Semi-Markov Processes and Reliability Analysis: Theoretical issues with applications

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Abstract

In Reliability Analysis the main focus is on stochastic processes and in particular semi-Markov processes since they allow for general lifetime distributions. In this thesis, we introduce a family of lifetime distributions, the H-class of distributions, which is closed under minimum. However, some distributions, that we are interested in, do not belong to the H-class, like the Modified Weibull Poisson distribution and therefore we approximate them, by members of the H-class. Moreover, using the multi-state systems and semi-Markov methodology, we produce maximum likelihood estimators for the parameters of the distribution of H-class.

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To my family, Panagiotis, Matina and Maria.

Contents

Introduction

Reliability theory consists of a set of models and methods that aim to solve problems that relate to the calculation, estimation and the optimization of operational probability or expected life or generally the lifetime distribution of a unit or a system of units. The reliability of a product (or system) can be defined as the probability that a product will perform a required function under specified conditions for a certain period of time.

Operation of a unit is called the maintenance of its characteristics within specific limits and under given conditions. While, failure of a unit or a system is called the possibility, after its occurrence, some characteristics of the unit exceed the allowed limits.

Stochastic modelling is an interesting and challenging area with applications in many different fields such as physics, psychology, medicine, biology, engineering, reliability and others. However, in this work, we focus on reliability inference, therefore we use multi-state systems that are represented as semi-Markov processes with several distributions for the sojourn times. The waiting times on each state before moving to another may not always have identical distributions, which complicates the situation further when using closed form estimators.

The idea is to develop a class of several lifetime distributions that covers most if not all of known extreme distributions. Several classes of distributions have been studied with the Exponential being the baseline one (Tahir, et al. (2015)). Recently other distributions have been used as the baseline as well. A family of distributions with Kumaraswamy been the baseline distribution has been used by Barbu, et al. (2021) for reliability inferences. Balasubramanian, et al. (1991) proposed a general class of distributions using a parent continuous distribution having the Modified Weibull Poisson (MWP) as the baseline distribution. The MWP distribution has been widely studied (see Ghorbani, et al. (2014), Wang, et al. (2015)).

The MWP is of special interest in Reliability theory since it combines zero-truncated Poisson distribution and Weibull distribution. Hence, Poisson will give the number of parts that have failed, Weibull will give the time point each part failed and the minimum among them, will provide the minimum time until the first failure. However, the problem is that MWP is not closed under minimum and this fact introduces a limitation since multi-state systems with the closeness under extremes have certain advantages.

The Gompertz distribution (Lenart, (2012)), as well as other families of distributions, like Weibull and Rayleigh, which belong in a class of distributions closed under minimum, are used as approximations for the MWP distribution which does not belong in the class of distributions, that mentioned previously.

In the current work we focus on a general class of distributions using a parent continuous distribution function and present some of its properties. Furthermore, for a multi-state system, we provide estimates for the parameters of the general class of distributions which are considered to vary over the states of the system. The asymptotic theory associated with the estimates obtained is also provided.

The manuscript is structured in 3 chapters. In the first chapter, we present the main definitions and properties of Order Statistics, some of which will be useful in subsequent Chapters. In the second chapter, we present the necessary theory of multi-state systems and some of reliability indices, such as Mean Time To Failure (MTTF), Mean Time Between Failure (MTBF), Mean Time To Repair (MTTR) and expected number of system failures. Also, we analyse the theory of Markov Models for discrete and continuous time and we focus on the study of semi-Markov Models, which are of great interest. Finally, we present the statistical estimation concept and some of the methods of estimation. In the third and last chapter, we introduce a special family of distributions closed under minimum, the H-class of distributions. We present the approximation of distributions which are not part of this family, by members of this family. Last but not least, we discuss the semi-Markov setting that is used in order to get the maximum likelihood estimators of the parameters involved.

