $s_3=1$ to $s_3=2$ changes in FR_1 of up to 24% were observed.

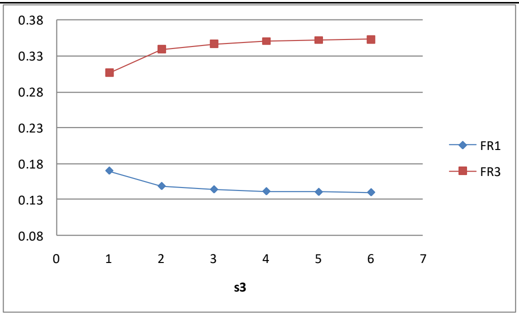
Effect of highest priority retailer – s_3

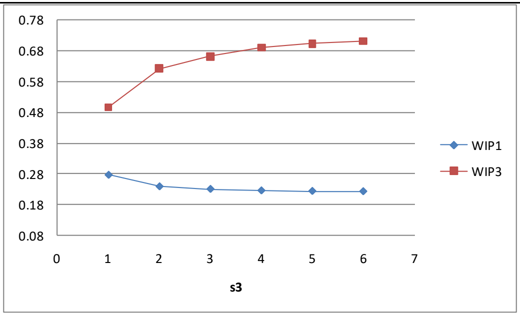
$s_d=0$, $Q_d=4$, $s_w=2$, $Q_w=2$, $s_1=1$, $Q_1=2$ $s_2=1$, $Q_2=2$, $Q_3=1$



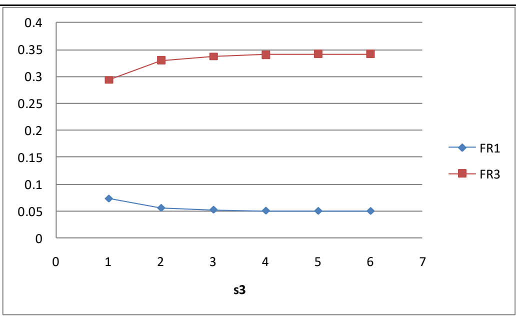


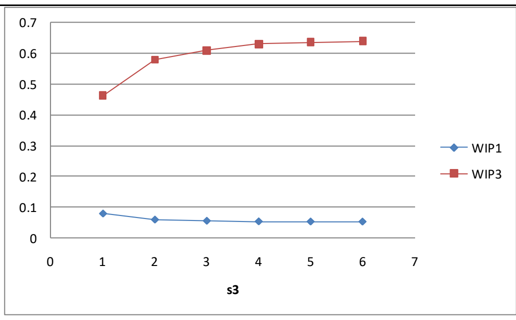
$s_d=0$, $Q_d=4$, $s_w=2$, $Q_w=4$, $s_1=1$, $Q_1=2$, $s_2=1$, $Q_2=2$, $Q_3=2$





$s_d=2$, $Q_d=2$, $s_w=2$, $Q_w=2$, $s_1=1$, $Q_1=1$, $s_2=1$, $Q_2=1$, $Q_3=2$

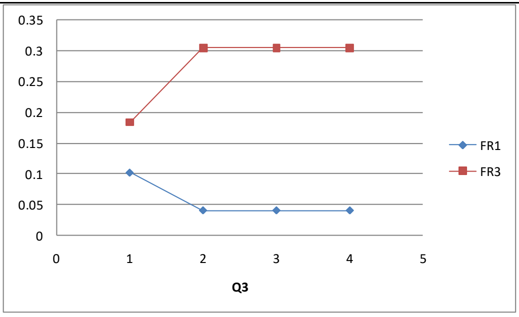


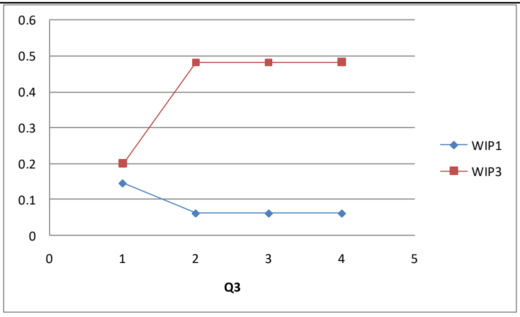


FR_1 decreases with increasing Q_3, with the decrease mirroring the decrease in the average wholesaler inventory. The effect decreases with increasing Q_3 values and beyond a point it becomes negligible. In general, Q_3 is more important than s_3 for FR_1. In the investigated scenarios, the greatest decrease in FR_1 for unitary increase in Q_3 was 60%.

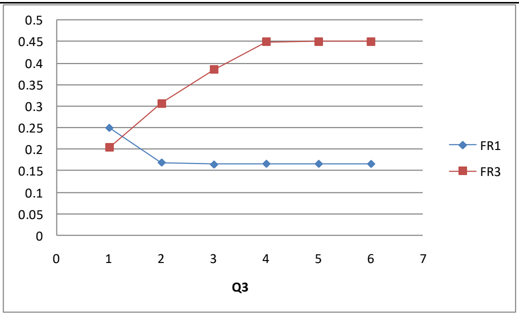
Effect of highest priority retailer – Q_3

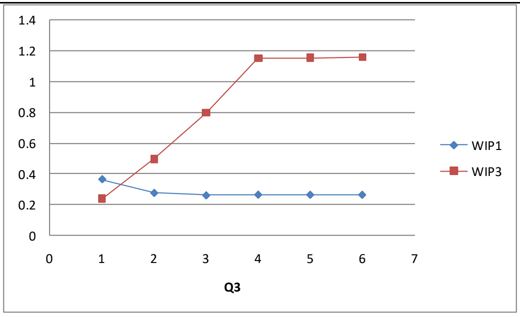
$s_d=0$, $Q_d=4$, $s_w=2$, $Q_w=2$, $s_1=1$, $Q_1=2$, $s_2=1$, $Q_2=2$, $s_3=1$



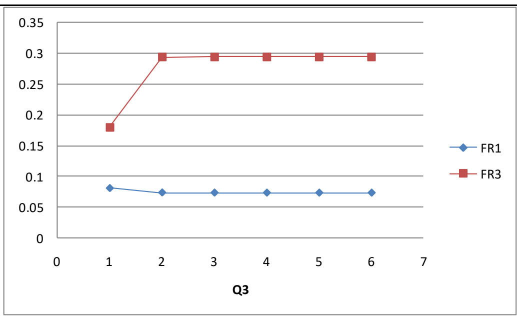


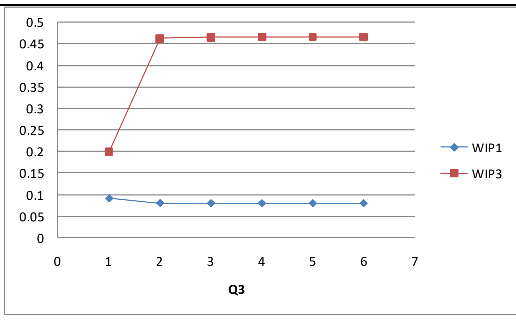
$s_d=0$, $Q_d=4$, $s_w=2$, $Q_w=4$, $s_1=1$, $Q_1=2$, $s_2=1$, $Q_2=2$, $s_3=1$





$s_d=2$, $Q_d=2$, $s_w=4$, $Q_w=2$, $s_1=1$, $Q_1=1$, $s_2=1$, $Q_2=1$, $s_3=1$



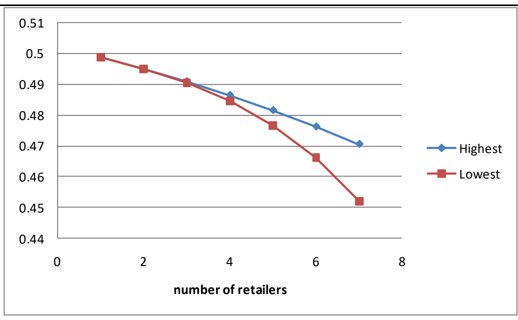


7.10.4 Effect of retailer addition to the system

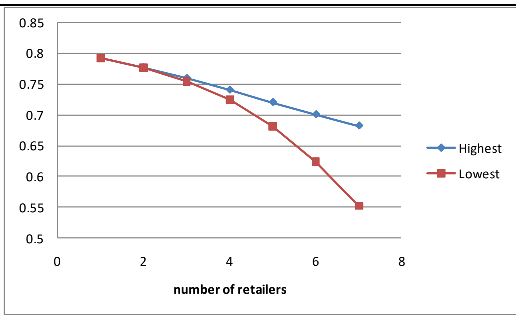
We investigate the behavior of the system with increasing number of Retailers supplied by the same Wholesaler. For better focus, all retailers are assumed to follow the same inventory control policy, while replenishment times are also the same for all retailers.

Order Fill rate of the highest and lowest priority retailers as a function of the number of retailers. $s_d=0$, $Q_d=2$, $s_w=0$, $Q_w=2$, $s_i=0$, $Q_i=1$,i=[1,7]. $\lambda=1$, $\mu_d=\mu_w=4$

$\mu(i)=1$, i=[1,7]

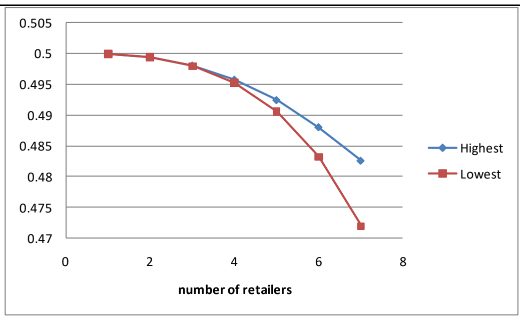


$\mu(i)=4$, i=[1,7]

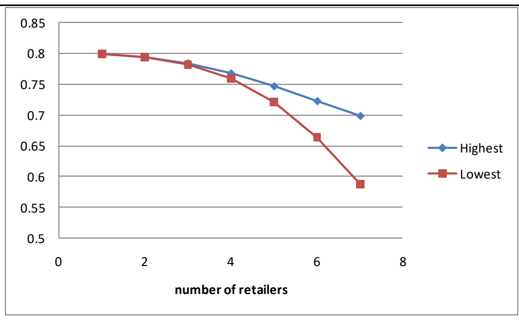


Order Fill rate of the highest and lowest priority retailers as a function of the number of retailers. s_d=0, Q_d=2, s_w=1, Q_w=2, s_i=0, Q_i=1 ,i=[1,7]. λ=1, μ_d=μ_w=4

μ(i)=1, i=[1,7]



μ(i)=4, i=[1,7]

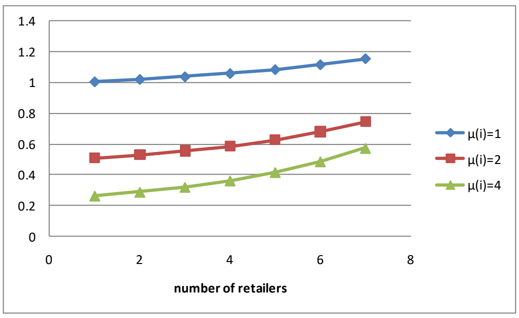


Increasing the number of the retailers causes the Fill rates to decrease as the available inventory at the wholesaler decreases. Initially the effect is almost the same for all retailers, irrespectively of their designated priority. The antagonism between the retailers becomes pronounced at some intermediate *n* value, where the stock-out probability for the wholesaler becomes important, and priorities start to have an effect on retailers' performance.

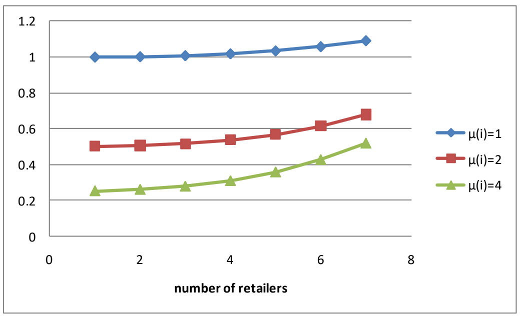
Considering total system performance, by increasing the number of retailers we increase the total output, but at the same time total lost sales also increase. The ratio of lost sales to output increases with *n*. For every extra retailer the marginal output increases, but marginal lost sales also increase. The changes in the ratio also depend on the other parameters of the system and they are more important for higher retailer transportation rates.

Ratio of lost sales to output as a function of the number of retailers.

s_d=0, Q_d=2, s_w=0, Q_w=2, s_i=0, Q_i=1 ,i=[1,7]. λ=1, μ_d=μ_w=4



s_d=0, Q_d=2, s_w=1, Q_w=2, s_i=0, Q_i=1 ,i=[1,7]. λ=1, μ_d=μ_w=4



7.10.5 Synopsis

The inventory policies of the upstream nodes can be used to enhance the performance of the system, but the limit of the attainable Fill rates depends on the retailers' respective policies. The unavoidable increase in average inventory is centered on the node whose policy is changed.

In the case of the Distribution Center policy, the inventory increase is observed at the DC and to a lesser extent at the Wholesaler. This may be preferable in cases where the cost of inventory increases as it moves downstream towards the end customers. Moreover, the benefits of more available inventory are shared amongst the retailers.

In a similar manner, the wholesaler's policy can also be used to enhance the performance of the retailers. Average inventories increase at the Wholesaler and the retailers. Under certain conditions and when Q_w is changed there may be a decrease in the upstream DC inventory, so the Wholesaler's policy can be used as a means to "transfer" available inventory downstream. In the investigated scenarios the presence of some safety stock at the DC and the Wholesaler is advantageous as it renders the system less sensitive to changes in the replenishment orders.

Both reorder points and reorder quantities have an impact on the performance measures. The effect of reorder points is in general more predictable. To choose the reorder point, the reorder quantities of the adjacent upstream and downstream nodes, and the possible values of inventory on hand must be taken into account. In certain scenarios small changes in the reorder point may be irrelevant. In regard with replenishment quantities, their effect is more dynamic. In some cases the performance measures exhibit a "jagged" pattern with local maxima and minima, so the fine tuning of the system may be beneficial.

The behavior of the system depends strongly on demand characteristics. Increased demand makes the system more sensitive to changing parameters. Moreover, reorder points become less important. If higher demand is to be met successfully, larger replenishment orders are needed. The interplay between the parameters becomes less important with higher demand as its relatively small effects are overridden by the lack of available inventory. High demand also keeps average inventories low over a wider parameter range.

The system has a dynamic nature. Our analysis indicates that the coordination of the inventory policies is possible. For a given desirable service level there can be found a combination of policies that minimizes total inventory in the system. There is also correlation in the performance of different nodes of the system. With regard to the retailers, higher priority retailers have a greater impact on the system, but the priority effect is not important as long as the stock-out probability for the wholesaler remains low. Adding retailers to the network causes fill rates to fall and this fall is more intense when higher initial service levels are concerned. The optimal number of retailers will depend on the specific costs parameters.

7.11 Conclusions

In this section we have presented a model based on Markov processes for the exact numerical analysis of a single product, three echelon, arborescent inventory system. A solution algorithm based on the properties of the infinitesimal generator matrix was developed, and the respective computer program was coded in Matlab. In comparison to existing models, we investigate a longer network, under more realistic, stochastic conditions, and we offer an exact solution without recourse to approximate methods.

The model was used to investigate the effect of the decision variables on the performance measures. Our analysis indicates a dynamic behaviour, especially when replenishment quantities are concerned. There is inter-dependence between the different members of the network and interplay between system parameters. More importantly, from a managerial point of view, our analysis also indicates that it is possible to coordinate the system. In many instances local optimal policies were observed, so it is possible to enhance performance with only minor adjustments in inventory policies.

As a further step, three directions of research are proposed. With regard to external demand, more general demand distributions can be investigated. Although the Poisson distribution is commonly used to model demand in inventory systems, it may not be appropriate to describe more complex, and more realistic cases. A compound Poisson distribution combining Poisson arrivals for customers with an empirical distribution for individual demand would offer more modelling flexibility and it would be relatively easy to integrate in the presented model. In a second direction, more general distributions can be used for replenishment times. The application of phase type distributions (Erlang, Coxian) to model times would allow for more realistic modelling, but it must be kept in mind that the total number of states, and so the computing requirements, would increase. Finally the development of a realistic cost function would allow the model to be used for optimization purposes. This could be done either through an exhaustive enumeration of possible policy combinations, or by combining the evaluative algorithm with an optimization heuristic.