Chapter 1

Order Statistics

1.1 Main Definitions and Properties

Definition 1.1.1: (Sahoo, P. (2013)) *Let $X_1, X_2, ..., X_n$ be observations from a random sample of size n from a distribution $f(x)$. Let $X_{(1)}$ denote the lowest of $\{X_1, X_2, ..., X_n\}$, $X_{(2)}$ denote the second smallest of $\{X_1, X_2, ..., X_n\}$, and similarly $X_{(k)}$ denote the k^{th} lowest of $\{X_1, X_2, ..., X_n\}$. Then the random variables $X_{(1)}, X_{(2)}, ..., X_{(n)}$ are called the order statistics of the sample $X_1, X_2, ..., X_n$. Especially, $X_{(k)}$ is called the k^{th}-order statistic of $X_1, X_2, ..., X_n$.*

Definition 1.1.2: (Casella, G. et al. (2021)) *The order statistics of a random sample $\{X_1, X_2, ..., X_n\}$ are the sample values placed in ascending order. They are designated by $X_{(1)}, ..., X_{(n)}$.*

The order statistics are random variables that satisfy the relation $X_{(1)} \leq X_{(2)} \leq ... \leq X_{(n)}$. Several statistics that are easily designated in terms of the order statistics, are the following.

The equation $R = X_{(n)} - X_{(1)}$, called *sample range*. The sample range is the distance between the lowest and largest observations. Also it is a measurement of the dispersion in the sample and should reflect the population dispersion.

The *sample median*, denoted by M, is a value such that approximately one-half of the observations are greater than M and one-half are less. In terms of order statistics, M is determined by

$$M = \begin{cases} X_{((n+1)/2)}, & if\ n\ is\ odd \\ \\ (X_{(n/2)} + X_{(n/2+1)})/2, & if\ n\ is\ even. \end{cases}$$

The median is a measurement of location that might be assumed instead to the sample mean. The benefit of the sample median over the sample mean is that it is less affected by extreme observations.

For each number p between 0 and 1, the $(100p)th$ sample percentile is the observation

such that approximately $n(1-p)$ of the observations are larger and np of the observations are less than this observation. The 25th percentile is the lower quartile, the 50th percentile is the sample median and the 75th percentile is the upper quartile. A measurement of dispersion that is commonly used is the interquartile range, which is the distance between the lower and upper quartiles.

Theorem 1.1.1: (Casella, G. et al. (2021)) Let $X_1, ..., X_n$ be a random sample from a discrete distribution with pmf $f_X(x_i) = p_i$, where $x_1 < x_2 < ...$ are the available values of X in ascending order. Define

$$P_0 = 0$$

$$P_1 = p_1$$

$$P_2 = p_1 + p_2$$

$$.$$

$$.$$

$$.$$

$$P_i = p_1 + p_2 + ... + p_i$$

$$.$$

$$.$$

$$.$$

Let $X_{(1)}, ..., X_{(n)}$ define the order statistics from the sample. So,

$$P(X_{(j)} \leq x_i) = \sum_{k=j}^{n} \binom{n}{k} P_i^k \ (1 - P_i)^{n-k}$$

and

$$P(X_{(j)} = x_i) = \sum_{k=j}^{n} \binom{n}{k} [P_i^k \ (1 - P_i)^{n-k} - P_{i-1}^k \ (1 - P_{i-1})^{n-k}].$$

1.2 Probability Density Function & Cumulative Distribution Function of the Minimum and Maximum

For $X_1, X_2, ..., X_n$ independent identically distributed continuous random variables with probability density function $f(x)$ and cumulative distribution function $F(x)$, the pdf of the k^{th} order statistic is

$$\begin{aligned} F_{(1)} &= P(X_{(1)} < x) = 1 - P(X_{(1)} > x) = 1 - P(X_1 > x, ..., X_n > x) \\ &= 1 - P(X_1 > x) \cdots P(X_n > x) = 1 - (1 - F(x))^n \end{aligned}$$

$$F_{(n)} = P(X_{(n)} < x) = 1 - P(X_{(n)} > x) = P(X_1 < x, ..., X_n < x)$$
$$= P(X_1 < x) \cdots P(X_n < x) = F(x)^n$$

and

$$f_{(1)}(x) = \frac{d}{dx}(1 - F(x))^n = n(1 - F(x))^{n-1} \frac{dF(x)}{dx} = nf(x)(1 - F(x))^{n-1}$$

$$f_{(n)}(x) = \frac{d}{dx}F(x)^n = nF(x)^{n-1} \frac{dF(x)}{dx} = nf(x)F(x)^{n-1}$$

Theorem 1.2.1: (Sahoo, P. (2013)) Let $X_1, X_2, ..., X_n$ be a random sample of size n from a distribution with pdf $f(x)$. Then the pdf of the k^{th} order statistic, $X_{(k)}$, is

$$g(x) = \frac{n!}{(k-1)!(n-k)!} [F(x)]^{k-1} f(x) [1 - F(x)]^{n-k},$$

where $F(x)$ is the cdf of $f(x)$.