7.12. References

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7.13 Appendix

7.13.1. Matlab algorithm

Below we present the program code for the three echelon arborescent system. The computer algorithm is given for Mathworks' Matlab, version R2018a (9.4.0.813654). The code essentially follows the lines of the analysis presented in chapter 7. Some parts of the model that are used repetitively have been defined as sub-routines (functions) that are stored separately and called whenever necessary. Such functions are given separately from the main body of the algorithm. Comments start with the symbol %.

Important note: Some lines of the algorithm have been omitted on purpose. Their position is denoted with […]

Main body algorithm

```
%{
--------------- Input ------------------------
n: The number of retailers
sdc: Reorder Point at the Distribution Centre (DC)
Qd: Replenishment order Quantity at the Distribution Centre(DC)
sw: Reorder Point at the Wholesaler (W)
Qwh: Replenishment order Quantity at the Wholesaler(W)
s(i): Reorder Point at Retailer (i)
Qr(i): Replenishment Order Quantity at Retailer (i)
md: Transportation rate for orders towards the DC
mw: Transportation rate for orders towards the Wholesaler
m(i): Transportation rate for orders towards retailer i
l(i): Arrival rate of external customers at Retailer i
s: The vector of reorder points for the Retailers. The first element
refers to Retailer 1, etc
Q: The vector of the replenishment quantities for the retailers
%}
n=7;
sdc=0;
Qd=2;
sw=0;
Qwh=2;
s=[0;0;0;0;0;0;0];
Qr=[1;1;1;1;1;1;1];
md=0.9;
mw=1.8;
m=[1;1.2;1.4;1.6;1.8;1;1];
l=[0.9;0.7;0.9;1.2;1.5;1;1];
if sdc<Qd & Qd<=Qwh
    sd=0;
else
    sd=sdc;
end
Qw=min(Qwh,sd+Qd);
for i=1:n
    Q(i)=min(Qr(i),sw+Qw);
end
% ---------- Infinitesimal Generator Matrix ---------------------
bsd=gcd(Qd,Qw);
NLd=floor(((sd+Qd)/bsd));
bsw=gcd(bsd,Q(1));
for i=2:n
    bsw=gcd(bsw,Q(i));
end
```

```
NLw=floor(((sw+Qw)/bsw));
nQ=zeros(n,1);
for i=1:n
    nQ(i)=floor(Q(i)/bsw);
end
nQw=floor(Qw/bsd);
nsw=floor(sw/bsw);
%                     * * *  Diagonal sub-matrices  * * *
% Iw=0
[B,Bl]=eIwz(n,s,Q,nQ,m,l,sw,Qw,md,mw,bsw );
% Iw>0
[C,Cl]=eIwp(n,s,Q,nQ,m,l,md,mw,bsw);
% Construction of the diagonal tier of the Infinitesimal Generator
Matrix
P=zeros(1);
% Id=0
%       Tw=0
%           Iw=0
P(1:Bl(n),1:Bl(n))=B+mw*eye(Bl(n));
%           Iw>0
for i=1:NLw
    P(Bl(n)+(i-1)*Cl(n)+1:Bl(n)+i*Cl(n),Bl(n)+(i-
1)*Cl(n)+1:Bl(n)+i*Cl(n))=C+mw*eye(Cl(n));
    lp=Bl(n)+i*Cl(n);
end
%       Tw>0
for i= 1:nQw
%           Iw=0
    P(lp+1:lp+Bl(n),lp+1:lp+Bl(n))=B;
    lp=lp+Bl(n);
%           Iw>0
    for j=1:nsw
        P(lp+1:lp+Cl(n),lp+1:lp+Cl(n))=C;
        lp=lp+Cl(n);
    end
end
L0=lp;
% Id>0
for i=1:NLd
%       Tw=0
    for j=1:NLw-nsw
        P(lp+1:lp+Cl(n),lp+1:lp+Cl(n))=C+mw*eye(Cl(n));
        lp=lp+Cl(n);
    end
%       Tw>0
    for i=1:nQw
%           Iw=0
        P(lp+1:lp+Bl(n),lp+1:lp+Bl(n))=B;
    lp=lp+Bl(n);
%           Iw>0
        for j=1:nsw
            P(lp+1:lp+Cl(n),lp+1:lp+Cl(n))=C;
            lp=lp+Cl(n);
        end
    end
end
% ns: number of states
ns=length(P);
% L1: the length of Basic Levels where Id>0
L1=(ns-L0)/NLd;
for i=L0+floor(sd/bsd)*L1+1:ns
```

```
    P(i,i)=P(i,i)+md;
end
%         * * * Upper-Diagonal sub-matrices * * *
b=Bl(n)+nsw*Cl(n);
% Id=0
%     Tw=0
%         Iw<=sw
if Qd>Qw
    step=L0+((Qd-Qw)/bsd-1)*L1+(NLw-nsw)*Cl(n)+(nQw-1)*b;
else
    step=Bl(n)+NLw*Cl(n)+((Qd/bsd)-1)*b;
end
P(1:b,step+1:step+b)=md*eye(b);
%         Iw>sw
P(b+1:b+(NLw-nsw)*Cl(n),L0+(Qd/bsd-1)*L1+1: L0+(Qd/bsd-1)*L1+(NLw-
nsw)*Cl(n))=md*eye((NLw-nsw)*Cl(n));
%    Tw>0
for i=1:nQw
    step=L0+(Qd/bsd-1)*L1+(NLw-nsw)*Cl(n)+(i-1)*b;
    P(Bl(n)+NLw*Cl(n)+(i-
1)*b+1:Bl(n)+NLw*Cl(n)+i*b,step+1:step+b)=md*eye(b);
end
% Id>0
for i=1:floor(sd/bsd)
    x=L0+(i-1)*L1;
    y=L0+(i+(Qd/bsd)-1)*L1;
    P(x+1:x+L1,y+1:y+L1)=md*eye(L1);
end
%       * * * Sub-Diagonal sub-matrices * * *
% Id=0
%     Iw=0
for Lw=1:nQw
    Tw=Lw*bsd;
    inD=indexDemand(Tw,Bl,s,Q,n);
    for state=1:Bl(n)
        if inD(state,n+1)==0
            sumQ=zeros(n,1);
            sum=1;
            sumQ(n)=sum;
                for i=1:n-1
                    sum=sum+inD(state,n-i+1)*nQ(n-i+1);
                    sumQ(n-i)=sum;
                end
            step=((s(1)+Q(1)+1)+min(Tw/bsw-sumQ(1),nQ(1)-
1)*(s(1)+1))*inD(state,1);
                for i=2:n
                    step=step+((s(i)+Q(i)+1)+min(Tw/bsw-
sumQ(i),nQ(i)-1)*(s(i)+1))*Bl(i-1)*inD(state,i);
                end
            x0=Bl(n)+NLw*Cl(n)+(Lw-1)*b+state;
            y0=step+state;
            P(x0,y0)=mw;
        else
            step=Bl(n)+(inD(state,n+1)/bsw-1)*Cl(n);
            nstate=state;
            for i=1:n-1
                p=ceil(nstate/Bl(n-i));
                nstate=nstate-(p-1)*Bl(n-i);
                step=step+(p-1-s(n-i+1)-1)*Cl(n-i)*(1-inD(state,n-
i+1));
            end
```

```
            step=step+(nstate-s(1)-1)*(1-inD(state,1));
            for i=2:n
                step=step+(Q(i)+(nQ(i)-1)*(s(i)+1))*Cl(i-
1)*inD(state,i);
            end
            step=step+(Q(1)+(nQ(1)-1)*(s(1)+1))*inD(state,1);
            nstate=state;
            for i=1:n-1
            p=ceil(nstate/Bl(n-i));
            nstate=nstate-(p-1)*Bl(n-i);
            step=step+(p-1)*Cl(n-i)*inD(state,n-i+1);
            end
       step=step+(nstate)*inD(state,1);
       x0=Bl(n)+NLw*Cl(n)+(Lw-1)*b+state;
       y0=step;
       P(x0,y0)=mw;
       end
    end
%     Iw>0
    x0=Bl(n)+NLw*Cl(n)+(Lw-1)*b+Bl(n);
    y0=Bl(n)+(Tw/bsw)*Cl(n);
    P(x0+1:x0+nsw*Cl(n),y0+1:y0+nsw*Cl(n))=mw*eye(nsw*Cl(n));
end
% Id>0
%     Iw=0
for Ld=1:NLd
    Id=Ld*bsd;
    for Lw=1:nQw
        Tw=Lw*bsd;
        inD=indexDemand(Tw,Bl,s,Q,n);
        for state=1:Bl(n)
            if inD(state,n+1)==0
                if Id>Qw
                    step=L0+((Id-Qw)/bsd-1)*L1;
                    step=[...]
                else
                    step=0;
                    step=[...]
                end
            sumQ=zeros(n,1);
            sum=1;
            sumQ(n)=sum;
                for i=1:n-1
                    sum=sum+inD(state,n-i+1)*nQ(n-i+1);
                    sumQ(n-i)=sum;
                end
            step=step+((s(1)+Q(1)+1)+min(Tw/bsw-sumQ(1),nQ(1)-
1)*(s(1)+1))*inD(state,1);
                for i=2:n
                    step=[...]
                end
            x0=L0+(Ld-1)*L1+(NLw-nsw)*Cl(n)+(Lw-1)*b+state;
            y0=step+state;
            P(x0,y0)=mw;
            else
                if inD(state,n+1)>sw
                    step= L0+(Ld-1)*L1+((inD(state,n+1)/bsw)-nsw-
1)*Cl(n);
                else
                    if Id>Qw
                        step=L0+((Id-Qw)/bsd-1)*L1;
```

```
                    step=step+(NLw-nsw)*Cl(n)+ ((Qw/bsd)-1)*b;
                else
                    step=0;
                    step=step+Bl(n)+NLw*Cl(n)+(Id/bsd-1)*b;
                end

            step=step+Bl(n)+(inD(state,n+1)/bsw-1)*Cl(n);
            end
            nstate=state;
            for i=1:n-1
                p=ceil(nstate/Bl(n-i));
                nstate=nstate-(p-1)*Bl(n-i);
                step=step+(p-1-s(n-i+1)-1)*Cl(n-i)*(1-
inD(state,n-i+1));
            end
            step=step+(nstate-s(1)-1)*(1-inD(state,1));
            for i=2:n
                step=step+(Q(i)+(nQ(i)-1)*(s(i)+1))*Cl(i-
1)*inD(state,i);
            end
            step=step+(Q(1)+(nQ(1)-1)*(s(1)+1))*inD(state,1);
            nstate=state;
            for i=1:n-1
                p=ceil(nstate/Bl(n-i));
                nstate=nstate-(p-1)*Bl(n-i);
                step=step+(p-1)*Cl(n-i)*inD(state,n-i+1);
            end
            step=step+(nstate)*inD(state,1);
            x0=L0+(Ld-1)*L1+(NLw-nsw)*Cl(n)+(Lw-1)*b+state;
            y0=step;
            P(x0,y0)=mw;
        end
    end
% Iw>0
        for i=1:nsw
            if Tw+i*bsw>sw
                step= L0+(Ld-1)*L1+((Tw+i*bsw)/bsw-nsw-1)*Cl(n) ;
            else
                if Id>Qw
                    step=L0+((Id-Qw)/bsd-1)*L1;
                    step=step+(NLw-nsw)*Cl(n)+ ((Qw/bsd)-1)*b;
                    step=step+Bl(n)+((Tw+i*bsw)/bsw-1)*Cl(n);
                else
                    step=0;
                    step=step+Bl(n)+NLw*Cl(n)+((Id/bsd)-
1)*b+Bl(n)+((Tw+i*bsw)/bsw-1)*Cl(n);
                end
            end
            x0=L0+(Ld-1)*L1+(NLw-nsw)*Cl(n)+(Lw-1)*b+Bl(n)+(i-
1)*Cl(n);
            y0=step;
            P(x0+1:x0+Cl(n),y0+1:y0+Cl(n))=mw*eye(Cl(n));
        end
    end
end
% Sub-Diagonal sub-matrices
%     Transitions to level where Iw>0
%         Retailer 1
k=min(nQ(1),floor(s(1)/bsw));
L=zeros((s(1)+1)*k,s(1));
for z=1:k
```

```
    for i=1:s(1)+1-z*bsw
        L((s(1)+1)*(z-1)+i,i+z*bsw-1)=m(1);
    end
end
% Id=0 Tw=0
for Level=nQ(1)+1:NLw
    for j=1:Cl(n)/Cl(1)
        x=Bl(n)+(Level-1)*Cl(n)+(j-1)*Cl(1)+1;
        y=[...]
        P(x,y)=l(1);
        if k>0 & s(1)>0
            P(x+Q(1):x+Q(1)+k*(s(1)+1)-1, y-s(1)+1:y)=L;
        end
    end
end
% Id=0 Tw>0
for Lw=1:nQw
    for Level=nQ(1)+1:nsw
        for j=1:Cl(n)/Cl(1)
            x=Bl(n)+NLw*Cl(n)+(Lw-1)*b+Bl(n)+(Level-1)*Cl(n)+(j-
1)*Cl(1)+1;
            y=Bl(n)+NLw*Cl(n)+(Lw-1)*b+Bl(n)+(Level-nQ(1)-
1)*Cl(n)+(j-1)*Cl(1)+(Q(1)+(nQ(1)-1)*(s(1)+1)+s(1))+1;
            P(x,y)=l(1);
            if k>0 & s(1)>0
                P(x+Q(1):x+Q(1)+k*(s(1)+1)-1,y-s(1)+1:y)=L;
            end
        end
    end
end
% Id>0
for Ld=1:NLd
 %   Tw=0
    for Level=max(nQ(1),nsw)+1:NLw
        for j=1:Cl(n)/Cl(1)
            x=L0+(Ld-1)*L1+(Level-nsw-1)*Cl(n)+(j-1)*Cl(1)+1;
            if Level-nQ(1)>nsw
                step=L0+(Ld-1)*L1+(Level-nQ(1)-nsw-1)*Cl(n);
            else
                Id=Ld*bsd;
                if Id>Qw
                    step=L0+((Id-Qw)/bsd-1)*L1  ;
                    step=step+(NLw-nsw)*Cl(n)+ ((Qw/bsd)-1)*b;
                    Iw0=Level-nQ(1);
                    step=step+Bl(n)+(Iw0-1)*Cl(n);
                else
                    step=0;
                    step=step+Bl(n)+NLw*Cl(n)+((Id/bsd)-1)*b;
                    Iw0=Level-nQ(1);
                    step=step+Bl(n)+(Iw0-1)*Cl(n);
                end
            end
            y=step+(j-1)*Cl(1)+(Q(1)+(nQ(1)-1)*(s(1)+1)+s(1))+1;
            P(x,y)=l(1);
            if k>0 & s(1)>0
                P(x+Q(1):x+Q(1)+k*(s(1)+1)-1, y-s(1)+1:y)=L;
            end
        end
    end
% Tw>0
    for Lw=1:nQw
```

```
          for Level=nQ(1)+1:nsw
              for j=1:Cl(n)/Cl(1)
                  x=L0+(Ld-1)*L1+(NLw-nsw)*Cl(n)+(Lw-1)*b+Bl(n)+(Level-
1)*Cl(n)+(j-1)*Cl(1)+1;
                  y=L0+(Ld-1)*L1+(NLw-nsw)*Cl(n)+(Lw-1)*b+Bl(n)+(Level-
nQ(1)-1)*Cl(n)+(j-1)*Cl(1)+(Q(1)+(nQ(1)-1)*(s(1)+1)+s(1))+1;
                  P(x,y)=l(1);
                      if k>0 & s(1)>0
                          P(x+Q(1):x+Q(1)+k*(s(1)+1)-1, y-s(1)+1:y)=L;
                      end
              end
          end
      end
end
%   Retailers 2 to n
for r=2:n
    k=min(nQ(r),floor(s(r)/bsw));
    L=esubMp(s,m,Cl,bsw,r,k);
%Id=0 Tw=0
    for Level=nQ(r)+1:NLw
        for j=1:Cl(n)/Cl(r)
            x=Bl(n)+(Level-1)*Cl(n)+(j-1)*Cl(r);
            y=Bl(n)+(Level-nQ(r)-1)*Cl(n)+(j-1)*Cl(r)+(Q(r)+(nQ(r)-
1)*(s(r)+1)+s(r))*Cl(r-1);
            P(x+1:x+Cl(r-1),y+1:y+Cl(r-1))=P(x+1:x+Cl(r-
1),y+1:y+Cl(r-1))+l(r)*eye(Cl(r-1));
            if (k>0) & (s(r)>0)
                x=x+Q(r)*Cl(r-1);
                y=y+Cl(r-1)-s(r)*Cl(r-1);
                P(x+1:x+(k*(s(r)+1))*Cl(r-1),y+1:y+s(r)*Cl(r-
1))=P(x+1:x+(k*(s(r)+1))*Cl(r-1),y+1:y+s(r)*Cl(r-1))+L;
            end
        end
     end
%Id=0, Tw>0
    for Lw=1:nQw
        for Level=nQ(r)+1:nsw
            for j=1:Cl(n)/Cl(r)
                x=Bl(n)+NLw*Cl(n)+(Lw-1)*b+Bl(n)+(Level-1)*Cl(n)+(j-
1)*Cl(r);
                y=Bl(n)+NLw*Cl(n)+(Lw-1)*b+Bl(n)+(Level-nQ(r)-
1)*Cl(n)+(j-1)*Cl(r)+(Q(r)+(nQ(r)-1)*(s(r)+1)+s(r))*Cl(r-1);
                P(x+1:x+Cl(r-1),y+1:y+Cl(r-1))=P(x+1:x+Cl(r-
1),y+1:y+Cl(r-1))+l(r)*eye(Cl(r-1));
                if (k>0) & (s(r)>0)
                    x=x+Q(r)*Cl(r-1);
                    y=y+Cl(r-1)-s(r)*Cl(r-1);
                    P(x+1:x+(k*(s(r)+1))*Cl(r-1),y+1:y+s(r)*Cl(r-
1))=P(x+1:x+(k*(s(r)+1))*Cl(r-1),y+1:y+s(r)*Cl(r-1))+L;
                end
            end
        end
    end
% Id>0
    for Ld=1:NLd
%      Tw=0
        for Level=max(nQ(r),nsw)+1:NLw
            for j=1:Cl(n)/Cl(r)
                x=L0+(Ld-1)*L1+(Level-nsw-1)*Cl(n)+(j-1)*Cl(r);
                if (Level-nQ(r))*bsw>sw
                    step=L0+(Ld-1)*L1+(Level-nQ(r)-nsw-1)*Cl(n);
```

```
                else
                    Id=Ld*bsd;
                    if Id>Qw
                        step=L0+((Id-Qw)/bsd-1)*L1 ;
                        step=step+(NLw-nsw)*Cl(n)+ ((Qw/bsd)-1)*b;
                        step=step+Bl(n)+(Level-nQ(r)-1)*Cl(n);
                    else
                        step=0;
                        step=step+Bl(n)+NLw*Cl(n)+((Id/bsd)-1)*b;
                        step=step+Bl(n)+(Level-nQ(r)-1)*Cl(n);
                    end
                end
                y=step+(j-1)*Cl(r)+(Q(r)+(nQ(r)-
1)*(s(r)+1)+s(r))*Cl(r-1);
                P(x+1:x+Cl(r-1),y+1:y+Cl(r-1))=P(x+1:x+Cl(r-
1),y+1:y+Cl(r-1))+l(r)*eye(Cl(r-1));
                if (k>0) & (s(r)>0)
                    x=x+Q(r)*Cl(r-1);
                    y=y+Cl(r-1)-s(r)*Cl(r-1);
                    P(x+1:x+(k*(s(r)+1))*Cl(r-1),y+1:y+s(r)*Cl(r-
1))=P(x+1:x+(k*(s(r)+1))*Cl(r-1),y+1:y+s(r)*Cl(r-1))+L;
                end
            end
        end
% Tw>0
        for Lw=1:nQw
            for Level=nQ(r)+1:nsw
                for j=1:Cl(n)/Cl(r)
                    x=L0+(Ld-1)*L1+(NLw-nsw)*Cl(n)+(Lw-
1)*b+Bl(n)+(Level-1)*Cl(n)+(j-1)*Cl(r);
                    y=L0+(Ld-1)*L1+(NLw-nsw)*Cl(n)+(Lw-
1)*b+Bl(n)+(Level-nQ(r)-1)*Cl(n)+(j-1)*Cl(r)+(Q(r)+(nQ(r)-
1)*(s(r)+1)+s(r))*Cl(r-1);
                    P(x+1:x+Cl(r-1),y+1:y+Cl(r-1))=P(x+1:x+Cl(r-
1),y+1:y+Cl(r-1))+l(r)*eye(Cl(r-1));
                    if (k>0) & (s(r)>0)
                        x=x+Q(r)*Cl(r-1);
                        y=y+Cl(r-1)-s(r)*Cl(r-1);
                        P(x+1:x+(k*(s(r)+1))*Cl(r-1),y+1:y+s(r)*Cl(r-
1))=P(x+1:x+(k*(s(r)+1))*Cl(r-1),y+1:y+s(r)*Cl(r-1))+L;
                    end
                end
            end
        end
    end
end
%         Transitions to level where Iw=0
sum=0;
sumQ(n)=sum;
for j=1:n-1
    sum=sum+(s(n-j+1)+1)*Bl(n-j);
    sumQ(n-j)=sum;
end
% Retailer 1
k=min(nQ(1),floor(s(1)/bsw));
L=zeros((s(1)+1)*k,s(1));
for z=1:k
    for i=1:s(1)+1-z*bsw
        L((s(1)+1)*(z-1)+i,i+z*bsw-1)=m(1);
    end
end
```

```
% Id=0, Tw=0
for Level=1:nQ(1)
    for i=1:Cl(n)/Cl(1)
        x=Bl(n)+(Level-1)*Cl(n)+(i-1)*Cl(1)+1;
        sum=0;
        for j=2:n-1
            p=ceil((i*Cl(1))/Cl(n-j+1));
            sum= sum+(p-1)*(s(n-j+1)+1)*Bl(n-j);
        end
        y=sumQ(1)+(s(1)+Q(1)+1)+(min(Level,nQ(1))-
1)*(s(1)+1)+s(1)+1+(i-1)*Bl(1)+sum;
        P(x,y)=l(1);
        if k>0 & s(1)>0
            P(x+Q(1):x+Q(1)+k*(s(1)+1)-1,y-
s(1)+1:y)=P(x+Q(1):x+Q(1)+k*(s(1)+1)-1,y-s(1)+1:y)+L;
        end
    end
end
% Id=0, Tw>0
for Lw=1:nQw
    for Level=1:min(nQ(1),nsw)
        for i=1:Cl(n)/Cl(1)
            x=Bl(n)+NLw*Cl(n)+(Lw-1)*b+Bl(n)+(Level-1)*Cl(n)+(i-
1)*Cl(1)+1;
            sum=0;
            for j=2:n-1
                p=ceil((i*Cl(1))/Cl(n-j+1));
                sum= sum+(p-1)*(s(n-j+1)+1)*Bl(n-j);
            end
            y=Bl(n)+NLw*Cl(n)+(Lw-1)*b+
sumQ(1)+(s(1)+Q(1)+1)+(min(Level,nQ(1))-1)*(s(1)+1)+s(1)+1+(i-
1)*Bl(1)+sum;
            P(x,y)=l(1);
            if k>0 & s(1)>0
                P(x+Q(1):x+Q(1)+k*(s(1)+1)-1,y-
s(1)+1:y)=P(x+Q(1):x+Q(1)+k*(s(1)+1)-1,y-s(1)+1:y)+L;
            end
        end
    end
end
% Id>0
for Ld=1:NLd
%     Tw=0
    for Level=nsw+1:nQ(1)
        for i=1:Cl(n)/Cl(1)
            x=L0+(Ld-1)*L1+(Level-nsw-1)*Cl(n)+(i-1)*Cl(1);
            sum=0;
            for j=2:n-1
                p=ceil((i*Cl(1))/Cl(n-j+1));
                sum= sum+(p-1)*(s(n-j+1)+1)*Bl(n-j);
            end
            Id=Ld*bsd;
            if Id>Qw
                step=L0+((Id-Qw)/bsd-1)*L1  ;
                step=step+(NLw-nsw)*Cl(n)+ ((Qw/bsd)-1)*b;
            else
                step=0;
                step=step+Bl(n)+NLw*Cl(n)+((Id/bsd)-1)*b;
            end
            y=step+ sumQ(1)+(s(1)+Q(1)+1)+(Level-
1)*(s(1)+1)+s(1)+1+(i-1)*Bl(1)+sum;
```

```
            P(x+1,y)=l(1);
            if k>0 & s(1)>0
                P(x+1+Q(1):x+1+Q(1)+k*(s(1)+1)-1,y-
s(1)+1:y)=P(x+1+Q(1):x++1+Q(1)+k*(s(1)+1)-1,y-s(1)+1:y)+L;
            end
        end
    end
%     Tw>0
    for Lw=1:nQw
        for Level=1:min(nQ(1),nsw)
            for i=1:Cl(n)/Cl(1)
                x=L0+(Ld-1)*L1+(NLw-nsw)*Cl(n)+(Lw-1)*b+Bl(n)+(Level-
1)*Cl(n)+(i-1)*Cl(1)+1;
                sum=0;
                for j=2:n-1
                    p=ceil((i*Cl(1))/Cl(n-j+1));
                    sum= sum+(p-1)*(s(n-j+1)+1)*Bl(n-j);
                end
                y=L0+(Ld-1)*L1+(NLw-nsw)*Cl(n)+(Lw-
1)*b+sumQ(1)+(s(1)+Q(1)+1)+(Level-1)*(s(1)+1)+s(1)+1+(i-1)*Bl(1)+sum;
                P(x,y)=l(1);
                if k>0 & s(1)>0
                    P(x+Q(1):x+Q(1)+k*(s(1)+1)-1,y-
s(1)+1:y)=P(x+Q(1):x+Q(1)+k*(s(1)+1)-1,y-s(1)+1:y)+L;
                end
            end
        end
    end
end
%               --- Retailers 2 to n ---
for r=2:n
    k=min(nQ(r),floor(s(r)/bsw));
    L=esubLz(s,l,Cl,Bl,r);
    M=esubMz(s,m,Cl,Bl,r,k,bsw);
% Id=0, Tw=0
    for Level=1:nQ(r)
        for z=1:Cl(n)/Cl(r)
            x=Bl(n)+(Level-1)*Cl(n)+(z-1)*Cl(r);
            sum=0;
            for i=r+1:n-1
                p=ceil((z*Cl(r))/Cl(n-i+r));
                sum= sum+(p-1)*(s(n-i+r)+1)*Bl(n-i+r-1);
            end
            y=[...]
            P(x+1:x+Cl(r-1),y+1:y+Bl(r-1))= P(x+1:x+Cl(r-
1),y+1:y+Bl(r-1))+L;
            if (s(r)+1)*k*Cl(r-1)>0 & s(r)*Cl(r-1)>0
                P(x+Q(r)*Cl(r-1)+1:x+Q(r)*Cl(r-1)+k*(s(r)+1)*Cl(r-
1),y-s(r)*Bl(r-1)+Bl(r-1)+1:y+Bl(r-1))= P(x+Q(r)*Cl(r-
1)+1:x+Q(r)*Cl(r-1)+k*(s(r)+1)*Cl(r-1),y-s(r)*Bl(r-1)+Bl(r-
1)+1:y+Bl(r-1))+M;
            end
        end
    end
% Id=0, Tw>0
    for Lw=1:nQw
        for Level=1:min(nQ(r),nsw)
            for z=1:Cl(n)/Cl(r)
               x=Bl(n)+NLw*Cl(n)+(Lw-1)*b+ Bl(n)+(Level-1)*Cl(n)+(z-
1)*Cl(r);
               sum=0;
```

```
                for i=r+1:n-1
                    p=ceil((z*Cl(r))/Cl(n-i+r));
                    sum= sum+(p-1)*(s(n-i+r)+1)*Bl(n-i+r-1);
                end
                y=[...]
                P(x+1:x+Cl(r-1),y+1:y+Bl(r-1))= P(x+1:x+Cl(r-
1),y+1:y+Bl(r-1))+L;
                if (s(r)+1)*k*Cl(r-1)>0 & s(r)*Cl(r-1)>0
                    P(x+Q(r)*Cl(r-1)+1:x+Q(r)*Cl(r-1)+k*(s(r)+1)*Cl(r-
1),y-s(r)*Bl(r-1)+Bl(r-1)+1:y+Bl(r-1))= P(x+Q(r)*Cl(r-
1)+1:x+Q(r)*Cl(r-1)+k*(s(r)+1)*Cl(r-1),y-s(r)*Bl(r-1)+Bl(r-
1)+1:y+Bl(r-1))+M;
                end
            end
        end
    end
% Id>0
    for Ld=1:NLd
% Id>0, Tw=0
        for Level=nsw+1:nQ(r)
            for z=1:Cl(n)/Cl(r)
                x=L0+(Ld-1)*L1+(Level-nsw-1)*Cl(n)+(z-1)*Cl(r);
                sum=0;
                for i=r+1:n-1
                    p=ceil((z*Cl(r))/Cl(n-i+r));
                    sum= sum+(p-1)*(s(n-i+r)+1)*Bl(n-i+r-1);
                end
                Id=Ld*bsd;
                if Id>Qw
                    step=L0+((Id-Qw)/bsd-1)*L1  ;
                    step=step+(NLw-nsw)*Cl(n)+ ((Qw/bsd)-1)*b;
                else
                    step=0;
                    step=step+Bl(n)+NLw*Cl(n)+((Id/bsd)-1)*b;
                end
                y= step+sumQ(r)+(s(r)+Q(r)+1)*Bl(r-
1)+(min(Level,nQ(r))-1)*(s(r)+1)*Bl(r-1)+ s(r)*Bl(r-1)+(z-
1)*Bl(r)+sum;
                P(x+1:x+Cl(r-1),y+1:y+Bl(r-1))= P(x+1:x+Cl(r-
1),y+1:y+Bl(r-1))+L;
                if (s(r)+1)*k*Cl(r-1)>0 & s(r)*Cl(r-1)>0
                    P(x+Q(r)*Cl(r-1)+1:x+Q(r)*Cl(r-
1)+k*(s(r)+1)*Cl(r-1),y-s(r)*Bl(r-1)+Bl(r-1)+1:y+Bl(r-1))=
P(x+Q(r)*Cl(r-1)+1:x+Q(r)*Cl(r-1)+k*(s(r)+1)*Cl(r-1),y-s(r)*Bl(r-
1)+Bl(r-1)+1:y+Bl(r-1))+M;
                end
            end
        end
% Id>0, Tw>0
        for Lw=1:nQw
            for Level=1:min(nsw,nQ(r))
                for z=1:Cl(n)/Cl(r)
                    x= L0+(Ld-1)*L1+(NLw-nsw)*Cl(n) +(Lw-
1)*b+Bl(n)+(Level-1)*Cl(n)+(z-1)*Cl(r);
                    sum=0;
                    for i=r+1:n-1
                        p=ceil((z*Cl(r))/Cl(n-i+r));
                        sum= sum+(p-1)*(s(n-i+r)+1)*Bl(n-i+r-1);
                    end
                    y=[...]
```

```
                    P(x+1:x+Cl(r-1),y+1:y+Bl(r-1))= P(x+1:x+Cl(r-
1),y+1:y+Bl(r-1))+L;
                    if (s(r)+1)*k*Cl(r-1)>0 & s(r)*Cl(r-1)>0
                        P(x+Q(r)*Cl(r-1)+1:x+Q(r)*Cl(r-
1)+k*(s(r)+1)*Cl(r-1),y-s(r)*Bl(r-1)+Bl(r-1)+1:y+Bl(r-1))=
P(x+Q(r)*Cl(r-1)+1:x+Q(r)*Cl(r-1)+k*(s(r)+1)*Cl(r-1),y-s(r)*Bl(r-
1)+Bl(r-1)+1:y+Bl(r-1))+M;
                    end
                end
            end
        end
    end
end
%   ----- Calculation of Stationary Probabilities Vector X ----------
-
R=P';
for i=1:ns
    R(ns,i)=1;
end
Y=zeros(ns,1);
Y(ns,1)=1;
X=linsolve(R,Y);
%  ---------------- Performance Measures ------------------
% WIPd: Average inventory on hand at the Distribution Centre
% utilizationd: utilization of the resource for transportation
towards the
wholesaler (P(Tw>0))
WIPd=0;
utilizationd=0;
for i=Bl(n)+NLw*Cl(n)+1:L0
    utilizationd=utilizationd+X(i);
end
for i=1:NLd
    sum=0;
    for j=(NLw-nsw)*Cl(n)+1:L1
        utilizationd=utilizationd+X(L0+(i-1)*L1+j);
    end
    for j=1:L1
        sum=sum+X(L0+(i-1)*L1+j);
    end
    WIPd=WIPd+i*bsd*sum;
end
% Average inventory on hand at the Wholesaler - WIPw
% Iw(i): the probability of Iw=i
Iw=0;
for Level=1:NLw
    sum=0;
    for i=1:Cl(n)
        sum=sum+X(Bl(n)+(Level-1)*Cl(n)+i);
    end
    Iw(Level)=sum;
end
for Lw=1:nQw
    for Level=1:nsw
        sum=0;
        for i=1:Cl(n)
            sum=sum+X(Bl(n)+NLw*Cl(n)+(Lw-1)*b+Bl(n)+(Level-
1)*Cl(n)+i);
        end
        Iw(Level)=Iw(Level)+sum;
    end
```

```
end
for Ld=1:NLd
    for Level=1:NLw-nsw
        sum=0;
        for i=1:Cl(n)
            sum=sum+X(L0+(Ld-1)*L1+(Level-1)*Cl(n)+i);
        end
        Iw(Level+nsw)=Iw(Level+nsw)+sum;
    end
    for Lw=1:nQw
        for Level=1:nsw
            sum=0;
            for i=1:Cl(n)
                sum=sum+X(L0+(Ld-1)*L1+(NLw-nsw)*Cl(n)+(Lw-
1)*b+Bl(n)+(Level-1)*Cl(n)+i);
            end
            Iw(Level)=Iw(Level)+sum;
        end
    end
end
WIPw=0;
for i=1:NLw
    WIPw=WIPw+i*bsw*Iw(i);
end
% WIP(i): Average inventory on hand at Retailer i,
% SO(i): stockout probability for retailer i
% utilization(i): utilization for transportation towards retailer i
% SOw: stockout probability for the wholesaler
%    Bl- block
% Id=0,Tw=0
lp=0;
[WIP,SO,utilization,SOw]=perfB(n,Bl,s,Q,nQ,X,lp);
% Id=0, Tw>0
for Lw=1:nQw
    lp=Bl(n)+NLw*Cl(n)+(Lw-1)*b;
    [p1,p2,p3,p4]=perfB(n,Bl,s,Q,nQ,X,lp);
    WIP=WIP+p1;
    SO=SO+p2;
    utilization=utilization+p3;
    SOw=SOw+p4;
end
%Id>0, Tw>0
for Ld=1:NLd
    for Lw=1:nQw
        lp=L0+(Ld-1)*L1+(NLw-nsw)*Cl(n)+(Lw-1)*b;
        [p1,p2,p3,p4]= perfB(n,Bl,s,Q,nQ,X,lp);
        WIP=WIP+p1;
        SO=SO+p2;
        utilization=utilization+p3;
        SOw=SOw+p4;
    end
end
%    Cl - Block
% Retailer 1
% Id=0, Tw=0
for Level=1:NLw
    lp=Bl(n)+(Level-1)*Cl(n);
    [p1,p2,p3]= perfC(n,Cl,s,Q,nQ,X,lp);
    WIP=WIP+p1;
    SO=SO+p2;
    utilization=utilization+p3;
```

```
end
% Id=0, Tw>0
for Lw=1:nQw
    for Level=1:nsw
        lp= Bl(n)+NLw*Cl(n)+(Lw-1)*b+Bl(n)+(Level-1)*Cl(n);
        [p1,p2,p3]= perfC(n,Cl,s,Q,nQ,X,lp);
        WIP=WIP+p1;
        SO=SO+p2;
        utilization=utilization+p3;
    end
end
% Id>0
for Ld=1:NLd
%      Tw=0
    for Level=1:NLw-nsw
        lp=L0+(Ld-1)*L1+(Level-1)*Cl(n);
        [p1,p2,p3]= perfC(n,Cl,s,Q,nQ,X,lp);
        WIP=WIP+p1;
        SO=SO+p2;
        utilization=utilization+p3;
    end
%     Tw>0
    for Lw=1:nQw
        for Level=1:nsw
            lp=L0+(Ld-1)*L1+(NLw-nsw)*Cl(n)+(Lw-1)*b+Bl(n)+(Level-
1)*Cl(n);
            [p1,p2,p3]= perfC(n,Cl,s,Q,nQ,X,lp);
            WIP=WIP+p1;
            SO=SO+p2;
            utilization=utilization+p3;
        end
    end
end
% FR(i): Fill rate at Retailer i
% Thr(i): Throughput at retailer i
% WIPtr(i): Average inventory in transit towards retailer i
% ARO(i): Average replenishment orders sent to retailer(i)
% WIPtrw: Average inventory in transit towards the Wholesaler
% AROd: Average number of units in replenishment orders sent to the
Wholesaler
% SOd: Stockout probability for the DC
Throughput=0;
for i=1:n
    FR(i)=1-SO(i);
    Thr(i)=l(i)*FR(i);
    WIPtr(i)=Thr(i)/m(i);
    ARO(i)=WIPtr(i)/utilization(i);
    Throughput=Throughput+Thr(i);
end
WIPtrw=Throughput/mw;
AROw=WIPtrw/utilizationd;
SOd=0;
for i=1:Bl(n)+nsw*Cl(n)
    SOd=SOd+X(i);
end
```