Proof: We prove the theorem assuming $f(x)$ continuous. In the case $f(x)$ is discrete the proof has to be modified appropriately. Let m be a positive real number and x be an arbitrary point in the domain of f. Let us divide the real line into three segments, namely

$$\mathbb{R} = (-\infty, x) \cup [x, x + m) \cap [x + m, \infty).$$

The probability, say p_1, of a sample value falls into the first interval $(-\infty, x]$ and is given by

$$p_1 = \int_{-\infty}^{x} f(t)dt = F(x).$$

Similarly, the probability p_2 of a sample value falls into the second interval $[x, x + m)$ is

$$p_2 = \int_{x}^{x+m} f(t)dt = F(x + m) - F(x).$$

In the same token, we can compute the probability p_3 of a sample value which falls into the third interval

$$p_3 = \int_{x+m}^{\infty} f(t)dt = 1 - F(x + m).$$

Then the probability, $P_m(x)$, that $(k - 1)$ sample values fall in the first interval,one falls in the second interval, and $(n - k)$ fall in the third interval is

$$P_h(x) = \binom{n}{k-1, 1, n-k} p_1^{k-1} p_2^1 p_3^{n-k} = \frac{n!}{(k-1)!(n-k)!} p_1^{k-1} p_2 p_3^{n-k}.$$

Hence the pdf $g(x)$ of the k^{th} statistics is given by

$$g(x) = \lim_{m \longrightarrow 0} \frac{P_m(x)}{m} = \lim_{m \longrightarrow 0} \left[\frac{n!}{(k-1)!(n-k)!} \; p_1^{k-1} \; \frac{p_2}{m} \; p_3^{n-k} \right]$$

$$= \frac{n!}{(k-1)!(n-k)!} \; [F(x)]^{k-1} \; \lim_{m \longrightarrow 0} \frac{F(x+m) - F(x)}{m} \; \lim_{m \longrightarrow 0} [1 - F(x+m)]^{n-k}$$

$$= \frac{n!}{(k-1)!(n-k)!} \; [F(x)]^{k-1} \; F'(x) \; [1 - F(x)]^{n-k}$$

$$= \frac{n!}{(k-1)!(n-k)!} \; [F(x)]^{k-1} \; f(x) \; [1 - F(x)]^{n-k}.$$

1.3 The Joint Distribution of the Minimum and Maximum

Consider the joint cumulative distribution function of the minimum and the maximum

$$F_{X_{(1)},X_{(n)}}(x, y) = P(X_{(1)} \leq x, X_{(n)} \leq y).$$

It is not clear how to write this in terms of the individual X_i. Assume instead the following relationship

$$(X_{(n)} \leq y) = P(X_{(1)} \leq x, X_{(n)} \leq y) + P(X_{(1)} > x, X_{(n)} \leq y) \tag{1.1}$$

We want to calculate the first term on the right-hand side. In terms of the last term, we have

$$P(X_{(1)} > x, X_{(n)} \leq y),$$

if $x \geq y$ the above expression is zero. In this instance, $P(X_{(1)} \leq x, X_{(n)} \leq y) = P(X_{(n)} \leq y)$ and (1.1) gives us only $P(X_{(n)} \leq y) = P(X_{(n)} \leq y)$. Considering that $x < y$. Then,

$$P(X_{(1)} > x, X_{(n)} \leq y) = P(x < X_1 \leq y, ..., x < X_n \leq y) \stackrel{iid}{=} [P(x < X_1 \leq y)]^n = [F(y) - F(x)]^n.$$

Hence, from (1.1), we have the following

$$F_{X_{(1)},X_{(n)}}(x, y) = P(X_{(1)} \leq x, X_{(n)} \leq y)$$

$$= P(X_{(n)} \leq y) - P(X_{(1)} > x, X_{(n)} \leq y)$$

$$= [F(y)]^n - [F(y) - F(x)]^n.$$

So, the joint probability density function is

$$f_{X_{(1)},X_{(n)}}(x,y) = \frac{d}{dx}\frac{d}{dy}\{[F(y)]^n - [F(y)-F(x)]^n\}$$

$$= \frac{d}{dx}\{n[F(y)]^{n-1}f(y) - n[F(y)-F(x)]^{n-1}f(y)\}$$

$$= n(n-1)[F(y)-F(x)]^{n-2}f(x)f(y)$$

For (x,y) in the support and $x < y$.