Defined functions

eIwz(n,s,Q,nQ,m,l,sw,Qw,md,mw,bsw)

```
function [ B1,Bl ] = Iwz( n,s,Q,nQ,m,l,sw,Qw,md,mw,bsw )
```

```
% Recursive process to construct the diagonal sub-matrix for all
retailers and for Iw=0
% Level 1 (seed)
z=s(1)+Q(1)+1;
B1=zeros(s(1)+Q(1)+1+nQ(1)*(s(1)+1));
B1(1,1)=-md-mw;
for i=2:z
    B1(i,i)=-md-mw-l(1);
    B1(i,i-1)=l(1);
end
for j=1:nQ(1)
    B1(z+(j-1)*(s(1)+1)+1,z+(j-1)*(s(1)+1)+1)=-md-mw-m(1);
    for i=2:s(1)+1
        B1(z+(j-1)*(s(1)+1)+i,z+(j-1)*(s(1)+1)+i)=-md-mw-m(1)-l(1);
        B1(z+(j-1)*(s(1)+1)+i,z+(j-1)*(s(1)+1)+i-1)=l(1);
    end
end
for j=1:nQ(1)
    for i=1:s(1)+1
        B1(z+(j-1)*(s(1)+1)+i,j*bsw+i)=m(1);
    end
end
%Bl:A vector recording the dimension of each level submatrix
Bl(1)=length(B1);
% Recursive construction of higher levels
for i=2:n
    B=B1;
    z=s(i)+Q(i)+1;
    B1=zeros((z+nQ(i)*(s(i)+1))*Bl(i-1));
    B1(1:Bl(i-1),1:Bl(i-1))=B;
    L=l(i)*eye(Bl(i-1));
    M=m(i)*eye(Bl(i-1));
    for k=2:z
        B1(1+(k-1)*Bl(i-1):k*Bl(i-1),1+(k-1)*Bl(i-1):k*Bl(i-1))=B-L;
        B1((k-1)*Bl(i-1)+1:k*Bl(i-1),(k-2)*Bl(i-1)+1:(k-1)*Bl(i-
1))=L;
    end
    for j=1:nQ(i)
        B1((z+(j-1)*(s(i)+1))*Bl(i-1)+1:(z+(j-1)*(s(i)+1)+1)*Bl(i-
1),(z+(j-1)*(s(i)+1))*Bl(i-1)+1:(z+(j-1)*(s(i)+1)+1)*Bl(i-1))=B-M;
        for k=1:s(i)
            B1((z+(j-1)*(s(i)+1)+k)*Bl(i-1)+1:(z+(j-
1)*(s(i)+1)+k+1)*Bl(i-1),(z+(j-1)*(s(i)+1)+k)*Bl(i-1)+1:(z+(j-
1)*(s(i)+1)+k+1)*Bl(i-1))=B-M-L;
            B1((z+(j-1)*(s(i)+1)+k)*Bl(i-1)+1:(z+(j-
1)*(s(i)+1)+k+1)*Bl(i-1),(z+(j-1)*(s(i)+1)+k-1)*Bl(i-1)+1:(z+(j-
1)*(s(i)+1)+k)*Bl(i-1))=L;
        end
    end
    for j=1:nQ(i)
        for k=1:s(i)+1
            B1((z+(j-1)*(s(i)+1)+k-1)*Bl(i-1)+1:(z+(j-
1)*(s(i)+1)+k)*Bl(i-1),(j*bsw+k-1)*Bl(i-1)+1:(j*bsw+k)*Bl(i-1))=M;
        end
    end
    Bl(i)=length(B1);
end
end
```

eIwp(n,s,Q,nQ,m,l,md,mw,bsw)

```
function [C1,Cl] = eIwp(n,s,Q,nQ,m,l,md,mw,bsw )
```

```
% Recursive process to construct the diagonal sub-matrix for all
retailers
% for Iw>0
% Level 1 (seed)
suml=0;
for i=1:n
    suml=suml+l(i);
end
C1=zeros(Q(1)+nQ(1)*(s(1)+1));
for i=1:Q(1)
    C1(i,i)=-md-mw-suml;
end
for i=2:Q(1)
    C1(i,i-1)=l(1);
end
for j=1:nQ(1)
    C1(Q(1)+(j-1)*(s(1)+1)+1,Q(1)+(j-1)*(s(1)+1)+1)=-md-mw-suml-
m(1)+l(1);
    for i=2:s(1)+1
        C1(Q(1)+(j-1)*(s(1)+1)+i,Q(1)+(j-1)*(s(1)+1)+i)=-md-mw-suml-
m(1);
        C1(Q(1)+(j-1)*(s(1)+1)+i,Q(1)+(j-1)*(s(1)+1)+i-1)=l(1);
    end
end
for j=1:nQ(1)
    x0=max(s(1)+1-j*bsw,0);
    y0=max(j*bsw-s(1)-1,0);
    for i=1:s(1)+1-x0
        C1(Q(1)+(j-1)*(s(1)+1)+x0+i,y0+i)=m(1);
    end
end
%Cl:A vector recording the dimension of each stage submatrix
Cl(1)=length(C1);
% Recursive construction of higher levels
for i=2:n
  C=C1;
    C1=zeros((Q(i)+nQ(i)*(s(i)+1))*Cl(i-1));
    C1(1:Cl(i-1),1:Cl(i-1))=C;
    L=l(i)*eye(Cl(i-1));
    M=m(i)*eye(Cl(i-1));
    for k=2:Q(i)
        […]
        […]
    end
    for j=1:nQ(i)
        C1((Q(i)+(j-1)*(s(i)+1))*Cl(i-1)+1:(Q(i)+(j-
1)*(s(i)+1)+1)*Cl(i-1),(Q(i)+(j-1)*(s(i)+1))*Cl(i-1)+1:(Q(i)+(j-
1)*(s(i)+1)+1)*Cl(i-1))=C+L-M;
        for k=1:s(i)
            C1((Q(i)+(j-1)*(s(i)+1)+k)*Cl(i-1)+1:(Q(i)+(j-
1)*(s(i)+1)+k+1)*Cl(i-1),(Q(i)+(j-1)*(s(i)+1)+k)*Cl(i-1)+1:(Q(i)+(j-
1)*(s(i)+1)+k+1)*Cl(i-1))=C-M;
            C1((Q(i)+(j-1)*(s(i)+1)+k)*Cl(i-1)+1:(Q(i)+(j-
1)*(s(i)+1)+k+1)*Cl(i-1),(Q(i)+(j-1)*(s(i)+1)+k-1)*Cl(i-
1)+1:(Q(i)+(j-1)*(s(i)+1)+k)*Cl(i-1))=L;
        end
    end
    for j=1:nQ(i)
        x0=max(s(i)+1-j*bsw,0);
        y0=max(j*bsw-s(i)-1,0);
        for k=1:s(i)+1-x0
```

```
            C1((Q(i)+(j-1)*(s(i)+1)+x0+k-1)*Cl(i-1)+1:(Q(i)+(j-
1)*(s(i)+1)+x0+k)*Cl(i-1),(y0+k-1)*Cl(i-1)+1:(y0+k)*Cl(i-1))=M;
        end
    end
    Cl(i)=length(C1);
end
end
```

indexDemand(Tw,Bl,s,Q,n)

```
function [inD] = indexDemand(Tw,Bl,s,Q,n)
%{
Function to construct a table with elements indicating to which of
the
retailers an incoming order at the wholesaler will be forwarded.
inA(i,j): A table with elements indicating which retailers ask for a
replenishment order in each state
inA(i,j)=1 if there is demand for a replenishment order,I(j)<=s(j)
inA(i,j)=0 if there is no demand for replenishment order, I(j)>s(j)
or T(j)>0
%}
inA=zeros(Bl(n),n);
% Retailer 1
block1=zeros(Bl(1),1);
for i=1:s(1)+1
    block1(i,1)=1;
end
for j=1:Bl(n)/Bl(1)
    inA((j-1)*Bl(1)+1:j*Bl(1),1:1)=block1;
end
% Recursive process for the rest of the retailers
for j=2:n
    block1=zeros(Bl(j),1);
    for i=1:(s(j)+1)*Bl(j-1)
        block1(i,1)=1;
    end
    for i=1:Bl(n)/Bl(j)
    inA((i-1)*Bl(j)+1:i*Bl(j),j:j)=block1;
    end
end
%{
inB(i,j): indicates whether in the i state the j retailer will
receive
product from an incoming order at the wholesaler
inB(i,j)=1: the j retailer will receive product, inB(i,j)=0 the j
retailer
will not receive product. Retailers that don't ask for replenishment
may
also have an index of 1.
%}
inB=zeros(Bl(n),n+1);
for i=1:Bl(n)
    td=0;
    for j=1:n
        if Tw>td
            inB(i,n-j+1)=1;
            td=td+inA(i,n-j+1)*Q(n-j+1);
        end
    end
    if Tw>td
        inB(i,n+1)=Tw-td;
    end
```

```
end
%{
inD(i,j):A table with elements indicating to which of the retailers
an incoming order at the wholesaler will be forwarded. The last
column indicates if Qw exceeds total demand. inD(i,j)=1:if at state i
a replenishment order arrives at the wholesaler some product will be
forwarded to retailer j
%}
inD=inB;
for j=1:n
    for i=1:Bl(n)
        inD(i,j)=inA(i,j)*inB(i,j);
    end
end
```

esubMp(s,m,Cl,bsw,r,k)

```
function [L] = esubMp(s,m,Cl,bsw,r,k)
L=zeros((s(r)+1)*k*Cl(r-1),s(r)*Cl(r-1));
for z=1:k
    for i=1:(s(r)+1-z*bsw)*Cl(r-1)
        L((s(r)+1)*(z-1)*Cl(r-1)+i,i+(z*bsw-1)*Cl(r-1))=m(r);
    end
end
```

esubLz(s,l,Cl,Bl,r)

```
function [L] = esubLz(s,l,Cl,Bl,r)
L=zeros(Cl(1),Bl(1));
for i=1:Cl(1)
    L(i,s(1)+1+i)=l(r);
end
for i=3:r
    L2=zeros(Cl(i-1),Bl(i-1));
    for j=1:Cl(i-1)/Cl(i-2)
        L2((j-1)*Cl(i-2)+1:j*Cl(i-2),(s(i-1)+1)*Bl(i-2)+(j-1)*Bl(i-
2)+1:(s(i-1)+1)*Bl(i-2)+(j-1)*Bl(i-2)+Bl(i-2))=L;
    end
    L=L2;
end
end
```

esubMz(s,m,Cl,Bl,r,k,bsw)

```
function [M] = esubMz(s,m,Cl,Bl,r,k,bsw)
M1=zeros(Cl(1),Bl(1));
for i=1:Cl(1)
    M1(i,s(1)+1+i)=m(r);
end
for i=3:r
    M2=zeros(Cl(i-1),Bl(i-1));
    for j=1:Cl(i-1)/Cl(i-2)
        M2((j-1)*Cl(i-2)+1:j*Cl(i-2),(s(i-1)+1)*Bl(i-2)+(j-1)*Bl(i-
2)+1:(s(i-1)+1)*Bl(i-2)+(j-1)*Bl(i-2)+Bl(i-2))=M1;
    end
    M1=M2;
end
M=zeros((s(r)+1)*k*Cl(r-1),s(r)*Bl(r-1));
for i=1:k
    for j=1:s(r)+1-i*bsw
        x=(i-1)*(s(r)+1)*Cl(r-1)+(j-1)*Cl(r-1);
        y=(i*bsw-1)*Bl(r-1)+(j-1)*Bl(r-1);
        M(x+1:x+Cl(r-1),y+1:y+Bl(r-1))=M1;
```

```
    end
end
end
```

perfB(n,Bl,s,Q,nQ,X,lp)

```
function [WIP,SO,utilization,SOw] = perfB(n,Bl,s,Q,nQ,X,lp)
% Bl- block
%    Retailer 1
sum=0;
sum2=0;
sum3=0;
sum4=0;
for i=1:Bl(n)/Bl(1)
% states where T1=0
% Stockout probability
    sum2=sum2+X(lp+(i-1)*Bl(1)+1);
% WIP
    for j=1:s(1)+Q(1)
        sum=sum+j*X(lp+(i-1)*Bl(1)+1+j);
    end
% Wholesaler Stockout
    for j=1:s(1)+1
        sum4=sum4+X(lp+(i-1)*Bl(1)+j);
    end
% states where T1>0
    for j=1:nQ(1)
        sum2=sum2+X(lp+(i-1)*Bl(1)+s(1)+Q(1)+1+(j-1)*(s(1)+1)+1);
        for z=1:s(1)
            sum=sum+z*X(lp+(i-1)*Bl(1)+s(1)+Q(1)+1+(j-
1)*(s(1)+1)+z+1);
            sum3=sum3+X(lp+(i-1)*Bl(1)+s(1)+Q(1)+1+(j-1)*(s(1)+1)+z);
        end
        sum3=sum3+X(lp+(i-1)*Bl(1)+s(1)+Q(1)+1+(j-
1)*(s(1)+1)+s(1)+1);
    end
end
WIP(1)=sum;
SO(1)=sum2;
utilization(1)=sum3;
SOw(1)=sum4;
% Retailers 2:n
for r=2:n
    sum=0;
    sum2=0;
    sum3=0;
    sum4=0;
    for i=1:Bl(n)/Bl(r)
% states where Tr=0
        for j=1:Bl(r-1)
            sum2=sum2+X(lp+(i-1)*Bl(r)+j);
        end
        for j=1:s(r)+Q(r)
            for z=1:Bl(r-1)
                sum=sum+j*X(lp+(i-1)*Bl(r)+Bl(r-1)+(j-1)*Bl(r-1)+z);
            end
        end
        for j=1:s(r)+1
            for z=1:Bl(r-1)
                sum4=sum4+X(lp+(i-1)*Bl(r)+(j-1)*Bl(r-1)+z);
            end
        end
```

```
% states where Tr>0
        for j=1:nQ(r)
            for z=1:Bl(r-1)
                sum2=sum2+X(lp+(i-1)*Bl(r)+(s(r)+Q(r)+1)*Bl(r-1)+(j-
1)*(s(r)+1)*Bl(r-1)+z);
            end
            for z=1:s(r)
                for y=1:Bl(r-1)
                    sum=sum+z*X(lp+(i-1)*Bl(r)+(s(r)+Q(r)+1)*Bl(r-
1)+(j-1)*(s(r)+1)*Bl(r-1)+Bl(r-1)+(z-1)*Bl(r-1)+y);
                    sum3=sum3+X(lp+(i-1)*Bl(r)+(s(r)+Q(r)+1)*Bl(r-
1)+(j-1)*(s(r)+1)*Bl(r-1)+(z-1)*Bl(r-1)+y);
                end
            end
            for y=1:Bl(r-1)
                sum3=sum3+X(lp+(i-1)*Bl(r)+(s(r)+Q(r)+1)*Bl(r-1)+(j-
1)*(s(r)+1)*Bl(r-1)+s(r)*Bl(r-1)+y);
            end
        end
    end
    WIP(r)=sum;
    SO(r)=sum2;
    utilization(r)=sum3;
    SOw(r)=sum4;
end
end
```

perfC(n,Cl,s,Q,nQ,X,lp)

```
function [WIP,SO,utilization] = perfC(n,Cl,s,Q,nQ,X,lp)
% Cl- block , Iw>0
%     Retailer 1
sum=0;
sum2=0;
sum3=0;
sum4=0;
for j=1:Cl(n)/Cl(1)
% States where T1=0
% WIP
    for z=1:Q(1)
        sum=sum+(z+s(1))*X(lp+(j-1)*Cl(1)+z);
    end
% States where T1>0
    for z=1:nQ(1)
% Stockout
        sum2=sum2+X(lp+(j-1)*Cl(1)+Q(1)+(z-1)*(s(1)+1)+1);
% WIP
        for y=1:s(1)
            sum=sum+y*X(lp+(j-1)*Cl(1)+Q(1)+(z-1)*(s(1)+1)+1+y);
            sum3=sum3+X(lp+(j-1)*Cl(1)+Q(1)+(z-1)*(s(1)+1)+y);
        end
% Utilization
        sum3=sum3+X(lp+(j-1)*Cl(1)+Q(1)+(z-1)*(s(1)+1)+s(1)+1);
    end
end
WIP(1)=sum;
SO(1)=sum2;
utilization(1)=sum3;
% Retailers 2:n
```

[…]

7.13.2. Validation Data

$n=2$, $s_d=2$, $Q_d=2$, $s_w=2$, $Q_w=2$, $\mu_d=0.4$ $\mu_w=0.6$, $\mu_1=1$, $\mu_2=2$, $\lambda_1=1.5$, $\lambda_2=2.5$. Simulation parameters: One replication of 2000000 time units with a warm up period of 10000 time units.

	Input				Matlab						Arena					
	s_1	Q_1	s_2	Q_2	FR_1	FR_2	WIP_1	WIP_2	WIP_w	WIP_d	FR_1	FR_2	WIP_1	WIP_2	WIP_w	WIP_d
1	0	1	0	1	0.193127	0.163348	0.193127	0.163348	0.256287	0.89998	0.193	0.163	0.193	0.164	0.256	0.899
2	0	1	0	2	0.083568	0.229886	0.083568	0.332957	0.132107	0.878607	0.084	0.23	0.084	0.333	0.132	0.878
3	0	1	0	3	0.08003	0.232081	0.08003	0.345843	0.118211	0.876566	0.08	0.232	0.08	0.346	0.118	0.877
4	0	1	0	4	0.079406	0.232466	0.079406	0.348525	0.116221	0.876255	0.079	0.232	0.079	0.349	0.117	0.878
5	0	1	1	1	0.177719	0.173415	0.177719	0.207884	0.151391	0.878432	0.178	0.174	0.178	0.208	0.152	0.88
6	0	1	1	2	0.061219	0.243618	0.061219	0.383223	0.078053	0.869611	0.061	0.244	0.061	0.383	0.078	0.87
7	0	1	1	3	0.059187	0.244865	0.059187	0.392109	0.070614	0.86877	0.059	0.245	0.059	0.393	0.071	0.869
8	0	1	1	4	0.058867	0.245061	0.058867	0.393665	0.069663	0.868646	0.059	0.245	0.059	0.394	0.069	0.868
9	0	1	2	1	0.173994	0.175775	0.173994	0.225417	0.129102	0.874891	0.174	0.176	0.174	0.225	0.129	0.874
10	0	1	2	2	0.040367	0.256287	0.040367	0.446473	0.041908	0.86493	0.04	0.256	0.04	0.447	0.042	0.866
11	0	1	2	3	0.03933	0.256919	0.03933	0.451853	0.038229	0.864602	0.039	0.257	0.039	0.452	0.038	0.866
12	0	1	2	4	0.039172	0.257016	0.039172	0.452707	0.037783	0.864552	0.039	0.257	0.039	0.452	0.038	0.865
13	0	2	0	1	0.213996	0.151217	0.247753	0.151217	0.206376	0.88967	0.214	0.151	0.248	0.151	0.206	0.89
14	0	2	0	2	0.118443	0.209466	0.177665	0.314199	0.044928	0.864017	0.118	0.21	0.178	0.314	0.045	0.868
15	0	2	0	3	0.117793	0.209852	0.176241	0.31626	0.042787	0.864133	0.118	0.21	0.176	0.316	0.043	0.866
16	0	2	0	4	0.117204	0.210223	0.175806	0.319316	0.04002	0.863628	0.117	0.21	0.176	0.319	0.04	0.862
17	0	2	1	1	0.190722	0.16574	0.210105	0.196144	0.128905	0.874867	0.191	0.166	0.21	0.196	0.129	0.874
18	0	2	1	2	0.092613	0.225015	0.138919	0.354247	0.023635	0.862554	0.093	0.225	0.139	0.354	0.024	0.864
19	0	2	1	3	0.092321	0.225188	0.138257	0.355381	0.022722	0.862621	0.092	0.225	0.138	0.355	0.023	0.86
20	0	2	1	4	0.092043	0.225361	0.138064	0.357099	0.021396	0.862416	0.092	0.225	0.138	0.357	0.021	0.864
21	0	2	2	1	0.1856	0.168905	0.202315	0.21293	0.111991	0.872256	0.186	0.169	0.202	0.213	0.112	0.873
22	0	2	2	2	0.063791	0.24233	0.095686	0.413746	0.010185	0.861886	0.064	0.242	0.096	0.414	0.01	0.864
23	0	2	2	3	0.063688	0.242391	0.095441	0.414245	0.009861	0.861915	0.064	0.242	0.095	0.413	0.01	0.863
24	0	2	2	4	0.063575	0.242461	0.095363	0.415043	0.009326	0.861842	0.064	0.242	0.096	0.415	0.009	0.862
25	0	3	0	1	0.21947	0.148001	0.270145	0.148001	0.193766	0.887789	0.219	0.148	0.27	0.148	0.193	0.889
26	0	3	0	2	0.11896	0.209155	0.179621	0.313521	0.044034	0.864036	0.119	0.209	0.179	0.313	0.044	0.864
27	0	3	0	3	0.11834	0.209526	0.178324	0.315628	0.0418	0.864066	0.118	0.21	0.179	0.316	0.042	0.865
28	0	3	0	4	0.117671	0.209943	0.177514	0.318622	0.039374	0.863635	0.118	0.21	0.177	0.318	0.039	0.863
29	0	3	1	1	0.194058	0.163759	0.222803	0.193121	0.123467	0.874257	0.194	0.164	0.223	0.193	0.123	0.877
30	0	3	1	2	0.092852	0.224871	0.139787	0.35386	0.023287	0.862557	0.093	0.225	0.14	0.354	0.023	0.863
31	0	3	1	3	0.092579	0.225033	0.139195	0.355	0.022334	0.862598	0.092	0.225	0.139	0.355	0.022	0.861
32	0	3	1	4	0.092259	0.225232	0.13883	0.356716	0.021132	0.862417	0.092	0.225	0.139	0.357	0.021	0.864
33	0	3	2	1	0.188459	0.167204	0.212935	0.209853	0.107958	0.871837	0.188	0.167	0.213	0.21	0.107	0.87
34	0	3	2	2	0.063881	0.242276	0.096001	0.413551	0.010079	0.861887	0.064	0.242	0.096	0.413	0.01	0.862
35	0	3	2	3	0.063786	0.242332	0.095785	0.414049	0.009738	0.861908	0.064	0.242	0.096	0.414	0.01	0.862
36	0	3	2	4	0.063656	0.242412	0.095642	0.414857	0.009244	0.861843	0.064	0.242	0.096	0.414	0.009	0.861
37	0	4	0	1	0.221065	0.147055	0.278291	0.147055	0.191262	0.887496	0.221	0.147	0.278	0.147	0.191	0.888
38	0	4	0	2	0.119542	0.208812	0.182463	0.313218	0.042946	0.863869	0.12	0.209	0.182	0.313	0.043	0.864
39	0	4	0	3	0.118851	0.209222	0.180722	0.315264	0.041053	0.863967	0.119	0.209	0.18	0.316	0.041	0.865
40	0	4	0	4	0.118194	0.209633	0.179967	0.318251	0.038594	0.863529	0.118	0.21	0.18	0.318	0.039	0.863
41	0	4	1	1	0.194855	0.163285	0.226695	0.192399	0.122475	0.87416	0.195	0.163	0.226	0.192	0.123	0.874
42	0	4	1	2	0.093129	0.224707	0.141054	0.353609	0.022901	0.862509	0.093	0.225	0.141	0.353	0.023	0.864
43	0	4	1	3	0.09282	0.22489	0.140272	0.354746	0.022058	0.862567	0.093	0.225	0.14	0.355	0.022	0.862

$n=5$, $Q_d=2$, $s_w=0$, $Q_w=2$, $s_1=0$, $Q_1=1$, $s_2=1$, $Q_2=1$, $s_3=0$, $s_4=0$, $\mu_d=2.5$, $\mu_w=3.6$, $\mu_1=1$, $\mu_2=1.2$, $\mu_3=1.4$, $\mu_4=1.6$, $\mu_5=1.8$, $\lambda_1=0.5$, $\lambda_2=0.7$ $\lambda_3=0.9$, $\lambda_4=1.2$ $\lambda_5=1.5$. Simulation parameters: One replication of 2000000 time units with a warm up period of 10000 time units.