Theorem 1.3.1: Let $X_{(1)},...,X_{(n)}$ denote the order statistics of a random sample, $X_1,...,X_n$, from a continuous population with cdf $F_X(x)$ and pdf $f_X(x)$. Hence, the joint probability density function of $X_{(i)}$ and $X_{(j)}$, $1 \le i < j \le n$, is

$$f_{X_{(i)},X_{(j)}}(v,u) = \frac{n!}{(i-1)!(j-1-i)!(n-j)!}\, f_X(v)f_X(u)\,[F_X(v)]^{i-1}\,[F_X(u)-F_X(v)]^{j-1-i}\,[1-F_X(u)]^{n-j}$$

for $-\infty < v < u < \infty$.

Perhaps the other most significant probability density function is $f_{X_{(1)},...,X_{(n)}}(x_1,...,x_n)$, the joint probability density function of all the order statistics, which is the following

$$f_{X_{(1)},...,X_{(n)}}(x_1,...,x_n) = \begin{cases} n!f_X(x_1)...f_X(x_n), & -\infty < x_1 < x_2 < \infty. \\ \\ 0, & otherwise \end{cases}$$

More informations about the *Joint Distribution of the Minimum and Maximum* can be found in Lectures: Order Statistics of Applied Mathematics, University of Colorado Boulder.

1.4 Limit Distributions of Maxima and Minima

Previously, we have shown that

$$F_{(1)} = P(X_{(1)} < x) = 1 - (1-F(x))^n$$
$$F_{(n)} = P(X_{(n)} < x) = F(x)^n$$

When n tends to infinity we get

$$\lim_{n\to\infty} F_{(1)}(x) = \lim_{n\to\infty} 1-(1-F(x))^n = \begin{cases} 0, & \text{if } F(x)=0 \\ \\ 1, & \text{if } F(x)>0. \end{cases}$$

$$\lim_{n\to\infty} F_{(n)}(x) = \lim_{n\to\infty} F(x)^n = \begin{cases} 1, & \text{if } F(x)=1 \\ \\ 0, & \text{if } F(x)<1. \end{cases}$$

Chapter 2

Multi-State Systems

2.1 Introduction to Multi-State Systems

In reliability theory, a simple system has two states: on (functional) or off (failed). Because of this assumption, the system's structural function is a binary function of binary variables, and the corresponding model is known as a binary reliability system.

In some multi-state systems, determining the state probability and output performance rate of an component is difficult. Some of the issues stem from data inaccuracy and insufficiency. In a broad sense, we define success as a happening or a type of behavior of these elements that meets a predefined criterion, whatever that criterion may be. If the criterion is broken, the result is a failure. The space shuttle Challenger (Vaughan, 1996), Three Mile Island (Chiles, 2002), the London Paddington train crash (Cullen, 2000), and the Gulf of Mexico oil disaster are only a few examples from the accident literature. Non-linearities may exist in practice, and they can have a major impact on system output performance.

We consider that the operation of safety is to identify and describe the functional faults that lead to accidents. This operation can be performed in two ways:
i) by asking why accidents occurred and looking for root causes, and
ii) by evaluating the output performance levels of several recognized effective engineering controls that are in operation.

Failure can have a variety of causes. These causes can be physical, human, logical, or financial. Clearly, a variety of criteria and elements can be used to determine what a failure is, including structure, output performance, cost, and even subjective intent. Nevertheless, no matter what kind of failure it is, if the consequences are severe, further investigation is required.