	Input					Matlab								Arena							
	sd	Q3	Q4	s5	Q5	FR1	FR3	FR5	WIP1	WIP3	WIP5	WIPw	WIPd	FR1	FR3	FR5	WIP1	WIP3	WIP5	WIPw	WIPd
1	2	1	1	0	1	0.624808	0.564852	0.498695	0.624808	0.564852	0.498695	0.774947	2.502065	0.624	0.566	0.498	0.625	0.565	0.498	0.775	2.501
2	2	1	1	0	2	0.614029	0.554899	0.616102	0.614029	0.554899	0.852735	0.751091	2.374281	0.614	0.554	0.616	0.614	0.555	0.852	0.751	2.375
3	2	1	1	1	1	0.613888	0.55577	0.664071	0.613888	0.55577	0.978973	0.699338	2.324867	0.614	0.557	0.665	0.615	0.557	0.980	0.700	2.326
4	2	1	1	1	2	0.601691	0.544131	0.763987	0.601691	0.544131	1.413704	0.687238	2.220037	0.602	0.545	0.764	0.601	0.544	1.414	0.687	2.219
5	2	1	1	2	1	0.608032	0.550997	0.743028	0.608032	0.550997	1.438498	0.661059	2.235519	0.608	0.551	0.743	0.608	0.550	1.437	0.661	2.235
6	2	1	1	2	2	0.590631	0.535105	0.859257	0.590631	0.535105	2.158342	0.643671	2.121687	0.591	0.535	0.860	0.591	0.534	2.160	0.643	2.120
7	2	1	1	3	1	0.604448	0.548039	0.788676	0.604448	0.548039	1.881012	0.638098	2.183086	0.605	0.548	0.789	0.605	0.548	1.883	0.638	2.185
8	2	1	1	3	2	0.584333	0.529974	0.9102	0.584333	0.529974	2.905608	0.618853	2.068146	0.584	0.530	0.910	0.584	0.530	2.903	0.618	2.066
9	2	1	2	0	1	0.618212	0.557228	0.49415	0.618212	0.557228	0.49415	0.76219	2.412259	0.618	0.558	0.494	0.618	0.558	0.494	0.762	2.412
10	2	1	2	0	2	0.599528	0.534376	0.619105	0.599528	0.534376	0.869763	0.730568	2.256926	0.600	0.535	0.620	0.599	0.535	0.870	0.730	2.257
11	2	1	2	1	1	0.607056	0.548827	0.658591	0.607056	0.548827	0.966824	0.682883	2.241327	0.607	0.549	0.659	0.607	0.549	0.966	0.683	2.240
12	2	1	2	1	2	0.583918	0.519581	0.764055	0.583918	0.519581	1.423786	0.662008	2.10525	0.584	0.520	0.764	0.584	0.520	1.424	0.662	2.104
13	2	1	2	2	1	0.601174	0.544309	0.737121	0.601174	0.544309	1.417056	0.644229	2.155814	0.601	0.545	0.738	0.602	0.545	1.418	0.644	2.158
14	2	1	2	2	2	0.569815	0.506953	0.859707	0.569815	0.506953	2.171839	0.615386	2.008125	0.570	0.507	0.860	0.569	0.507	2.172	0.615	2.008
15	2	1	2	3	1	0.597636	0.541551	0.782415	0.597636	0.541551	1.84812	0.621061	2.105918	0.598	0.541	0.783	0.598	0.541	1.850	0.621	2.106
16	2	1	2	3	2	0.56193	0.499936	0.910392	0.56193	0.499936	2.916437	0.589271	1.956005	0.562	0.499	0.911	0.562	0.499	2.915	0.589	1.955
17	2	2	1	0	1	0.619803	0.656089	0.495339	0.619803	0.867559	0.495339	0.76614	2.435666	0.620	0.656	0.496	0.620	0.867	0.496	0.767	2.436
18	2	2	1	0	2	0.607511	0.652933	0.618805	0.607511	0.875711	0.866056	0.742693	2.298482	0.608	0.653	0.619	0.608	0.875	0.867	0.743	2.298
19	2	2	1	1	1	0.608126	0.643769	0.659784	0.608126	0.844471	0.969499	0.686971	2.261684	0.608	0.644	0.660	0.608	0.844	0.970	0.688	2.261
20	2	2	1	1	2	0.594343	0.642027	0.764667	0.594343	0.860149	1.42312	0.676204	2.147094	0.594	0.642	0.764	0.594	0.859	1.422	0.675	2.144
21	2	2	1	2	1	0.602035	0.637371	0.738269	0.602035	0.832645	1.421398	0.648014	2.174518	0.603	0.638	0.738	0.602	0.832	1.421	0.647	2.173
22	2	2	1	2	2	0.582598	0.632124	0.860141	0.582598	0.845204	2.17152	0.630923	2.050102	0.583	0.632	0.860	0.582	0.845	2.170	0.631	2.050
23	2	2	1	3	1	0.598353	0.633418	0.783562	0.598353	0.825403	1.854502	0.624642	2.123497	0.599	0.633	0.784	0.599	0.825	1.855	0.625	2.124
24	2	2	1	3	2	0.576014	0.626276	0.910855	0.576014	0.836045	2.91857	0.60543	1.997912	0.577	0.627	0.910	0.577	0.837	2.917	0.606	1.998
25	2	2	2	0	1	0.611764	0.660481	0.489744	0.611764	0.895799	0.489744	0.752129	2.335431	0.611	0.660	0.490	0.612	0.897	0.489	0.752	2.337
26	2	2	2	0	2	0.591441	0.665804	0.622446	0.591441	0.952316	0.887219	0.717183	2.156142	0.591	0.666	0.622	0.592	0.952	0.887	0.716	2.156
27	2	2	2	1	1	0.600081	0.645899	0.653374	0.600081	0.864018	0.955298	0.668928	2.170709	0.601	0.646	0.653	0.600	0.863	0.955	0.670	2.170
28	2	2	2	1	2	0.575476	0.654518	0.764508	0.575476	0.940874	1.434779	0.644514	2.008415	0.575	0.654	0.765	0.575	0.939	1.435	0.644	2.008
29	2	2	2	2	1	0.594067	0.639149	0.731456	0.594067	0.85055	1.396884	0.629849	2.087928	0.595	0.640	0.731	0.594	0.851	1.396	0.630	2.089
30	4	1	1	0	1	0.632656	0.571812	0.505362	0.632656	0.571812	0.505362	0.820717	4.319622	0.633	0.572	0.505	0.633	0.572	0.505	0.820	4.320
31	4	1	1	0	2	0.624972	0.564183	0.621029	0.624972	0.564183	0.856281	0.809903	4.1328	0.625	0.563	0.621	0.625	0.564	0.856	0.811	4.133
32	4	1	1	1	1	0.624776	0.564959	0.674826	0.624776	0.564959	1.001661	0.75506	4.063401	0.625	0.565	0.675	0.625	0.565	1.003	0.755	4.063
33	4	1	1	1	2	0.615906	0.555873	0.771292	0.615906	0.555873	1.43196	0.755078	3.900098	0.616	0.556	0.771	0.616	0.556	1.432	0.756	3.904
34	4	1	1	2	1	0.620445	0.561271	0.756289	0.620445	0.561271	1.485254	0.721639	3.927329	0.621	0.561	0.756	0.621	0.561	1.485	0.721	3.924
35	4	1	1	2	2	0.607955	0.548987	0.865604	0.607955	0.548987	2.188235	0.717654	3.74542	0.608	0.549	0.866	0.608	0.549	2.188	0.718	3.744
36	4	1	1	3	1	0.617725	0.558944	0.803642	0.617725	0.558944	1.959679	0.701272	3.844723	0.618	0.559	0.804	0.617	0.559	1.958	0.701	3.845
37	4	1	1	3	2	0.603305	0.544989	0.915832	0.603305	0.544989	2.953637	0.69592	3.659088	0.604	0.545	0.916	0.603	0.545	2.955	0.697	3.665
38	4	1	2	0	1	0.62796	0.565941	0.502064	0.62796	0.565941	0.502064	0.816964	4.186968	0.627	0.566	0.502	0.627	0.566	0.502	0.816	4.189
39	4	1	2	0	2	0.61509	0.548849	0.625424	0.61509	0.548849	0.875409	0.802582	3.952247	0.615	0.548	0.626	0.615	0.548	0.876	0.803	3.953
40	4	1	2	1	1	0.619718	0.559421	0.670702	0.619718	0.559421	0.99234	0.746172	3.932055	0.620	0.559	0.671	0.619	0.560	0.993	0.745	3.928
41	4	1	2	1	2	0.603562	0.537289	0.772813	0.603562	0.537289	1.446623	0.742078	3.714506	0.603	0.537	0.772	0.604	0.537	1.446	0.742	3.717
42	4	1	2	2	1	0.615244	0.555826	0.7518	0.615244	0.555826	1.468512	0.711566	3.798165	0.615	0.555	0.753	0.615	0.555	1.470	0.712	3.798
43	4	1	2	2	2	0.593284	0.527547	0.867176	0.593284	0.527547	2.208469	0.700641	3.556638	0.594	0.527	0.867	0.594	0.527	2.207	0.700	3.554
44	4	1	2	3	1	0.612468	0.55358	0.798867	0.612468	0.55358	1.933562	0.690413	3.717517	0.612	0.554	0.799	0.613	0.553	1.933	0.691	3.716

7.13.3. Numerical Results Data

Effect of Q_d for balanced systems. s_w=2, s_1=1, Q_1=2, s_2=1

	s_d	Q_d	Q_w	Q_2	FR_1	FR_2	WIP_1	WIP_2	WIP_w	WIP_d	$WIPtr_1$	$WIPtr_2$	$WIPtr_w$	ARO_1	ARO_2	ARO_w	SOw_1	SOw_2	SO_d	WIPtotal
1	2	1	1	1	0.7052	0.6278	1.2901	0.9139	1.3623	1.7217	0.7052	0.6278	0.6665	1.5713	1.0000	1.0000	0.0945	0.0861	0.0850	7.2877
2	2	2	1	1	0.7286	0.6376	1.3586	0.9349	1.5333	2.7199	0.7286	0.6376	0.6831	1.6376	1.0000	1.0000	0.0670	0.0651	0.0192	8.5961
3	2	3	1	1	0.7309	0.6386	1.3656	0.9371	1.5537	3.3019	0.7309	0.6386	0.6848	1.6455	1.0000	1.0000	0.0644	0.0629	0.0111	9.2126
4	2	4	1	1	0.7317	0.6390	1.3680	0.9378	1.5609	3.8118	0.7317	0.6390	0.6853	1.6482	1.0000	1.0000	0.0636	0.0622	0.0083	9.7344
5	2	5	1	1	0.7321	0.6392	1.3694	0.9382	1.5651	4.3166	0.7321	0.6392	0.6856	1.6498	1.0000	1.0000	0.0631	0.0618	0.0066	10.2463
6	2	6	1	1	0.7324	0.6393	1.3703	0.9385	1.5678	4.8189	0.7324	0.6393	0.6859	1.6508	1.0000	1.0000	0.0628	0.0615	0.0055	10.7531
7	2	7	1	1	0.7326	0.6394	1.3709	0.9387	1.5698	5.3202	0.7326	0.6394	0.6860	1.6516	1.0000	1.0000	0.0626	0.0613	0.0047	11.2576
8	2	8	1	1	0.7327	0.6395	1.3714	0.9389	1.5712	5.8211	0.7327	0.6395	0.6861	1.6521	1.0000	1.0000	0.0624	0.0611	0.0041	11.7609
9	2	1	2	2	0.7360	0.7451	1.3893	1.4040	1.7959	1.3596	0.7360	0.7451	0.7406	1.7391	1.7068	1.5176	0.0705	0.0534	0.1162	8.1706
10	2	2	2	2	0.7910	0.7931	1.5726	1.5777	2.9266	3.0076	0.7910	0.7931	0.7920	2.0000	2.0000	2.0000	0.0160	0.0129	0.0194	11.4606
11	2	3	2	2	0.7849	0.7874	1.5478	1.5538	2.6771	3.1256	0.7849	0.7874	0.7862	1.9432	1.9428	1.9259	0.0186	0.0148	0.0116	11.2627
12	2	4	2	2	0.7921	0.7939	1.5758	1.5801	2.9671	4.1976	0.7921	0.7939	0.7930	2.0000	2.0000	2.0000	0.0141	0.0115	0.0066	12.6995
13	2	5	2	2	0.7878	0.7899	1.5585	1.5637	2.7849	4.1737	0.7878	0.7899	0.7888	1.9629	1.9625	1.9562	0.0164	0.0132	0.0062	12.4474
14	2	6	2	2	0.7922	0.7940	1.5762	1.5805	2.9740	5.2102	0.7922	0.7940	0.7931	2.0000	2.0000	2.0000	0.0138	0.0113	0.0043	13.7202
15	2	7	2	2	0.7890	0.7910	1.5634	1.5682	2.8363	5.1889	0.7890	0.7910	0.7900	1.9722	1.9720	1.9688	0.0155	0.0126	0.0041	13.5269
16	2	8	2	2	0.7923	0.7940	1.5764	1.5806	2.9774	6.2131	0.7923	0.7940	0.7932	2.0000	2.0000	2.0000	0.0137	0.0112	0.0032	14.7270
17	3	1	2	2	0.7553	0.7619	1.4521	1.4637	2.1432	1.9942	0.7553	0.7619	0.7586	1.8197	1.7981	1.6696	0.0503	0.0385	0.0782	9.3291
18	3	2	2	2	0.7910	0.7931	1.5726	1.5777	2.9266	3.0076	0.7910	0.7931	0.7920	2.0000	2.0000	2.0000	0.0160	0.0129	0.0194	11.4606
19	3	3	2	2	0.7905	0.7924	1.5694	1.5741	2.9012	4.1222	0.7905	0.7924	0.7915	1.9858	1.9858	1.9846	0.0149	0.0121	0.0056	12.5413
20	3	4	2	2	0.7921	0.7939	1.5758	1.5801	2.9671	4.1976	0.7921	0.7939	0.7930	2.0000	2.0000	2.0000	0.0141	0.0115	0.0066	12.6995
21	3	5	2	2	0.7916	0.7934	1.5734	1.5778	2.9449	5.2096	0.7916	0.7934	0.7925	1.9934	1.9933	1.9932	0.0141	0.0115	0.0030	13.6831
22	3	6	2	2	0.7922	0.7940	1.5762	1.5805	2.9740	5.2102	0.7922	0.7940	0.7931	2.0000	2.0000	2.0000	0.0138	0.0113	0.0043	13.7202
23	3	7	2	2	0.7919	0.7936	1.5745	1.5788	2.9571	6.2176	0.7919	0.7936	0.7927	1.9953	1.9953	1.9952	0.0139	0.0113	0.0021	14.7062
24	3	8	2	2	0.7923	0.7940	1.5764	1.5806	2.9774	6.2131	0.7923	0.7940	0.7932	2.0000	2.0000	2.0000	0.0137	0.0112	0.0032	14.7270

Effect of s_w for balanced systems. s_1=1, Q_1=2, s_2=1

	s_d	Q_d	s_w	Q_w	s_2	Q_2	FR_1	FR_2	WIP_1	WIP_2	WIP_w	WIP_d	$WIPtr_1$	$WIPtr_2$	$WIPtr_w$	ARO_1	ARO_2	AROw	SOw_1	SOw_2	SO_d	WIPtotal
1	2	2	0	1	1	1	0.5709	0.6026	0.8028	0.8561	0.4056	2.8635	0.5709	0.6026	0.5867	1.0000	1.0000	1.0000	0.1972	0.1439	0.0076	6.6882
2	2	2	1	1	1	1	0.6869	0.6193	1.2148	0.8938	0.9088	2.7667	0.6869	0.6193	0.6531	1.4481	1.0000	1.0000	0.1040	0.1062	0.0145	7.7435
3	2	2	2	1	1	1	0.7286	0.6376	1.3586	0.9349	1.5333	2.7199	0.7286	0.6376	0.6831	1.6376	1.0000	1.0000	0.0670	0.0651	0.0192	8.5961
4	2	2	3	1	1	1	0.7521	0.6475	1.4373	0.9571	2.2428	2.6935	0.7521	0.6475	0.6998	1.7458	1.0000	1.0000	0.0446	0.0429	0.0225	9.4301
5	2	2	4	1	1	1	0.7672	0.6535	1.4890	0.9705	3.0071	2.6768	0.7672	0.6535	0.7104	1.8226	1.0000	1.0000	0.0305	0.0295	0.0246	10.2746
6	2	2	5	1	1	1	0.7772	0.6575	1.5228	0.9795	3.8178	2.6659	0.7772	0.6575	0.7174	1.8749	1.0000	1.0000	0.0212	0.0205	0.0260	11.1382
7	2	2	6	1	1	1	0.7840	0.6602	1.5458	0.9856	4.6664	2.6586	0.7840	0.6602	0.7221	1.9112	1.0000	1.0000	0.0149	0.0144	0.0269	12.0228
8	2	2	7	1	1	1	0.7887	0.6621	1.5617	0.9898	5.5460	2.6535	0.7887	0.6621	0.7254	1.9367	1.0000	1.0000	0.0105	0.0102	0.0276	12.9272
9	2	2	8	1	1	1	0.7920	0.6634	1.5728	0.9928	6.4507	2.6500	0.7920	0.6634	0.7277	1.9548	1.0000	1.0000	0.0075	0.0072	0.0280	13.8493
10	2	2	0	2	1	1	0.7207	0.6280	1.2909	0.9133	0.9839	3.2117	0.7207	0.6280	0.6744	1.4874	1.0000	2.0000	0.0553	0.0867	0.0070	8.4229
11	2	2	1	2	1	1	0.7706	0.6445	1.4872	0.9504	1.6949	3.1569	0.7706	0.6445	0.7075	1.7971	1.0000	2.0000	0.0225	0.0496	0.0095	9.4120
12	2	2	2	2	1	1	0.7880	0.6577	1.5548	0.9799	2.6054	3.1247	0.7880	0.6577	0.7229	1.9196	1.0000	2.0000	0.0101	0.0201	0.0116	10.4333
13	2	2	3	2	1	1	0.7944	0.6628	1.5786	0.9913	3.5472	3.1137	0.7944	0.6628	0.7286	1.9604	1.0000	2.0000	0.0045	0.0087	0.0126	11.4166
14	2	2	4	2	1	1	0.7975	0.6649	1.5905	0.9960	4.5211	3.1075	0.7975	0.6649	0.7312	1.9823	1.0000	2.0000	0.0020	0.0040	0.0134	12.4087
15	2	2	5	2	1	1	0.7989	0.6659	1.5957	0.9982	5.5064	3.1051	0.7989	0.6659	0.7324	1.9921	1.0000	2.0000	0.0009	0.0018	0.0137	13.4026