For example, a malfunctioning valve in a fluid control network may be "stuck-open" or "stuck-closed," while a gadget in a safety monitoring system will malfunction if it "fails to detect breakdown" or "initiates a false alert." The term "three-state device" refers to a structure whose components can fail in two different ways. By assigning m$\geq$ 2 failure modes to each component, a natural extension of three-state devices can be created simply. Multi state system is the name given to the resulting structure.

2.1.1 Main Definitions and Properties

Generic Multi-State System Model

To study MSS behavior one has to know the features of its elements. Any system element j can have k_j different states corresponding to the output performance rates, represented by the set

$$\mathbf{o}_j = \{o_{j1}, ..., o_{jk_j}\}$$

where o_{ji} is the output performance rate of element j in the state i, $i \in \{1, 2, ..., k_j\}$.

The output performance rate $O_j(t)$ of element j at any instant $t \geq 0$ is a random variable that takes its values from $o_j : O_j(t) \in \mathbf{o}_j$. Hence, the output performance rate of element j is determined as a stochastic process, for the interval $[0, T]$, where T is the MSS operation duration.

The conventional multi-state system reliability theory has two fundamental assumptions:
i) every state probability of an element which compounded a multi-state system can be completely described by probability measures and
ii) the state output performance rate of an element which compounded a multi-state system can be exactly designated.

In some instances, the output performance of element cannot be measured only by a single value, but by more complex mathematical objects, most commonly vectors. Especially, the element output performance is determined as a vector stochastic process $O_j(t)$.

The probabilities related with the various states of the system element j at any instant t can be represented by the set

$$\mathbf{p}_j(t) = \{p_{j1}(t), ..., p_{jk_j}(t)\}, \tag{2.1}$$

where

$$p_j(t) = \mathbb{P}\{O_j(t) = o_{ji}\} \tag{2.2}$$

Observe that since the element states compose the entire set of mutually exclusive happenings (i.e., element j can always be in one and only one of k_j states), $\sum_{i=1}^{k_j} p_{ji}(t) = 1$, for any $t : 0 \leq t \leq T$.

Expression (2.2) determines the probability mass function for a discrete random variable $O_j(t)$ at any instant t. The collection of pairs $o_{ji},\ p_{ji}(t),\ i = 1, ..., k$ completely defines the probability distribution of output performance of the element j at any instant t.

Note that the output performance distribution (**PD**) can be used to illustrate the behavior of binary elements (those with only total failures). Indeed, assume a binary

element b with a nominal output performance (output performance rate corresponding to a fully operable state) o^* and the probability that the element is in the completely operable state $p(t)$. Considering that the element's output performance rate in a state of total failure is 0, one gets its **PD** as follows: $\mathbf{o}_b = \{0, o^*\},\ \mathbf{p}_b(t) = \{1 - p(t),\ p(t)\}$.

When the MSS consists of n elements, the output performance rates of this system are unambiguously defined by the output performance rates of these elements. At each moment, the system elements have several output performance rates corresponding to their states. The states of its elements define the state of the entire system. Consider that the entire system has K different states and that o_i is the entire system output performance rate in state $i,\ i \in \{1, \ldots, K\}$. The entire MSS output performance rate at time t is a random variable that takes values from the set $\{o_1, \ldots, o_K\}$.

Definition 1.1 (Lisnianski, A. et al.(2010)) *Let* $L^n = \{o_{11}, ..., o_{1k_1}\} \times ... \times \{o_{n1}, ..., o_{nk_n}\}$ *be the space of possible combinations of output performance rates for all of the MSS elements and* $M = \{o_1, \ldots, o_K\}$ *be the space of possible values of the output performance rate for the entire system.*

The transform $f(O_1(t), ..., O_n(t)) : L^n \to M$ *which maps the space of the elements' output performance rates into the space of the system's output performance rates, is called the MSS structure function.*

Observe that the MSS structure function is a binary structure function's extension. The only difference is in the determination of the state spaces: the binary structure function is mapped as $\{0, 1\}^n \to \{0, 1\}$, while in the MSS, one deals with much more complex spaces.