16	2	2	6	2	1	1	0.7995	0.6663	1.5981	0.9992	6.4993	3.1039	0.7995	0.6663	0.7329	1.9964	1.0000	2.0000	0.0004	0.0008	0.0139	14.3992
17	2	2	7	2	1	1	0.7998	0.6665	1.5991	0.9996	7.4957	3.1034	0.7998	0.6665	0.7331	1.9984	1.0000	2.0000	0.0002	0.0004	0.0140	15.3973
18	2	2	8	2	1	1	0.7999	0.6666	1.5996	0.9998	8.4939	3.1032	0.7999	0.6666	0.7332	1.9993	1.0000	2.0000	0.0001	0.0002	0.0140	16.3963
19	2	2	0	2	1	2	0.7637	0.7718	1.4892	1.5088	1.2022	3.0542	0.7637	0.7718	0.7677	2.0000	2.0000	2.0000	0.0645	0.0526	0.0151	9.5576
20	2	2	1	2	1	2	0.7637	0.7718	1.4892	1.5088	1.2022	3.0542	0.7637	0.7718	0.7677	2.0000	2.0000	2.0000	0.0645	0.0526	0.0151	9.5576
21	2	2	2	2	1	2	0.7910	0.7931	1.5726	1.5777	2.9266	3.0076	0.7910	0.7931	0.7920	2.0000	2.0000	2.0000	0.0160	0.0129	0.0194	11.4606
22	2	2	3	2	1	2	0.7910	0.7931	1.5726	1.5777	2.9266	3.0076	0.7910	0.7931	0.7920	2.0000	2.0000	2.0000	0.0160	0.0129	0.0194	11.4606
23	2	2	4	2	1	2	0.7977	0.7982	1.5929	1.5942	4.8455	2.9955	0.7977	0.7982	0.7979	2.0000	2.0000	2.0000	0.0041	0.0033	0.0211	13.4219
24	2	2	5	2	1	2	0.7977	0.7982	1.5929	1.5942	4.8455	2.9955	0.7977	0.7982	0.7979	2.0000	2.0000	2.0000	0.0041	0.0033	0.0211	13.4219
25	2	2	6	2	1	2	0.7994	0.7995	1.5981	1.5985	6.8205	2.9922	0.7994	0.7995	0.7995	2.0000	2.0000	2.0000	0.0011	0.0009	0.0216	15.4077
26	2	2	7	2	1	2	0.7994	0.7995	1.5981	1.5985	6.8205	2.9922	0.7994	0.7995	0.7995	2.0000	2.0000	2.0000	0.0011	0.0009	0.0216	15.4077
27	2	2	8	2	1	2	0.7998	0.7999	1.5995	1.5996	8.8128	2.9914	0.7998	0.7999	0.7999	2.0000	2.0000	2.0000	0.0003	0.0002	0.0217	17.4029
28	0	4	0	2	1	2	0.7603	0.7692	1.4797	1.5010	1.1642	2.2869	0.7603	0.7692	0.7647	2.0000	2.0000	2.0000	0.0701	0.0572	0.0355	8.7261
29	0	4	1	2	1	2	0.7603	0.7692	1.4797	1.5010	1.1642	2.2869	0.7603	0.7692	0.7647	2.0000	2.0000	2.0000	0.0701	0.0572	0.0355	8.7261
30	0	4	2	2	1	2	0.7897	0.7920	1.5688	1.5745	2.8650	2.2653	0.7897	0.7920	0.7909	2.0000	2.0000	2.0000	0.0182	0.0147	0.0399	10.6462
31	0	4	3	2	1	2	0.7897	0.7920	1.5688	1.5745	2.8650	2.2653	0.7897	0.7920	0.7909	2.0000	2.0000	2.0000	0.0182	0.0147	0.0399	10.6462
32	0	4	4	2	1	2	0.7973	0.7979	1.5917	1.5932	4.7740	2.2597	0.7973	0.7979	0.7976	2.0000	2.0000	2.0000	0.0048	0.0039	0.0411	12.6113
33	0	4	5	2	1	2	0.7973	0.7979	1.5917	1.5932	4.7740	2.2597	0.7973	0.7979	0.7976	2.0000	2.0000	2.0000	0.0048	0.0039	0.0411	12.6113
34	0	4	6	2	1	2	0.7993	0.7994	1.5978	1.5982	6.7455	2.2582	0.7993	0.7994	0.7993	2.0000	2.0000	2.0000	0.0013	0.0011	0.0414	14.5978
35	0	4	7	2	1	2	0.7993	0.7994	1.5978	1.5982	6.7455	2.2582	0.7993	0.7994	0.7993	2.0000	2.0000	2.0000	0.0013	0.0011	0.0414	14.5978
36	0	4	8	2	1	2	0.7998	0.7998	1.5994	1.5995	8.7368	2.2578	0.7998	0.7998	0.7998	2.0000	2.0000	2.0000	0.0003	0.0003	0.0415	16.5930

Effect of Q_w for balanced systems

	sd	Qd	sw	Qw	s1	Q1	s2	Q2	FR1	FR2	WIP1	WIP2	WIPw	WIPd	WIPtr1	WIPtr2	WIPtrw	ARO1	ARO2	AROw	SOw1	SOw2	Sod	WIPtotal
1	2	4	2	1	1	2	1	2	0.7130	0.7240	1.2973	1.3158	1.4022	3.7782	0.7130	0.7240	0.7185	1.5369	1.5243	1.0000	0.0763	0.0578	0.0102	9.9489
2	2	4	2	2	1	2	1	2	0.7921	0.7939	1.5758	1.5801	2.9671	4.1976	0.7921	0.7939	0.7930	2.0000	2.0000	2.0000	0.0141	0.0115	0.0066	12.6995
3	2	4	2	3	1	2	1	2	0.7872	0.7884	1.5497	1.5537	3.0514	3.7394	0.7872	0.7884	0.7878	1.9032	1.9145	2.9241	0.0089	0.0084	0.0083	12.2577
4	2	4	2	4	1	2	1	2	0.7976	0.7976	1.5923	1.5923	4.1618	3.2024	0.7976	0.7976	0.7976	2.0000	2.0000	4.0000	0.0044	0.0044	0.0068	12.9417
5	2	4	2	5	1	2	1	2	0.7976	0.7976	1.5923	1.5923	4.1618	3.2024	0.7976	0.7976	0.7976	2.0000	2.0000	4.0000	0.0044	0.0044	0.0068	12.9417
6	2	4	2	6	1	2	1	2	0.7976	0.7976	1.5923	1.5923	4.1618	3.2024	0.7976	0.7976	0.7976	2.0000	2.0000	4.0000	0.0044	0.0044	0.0068	12.9417
7	4	4	2	1	1	2	1	2	0.7153	0.7259	1.3043	1.3221	1.4231	5.7668	0.7153	0.7259	0.7206	1.5437	1.5314	1.0000	0.0736	0.0557	0.0013	11.9781
8	4	4	2	2	1	2	1	2	0.7925	0.7941	1.5768	1.5810	2.9836	6.1814	0.7925	0.7941	0.7933	2.0000	2.0000	2.0000	0.0135	0.0110	0.0011	14.7028
9	4	4	2	3	1	2	1	2	0.7880	0.7891	1.5526	1.5563	3.0941	5.6395	0.7880	0.7891	0.7886	1.9073	1.9179	2.9718	0.0081	0.0077	0.0014	14.2082
10	4	4	2	4	1	2	1	2	0.7979	0.7979	1.5930	1.5930	4.1893	7.1741	0.7979	0.7979	0.7979	2.0000	2.0000	4.0000	0.0040	0.0040	0.0002	16.9430
11	4	4	2	5	1	2	1	2	0.7939	0.7939	1.5752	1.5752	4.1683	5.5642	0.7939	0.7939	0.7939	1.9526	1.9526	4.9433	0.0043	0.0043	0.0010	15.2645
12	4	4	2	6	1	2	1	2	0.7986	0.7986	1.5954	1.5954	5.1912	5.9391	0.7986	0.7986	0.7986	2.0000	2.0000	5.9576	0.0027	0.0027	0.0005	16.7168
13	4	4	2	7	1	2	1	2	0.7949	0.7949	1.5793	1.5793	4.7354	5.2154	0.7949	0.7949	0.7949	1.9603	1.9603	5.9241	0.0036	0.0036	0.0013	15.4939
14	4	4	2	8	1	2	1	2	0.7989	0.7989	1.5966	1.5966	6.1986	6.8003	0.7989	0.7989	0.7989	2.0000	2.0000	7.9748	0.0020	0.0020	0.0001	18.5890
15	2	6	2	1	1	2	1	1	0.7324	0.6393	1.3703	0.9385	1.5678	4.8189	0.7324	0.6393	0.6859	1.6508	1.0000	1.0000	0.0628	0.0615	0.0055	10.7531
16	2	6	2	2	1	2	1	1	0.7888	0.6583	1.5576	0.9813	2.6290	5.2789	0.7888	0.6583	0.7236	1.9238	1.0000	2.0000	0.0093	0.0187	0.0027	12.6175
17	2	6	2	3	1	2	1	1	0.7946	0.6615	1.5793	0.9883	3.2106	3.7909	0.7946	0.6615	0.7280	1.9687	1.0000	3.0000	0.0054	0.0117	0.0074	11.7531
18	2	6	2	4	1	2	1	1	0.7962	0.6632	1.5854	0.9923	3.7562	5.2846	0.7962	0.6632	0.7297	1.9771	1.0000	3.9834	0.0037	0.0077	0.0016	13.8076
19	2	6	2	5	1	2	1	1	0.7966	0.6635	1.5869	0.9930	4.0917	5.2756	0.7966	0.6635	0.7301	1.9797	1.0000	4.4881	0.0033	0.0070	0.0025	14.1374
20	2	6	2	6	1	2	1	1	0.7975	0.6644	1.5905	0.9950	4.7764	5.2690	0.7975	0.6644	0.7310	1.9852	1.0000	6.0000	0.0024	0.0050	0.0004	14.8239
21	2	6	2	7	1	2	1	1	0.7975	0.6644	1.5905	0.9950	4.7764	5.2690	0.7975	0.6644	0.7310	1.9852	1.0000	6.0000	0.0024	0.0050	0.0004	14.8239
22	2	6	2	8	1	2	1	1	0.7975	0.6644	1.5905	0.9950	4.7764	5.2690	0.7975	0.6644	0.7310	1.9852	1.0000	6.0000	0.0024	0.0050	0.0004	14.8239
23	6	2	2	1	1	2	1	1	0.7337	0.6399	1.3744	0.9398	1.5804	6.6741	0.7337	0.6399	0.6868	1.6556	1.0000	1.0000	0.0614	0.0602	0.0004	12.6291
24	6	2	2	2	1	2	1	1	0.7890	0.6585	1.5582	0.9816	2.6350	7.0901	0.7890	0.6585	0.7237	1.9247	1.0000	2.0000	0.0091	0.0184	0.0003	14.4361
25	6	2	2	3	1	2	1	1	0.7949	0.6618	1.5804	0.9891	3.2292	6.4067	0.7949	0.6618	0.7284	1.9698	1.0000	2.9902	0.0050	0.0109	0.0005	14.3904

26	6	2	2	4	1	2	1	1	0.7963	0.6633	1.5856	0.9924	3.7643	6.8083	0.7963	0.6633	0.7298	1.9774	1.0000	3.9894	0.0036	0.0076	0.0003	15.3400
27	6	2	2	5	1	2	1	1	0.7970	0.6640	1.5885	0.9939	4.2640	6.1303	0.7970	0.6640	0.7305	1.9819	1.0000	4.9365	0.0029	0.0061	0.0004	15.1682
28	6	2	2	6	1	2	1	1	0.7975	0.6644	1.5904	0.9949	4.7586	6.4959	0.7975	0.6644	0.7310	1.9850	1.0000	5.9360	0.0024	0.0051	0.0002	16.0327
29	6	2	2	7	1	2	1	1	0.7978	0.6647	1.5917	0.9956	5.2378	5.8190	0.7978	0.6647	0.7313	1.9869	1.0000	6.8693	0.0021	0.0044	0.0003	15.8379
30	6	2	2	8	1	2	1	1	0.7981	0.6650	1.5928	0.9962	5.7446	6.1599	0.7981	0.6650	0.7315	1.9886	1.0000	7.8807	0.0018	0.0038	0.0002	16.6881

Effect of s_1 for balanced systems

	sd	Qd	sw	Qw	s1	Q1	s2	Q2	FR1	FR2	WIP1	WIP2	WIPw	WIPd	WIPtr1	WIPtr2	WIPtrw	ARO1	ARO2	AROw	SOw1	SOw2	Sod	WIPtotal
1	4	2	2	4	0	2	1	1	0.6631	0.6639	0.9914	0.9937	3.8106	4.9540	0.6631	0.6639	0.6635	1.9808	1.0000	3.9452	0.0022	0.0063	0.0007	12.7402
2	4	2	2	4	1	2	1	1	0.7961	0.6631	1.5849	0.9921	3.7352	4.8305	0.7961	0.6631	0.7296	1.9761	1.0000	3.9196	0.0038	0.0079	0.0013	13.3314
3	4	2	2	4	2	2	1	1	0.8849	0.6626	2.4165	0.9908	3.6807	4.7438	0.8849	0.6626	0.7737	1.9717	1.0000	3.8978	0.0051	0.0092	0.0020	14.1530
4	4	2	2	4	3	2	1	1	0.9299	0.6621	3.2261	0.9898	3.6504	4.6966	0.9299	0.6621	0.7960	1.9675	1.0000	3.8816	0.0063	0.0102	0.0026	14.9509
5	4	2	2	4	4	2	1	1	0.9571	0.6617	4.1042	0.9889	3.6294	4.6657	0.9571	0.6617	0.8094	1.9628	1.0000	3.8676	0.0073	0.0111	0.0031	15.8164
6	4	2	2	4	5	2	1	1	0.9733	0.6615	5.0045	0.9885	3.6172	4.6466	0.9733	0.6615	0.8174	1.9607	1.0000	3.8576	0.0080	0.0115	0.0035	16.7089
7	4	2	2	4	6	2	1	1	0.9832	0.6614	5.9336	0.9881	3.6086	4.6343	0.9832	0.6614	0.8223	1.9590	1.0000	3.8503	0.0085	0.0119	0.0038	17.6314
8	4	2	2	4	7	2	1	1	0.9895	0.6612	6.8809	0.9878	3.6032	4.6267	0.9895	0.6612	0.8254	1.9580	1.0000	3.8457	0.0088	0.0122	0.0040	18.5747
9	4	2	2	4	8	2	1	1	0.9933	0.6612	7.8434	0.9877	3.5997	4.6218	0.9933	0.6612	0.8273	1.9573	1.0000	3.8426	0.0091	0.0123	0.0042	19.5343
10	4	2	2	4	0	2	1	2	0.6648	0.7982	0.9972	1.5941	4.2214	4.8219	0.6648	0.7982	0.7315	2.0000	2.0000	3.9104	0.0028	0.0034	0.0016	13.8292
11	4	2	2	4	1	2	1	2	0.7975	0.7976	1.5921	1.5923	4.1351	4.6919	0.7975	0.7976	0.7976	2.0000	2.0000	3.8778	0.0046	0.0044	0.0028	14.4041
12	4	2	2	4	2	2	1	2	0.8866	0.7971	2.4307	1.5908	4.0731	4.5997	0.8866	0.7971	0.8419	2.0000	2.0000	3.8503	0.0063	0.0053	0.0039	15.2199
13	4	2	2	4	3	2	1	2	0.9314	0.7968	3.2455	1.5896	4.0379	4.5498	0.9314	0.7968	0.8641	2.0000	2.0000	3.8313	0.0076	0.0060	0.0048	16.0152
14	4	2	2	4	4	2	1	2	0.9584	0.7964	4.1314	1.5885	4.0132	4.5170	0.9584	0.7964	0.8774	2.0000	2.0000	3.8150	0.0088	0.0066	0.0055	16.8825
15	4	2	2	4	5	2	1	2	0.9744	0.7963	5.0373	1.5879	3.9983	4.4972	0.9744	0.7963	0.8853	2.0000	2.0000	3.8043	0.0096	0.0070	0.0061	17.7767
16	4	2	2	4	6	2	1	2	0.9842	0.7961	5.9724	1.5875	3.9881	4.4844	0.9842	0.7961	0.8901	2.0000	2.0000	3.7963	0.0103	0.0072	0.0065	18.7027
17	4	2	2	4	7	2	1	2	0.9901	0.7960	6.9243	1.5872	3.9818	4.4765	0.9901	0.7960	0.8931	2.0000	2.0000	3.7913	0.0107	0.0074	0.0068	19.6490
18	4	2	2	4	8	2	1	2	0.9939	0.7960	7.8906	1.5870	3.9776	4.4716	0.9939	0.7960	0.8949	2.0000	2.0000	3.7880	0.0110	0.0075	0.0070	20.6115
19	2	4	2	2	0	2	1	2	0.6609	0.7954	0.9914	1.5850	3.0786	4.2649	0.6609	0.7954	0.7281	2.0000	2.0000	2.0000	0.0086	0.0087	0.0047	12.1043
20	2	4	2	2	1	2	1	2	0.7921	0.7939	1.5758	1.5801	2.9671	4.1976	0.7921	0.7939	0.7930	2.0000	2.0000	2.0000	0.0141	0.0115	0.0066	12.6995
21	2	4	2	2	2	2	1	2	0.8813	0.7926	2.4015	1.5760	2.8863	4.1516	0.8813	0.7926	0.8370	2.0000	2.0000	2.0000	0.0196	0.0139	0.0082	13.5263
22	2	4	2	2	3	2	1	2	0.9266	0.7917	3.2006	1.5731	2.8413	4.1281	0.9266	0.7917	0.8592	2.0000	2.0000	2.0000	0.0233	0.0155	0.0093	14.3207
23	2	4	2	2	4	2	1	2	0.9545	0.7909	4.0706	1.5706	2.8093	4.1135	0.9545	0.7909	0.8727	2.0000	2.0000	2.0000	0.0266	0.0170	0.0101	15.1821
24	2	4	2	2	5	2	1	2	0.9713	0.7905	4.9607	1.5692	2.7906	4.1048	0.9713	0.7905	0.8809	2.0000	2.0000	2.0000	0.0288	0.0178	0.0106	16.0679
25	2	4	2	2	6	2	1	2	0.9817	0.7902	5.8815	1.5682	2.7783	4.0993	0.9817	0.7902	0.8860	2.0000	2.0000	2.0000	0.0306	0.0184	0.0110	16.9851
26	2	4	2	2	7	2	1	2	0.9883	0.7900	6.8212	1.5676	2.7707	4.0958	0.9883	0.7900	0.8892	2.0000	2.0000	2.0000	0.0316	0.0187	0.0112	17.9228
27	2	4	2	2	8	2	1	2	0.9925	0.7899	7.7775	1.5672	2.7658	4.0936	0.9925	0.7899	0.8912	2.0000	2.0000	2.0000	0.0323	0.0189	0.0114	18.8776
28	2	2	2	2	0	1	1	2	0.4934	0.7928	0.4934	1.5722	2.7278	3.2432	0.4934	0.7928	0.6431	1.0000	1.9546	2.0000	0.0131	0.0065	0.0070	9.9660
29	2	2	2	2	1	1	1	2	0.6557	0.7902	0.9766	1.5624	2.6027	3.1246	0.6557	0.7902	0.7230	1.0000	1.9416	2.0000	0.0234	0.0094	0.0116	10.4351
30	2	2	2	2	2	1	1	2	0.7360	0.7882	1.4484	1.5547	2.5363	3.0624	0.7360	0.7882	0.7621	1.0000	1.9307	2.0000	0.0310	0.0115	0.0149	10.8880
31	2	2	2	2	3	1	1	2	0.7836	0.7868	1.9079	1.5497	2.4919	3.0233	0.7836	0.7868	0.7852	1.0000	1.9250	2.0000	0.0365	0.0131	0.0175	11.3284
32	2	2	2	2	4	1	1	2	0.8149	0.7859	2.3552	1.5464	2.4623	2.9972	0.8149	0.7859	0.8004	1.0000	1.9212	2.0000	0.0406	0.0142	0.0193	11.7623
33	2	2	2	2	5	1	1	2	0.8370	0.7852	2.7906	1.5439	2.4409	2.9787	0.8370	0.7852	0.8111	1.0000	1.9185	2.0000	0.0436	0.0151	0.0207	12.1876
34	2	2	2	2	6	1	1	2	0.8535	0.7847	3.2147	1.5420	2.4250	2.9650	0.8535	0.7847	0.8191	1.0000	1.9165	2.0000	0.0459	0.0157	0.0217	12.6040
35	2	2	2	2	7	1	1	2	0.8661	0.7843	3.6276	1.5406	2.4127	2.9543	0.8661	0.7843	0.8252	1.0000	1.9149	2.0000	0.0477	0.0162	0.0225	13.0110
36	2	2	2	2	8	1	1	2	0.8762	0.7840	4.0296	1.5395	2.4030	2.9459	0.8762	0.7840	0.8301	1.0000	1.9137	2.0000	0.0491	0.0166	0.0231	13.4083

Effect of Q_1 for balanced systems

	sd	Qd	sw	Qw	s1	Q1	s2	Q2	FR1	FR2	WIP1	WIP2	WIPw	WIPd	WIPtr1	WIPtr2	WIPtrw	ARO1	ARO2	AROw	SOw1	SOw2	Sod	WIPtotal
1	2	2	6	2	2	1	1	2	0.7493	0.7994	1.4974	1.5975	6.3990	3.0360	0.7493	0.7994	0.7743	1.0000	1.9961	2.0000	0.0016	0.0006	0.0181	14.8529
2	2	2	6	2	2	2	1	2	0.8881	0.7992	2.4401	1.5976	6.6925	2.9167	0.8881	0.7992	0.8437	2.0000	2.0000	2.0000	0.0020	0.0014	0.0273	16.1779
3	2	2	6	2	2	3	1	2	0.9210	0.7981	2.9783	1.5931	5.9804	2.8503	0.9210	0.7981	0.8595	2.9590	1.9924	2.0000	0.0025	0.0024	0.0373	15.9807
4	2	2	6	2	2	4	1	2	0.9395	0.7986	3.5061	1.5954	6.3474	2.8141	0.9395	0.7986	0.8691	3.9358	2.0000	2.0000	0.0017	0.0027	0.0422	16.8701
5	2	2	6	2	2	5	1	2	0.9498	0.7966	3.9892	1.5877	5.6788	2.7881	0.9498	0.7966	0.8732	4.8189	1.9880	2.0000	0.0020	0.0046	0.0521	16.6634
6	2	2	6	2	2	6	1	2	0.9580	0.7971	4.4992	1.5907	6.0091	2.7676	0.9580	0.7971	0.8776	5.7872	2.0000	2.0000	0.0014	0.0054	0.0589	17.4993
7	2	2	6	2	2	7	1	2	0.9618	0.7911	4.8706	1.5674	5.5097	2.7615	0.9618	0.7911	0.8765	6.4140	1.9617	2.0000	0.0015	0.0108	0.0631	17.3386
8	2	2	6	2	2	8	1	2	0.9663	0.7916	5.3309	1.5727	5.7507	2.7491	0.9663	0.7916	0.8789	7.2714	2.0000	2.0000	0.0010	0.0157	0.0677	18.0403
9	0	4	4	4	3	1	1	1	0.7996	0.6663	1.9973	0.9991	5.7367	3.2671	0.7996	0.6663	0.7329	1.0000	1.0000	4.0000	0.0011	0.0009	0.0039	14.1989
10	0	4	4	4	3	2	1	1	0.9327	0.6659	3.2595	0.9982	5.6459	3.2007	0.9327	0.6659	0.7993	1.9940	1.0000	4.0000	0.0011	0.0018	0.0070	15.5022
11	0	4	4	4	3	3	1	1	0.9594	0.6655	3.9486	0.9975	5.6230	3.1875	0.9594	0.6655	0.8125	2.9804	1.0000	4.0000	0.0008	0.0025	0.0094	16.1940
12	0	4	4	4	3	4	1	1	0.9692	0.6644	4.4682	0.9950	5.5434	3.1832	0.9692	0.6644	0.8168	3.9595	1.0000	4.0000	0.0006	0.0050	0.0169	16.6401
13	0	4	4	4	3	5	1	1	0.9748	0.6619	4.9503	0.9893	5.4614	3.1816	0.9748	0.6619	0.8184	4.8687	1.0000	4.0000	0.0004	0.0107	0.0256	17.0377
14	0	4	4	4	3	6	1	1	0.9779	0.6597	5.3273	0.9843	5.4597	3.1812	0.9779	0.6597	0.8188	5.5461	1.0000	4.0000	0.0003	0.0157	0.0334	17.4088
15	0	4	4	4	3	7	1	1	0.9793	0.6575	5.5748	0.9794	5.4517	3.1816	0.9793	0.6575	0.8184	5.9400	1.0000	4.0000	0.0003	0.0206	0.0397	17.6425
16	0	4	4	4	3	8	1	1	0.9798	0.6559	5.6930	0.9758	5.4343	3.1822	0.9798	0.6559	0.8178	6.1000	1.0000	4.0000	0.0003	0.0242	0.0424	17.7388
17	0	4	4	4	1	1	1	3	0.6658	0.8563	0.9981	2.1371	5.6853	3.2390	0.6658	0.8563	0.7610	1.0000	2.9855	4.0000	0.0019	0.0005	0.0075	14.3425
18	0	4	4	4	1	2	1	3	0.7982	0.8558	1.5933	2.1340	5.5525	3.1730	0.7982	0.8558	0.8270	1.9905	2.9793	4.0000	0.0020	0.0009	0.0141	14.9338
19	0	4	4	4	1	3	1	3	0.8540	0.8542	2.1203	2.1211	5.4653	3.1459	0.8540	0.8542	0.8541	2.9452	2.9488	4.0000	0.0015	0.0014	0.0183	15.4150
20	0	4	4	4	1	4	1	3	0.8846	0.8517	2.6188	2.1080	5.3603	3.1319	0.8846	0.8517	0.8681	3.8563	2.9018	4.0000	0.0011	0.0023	0.0205	15.8235
21	0	4	4	4	1	5	1	3	0.9040	0.8509	3.1027	2.1087	5.2893	3.1225	0.9040	0.8509	0.8775	4.7371	2.9325	4.0000	0.0008	0.0060	0.0311	16.2557
22	0	4	4	4	1	6	1	3	0.9129	0.8496	3.4216	2.1035	5.1062	3.1187	0.9129	0.8496	0.8813	5.2689	2.9417	4.0000	0.0007	0.0089	0.0371	16.3937
23	0	4	4	4	1	7	1	3	0.9164	0.8490	3.5980	2.1040	5.0015	3.1173	0.9164	0.8490	0.8827	5.5159	2.9496	4.0000	0.0007	0.0106	0.0410	16.4689
24	0	4	4	4	1	8	1	3	0.9179	0.8492	3.6958	2.1075	4.9824	3.1164	0.9179	0.8492	0.8836	5.6255	2.9821	4.0000	0.0007	0.0125	0.0447	16.5529

8. Conclusions

In this thesis three different kinds of production-inventory systems were investigated. The systems were modeled as Continuous time Markov Chains and algorithms were provided for the numerical evaluation of performance measures. The respective computer programs were built in Matlab and extensive numerical experiments were carried out in order to understand the effects of the input parameters on the performance measures and the overall system behavior. The conclusions of our analysis and proposals of possible managerial concern are given in each respective chapter. Here we give some more abstract inferences, drawn from the sum of our investigation. Moreover, we propose directions of further research that would build on our analysis and underline the contribution of our work.

8.1 General conclusions

Production-inventory systems are of dynamic nature and are characterized by the interrelation between their parameters. Even relatively simple and straightforward systems, such as the linear push-pull system, have complex dynamics that pose difficulties in their analysis. On the one hand, this interplay of parameters should be a source of concern when performance measures are estimated. Particularly where optimal solutions are sought, a sensitivity analysis encompassing a wide range of variables must follow. On the other hand, this complexity highlights the need for studied simplifications, especially when large systems with many parameters are concerned.

Somewhat contrary to the above, under certain conditions, the behavior of many performance measures with changing decision variables values were found to be described quite accurately by simple relations (for example linear or logarithmic). In most cases this presupposed that most of the other parameters were kept constant, and that only a limited range of values was investigated. However, the potential remains that for practical problems of limited scope, complex models may not be necessary and that satisfactory results may be obtained without recourse to extensive theoretical research.

A final general conclusion has to do with the deleterious effect of variability. We focused mainly on external demand variability and in almost every case the increase in variability impaired performance in terms of inventory-fill rate balance. Moreover, increased demand uncertainty made the systems more unpredictable, while the effects were observable even in the upstream members of the supply networks.

8.2 Further research

The proposed algorithms are evaluative tools, estimating the performance measures of systems of given structure for various combinations of input parameters. Any investigation of optimal solutions was made through an exhaustive enumeration of all possible policies over a prescribed range of the decision variables, and was based on the balance between inventory levels and service levels. The developed models can be used as the evaluative part of a more general optimization model. The proposed performance measures cover a wide range of metrics and are available for all members of the supply networks, so the algorithm offers a good basis for realistic and flexible cost estimation. Moreover, short computation times and the analytic nature of the models facilitate their integration with heuristic algorithms that would offer a more structured approach to optimal solutions.

With regard to the limitations on model size, it must be noted that during the development of the computer algorithms the main concern was transparency and tractability with the corresponding theory, so little attention was given to algorithmic and computational efficiency. Significant gains in computational time and required memory can be made by "rephrasing" and "shortcutting" the existing computer code. The employment of some Matlab features, such as the use of sparse matrices, could also be useful. Alternatively, if deemed advantageous, the proposed algorithms can be modeled in some other programming language. A different approach for the handling of large models would be to try to represent the systems as stochastic automata networks, but in such a case extensive remodeling would be required.

Finally, the proposed algorithms can be used as a starting point for the study of more generalized versions of the investigated systems. A possible approach would be to relax the assumption of exponentially distributed times and reinterpret the proposed transition matrices as describing the embedded Markov chains of a semi Markov process.