Thus, we can determined a generic model of the MSS.
Models of output performance stochastic processes should be combined in this generic MSS model.

$$O_j(t),\ j = 1, 2, ..., n \tag{2.3}$$

for each system element j and of the system structure function that gives the stochastic process corresponding to the output performance of the integrated MSS

$$O(t) = f(O_1(t), ..., O_n(t)). \tag{2.4}$$

In actuality, output performance stochastic processes $O_j(t)$ can be given in several forms. Consider, the output performance probability distributions for all of the system elements can be produced at every time t throughout the operation duration $[0, T]$. Thus, these probability distributions present the MSS.

$$\mathbf{o}_j, \mathbf{p}_j(t),\ 1 \le j \le n, \tag{2.5}$$

and structure function of the system:

$$O(t) = f(O_1(t), ..., O_n(t)). \tag{2.6}$$

Does not matter how the structure function is determined. For any given set $\{O_1(t), ..., O_n(t)\}$, it can be represented in analytical form or in a table or be described as an algorithm for defining the system output performance $O(t)$ without ambiguity. We will assume some of the available representations of MSS structure functions as examples, in the next subsection.

Example (Lisnianski, A. et al.(2010)) Assume a 2-out-of-3 MSS. This system consists of three binary elements with the output performance rates $O_i(t) \in \{o_{i1}, o_{i2}\} = \{0, 1\}$, for $i = 1, 2, 3$, where

$$o_{i1} = \begin{cases} 0, & \text{if element } i \text{ is in a state of total failure,} \\ 1, & \text{if element } i \text{ operates perfectly.} \end{cases}$$

The system output performance rate $O(t)$ at any instant t is

$$O(t) = \begin{cases} 0, & \text{if there is more than one failed element,} \\ 1, & \text{if there is only one failed element,} \\ 2, & \text{if all the elements operate perfectly.} \end{cases}$$

The values of the system structure function $O(t) = f(O_1(t), O_2(t), O_3(t))$ for all the possible system states are presented in Table 2.1.

$O_1(t)$	$O_2(t)$	$O_3(t)$	$f(O_1(t), O_2(t), O_3(t))$
1	1	1	2
1	1	0	1
1	0	1	1
1	0	0	0
0	1	1	1
0	1	0	0
0	0	1	0
0	0	0	0

Table 2.1: Structure function for 2-out of-3 system.

Main Properties of Multi-State Systems

The main properties of multi-state systems are *Relevancy of systems Elements*, *Coherency* and *Homogeneity*. Below we will define them one by one. A more extensive collection with further analysis of these properties can be found in Lisnianski, A. et al.(2010).

• **Relevancy of systems Elements**

In the binary context, the relevancy of a system element means that the state of an entire system totally dependent on the state of this element in particular cases. In terms of the system structure function, the relevancy of element j means that there are such $O_1(t), \ldots, O_n(t)$ that

$$\begin{aligned} &f(O_1(t), ..., O_{j-1}(t), 1, O_{j+1}(t), ..., O_n(t)) = 1, \\ &f(O_1(t), ..., O_{j-1}(t), 0, O_{j+1}(t), ..., O_n(t)) = 0. \end{aligned} \tag{2.7}$$

It should be noted that for the binary systems $O_j(t) \in \{0, 1\}$ for $1 \leq j \leq n$.

When the MSS is assumed, an element is relevant if several changes in its state cause changes in the complete system state without changes in the states of the existing elements. In terms of the MSS structure function, the relevancy of element j means that there are such $O_1(t), \ldots, O_n(t)$ that for some $o_{jk} \neq o_{jm}$

$$\begin{aligned} &f(O_1(t), ..., O_{j-1}(t), o_{jk}, O_{j+1}(t), ..., O_n(t)) \neq \\ &f(O_1(t), ..., O_{j-1}(t), o_{jm}, O_{j+1}(t), ..., O_n(t)). \end{aligned} \tag{2.8}$$

• **Coherency**

In the binary system context coherency means that:

· All elements of the system are relevant.
· The fault of all the elements causes the fault of the entire system.
· The entire system operates as a result of the operation of all the elements.
· No extra failure can make the system operate again, once it has failed.
· No repair or addition of elements can cause the system to fail while it is in operation.

For MSSs these requirements are met in systems with monotonic structure functions:

$$\begin{aligned} &f(O_1(t), ..., O_n(t)) = 1, \text{ if } O_j(t) = 1 \text{ for } 1 \leq j \leq n, \\ &f(O_1(t), ..., O_n(t)) = 0, \text{ if } O_j(t) = 0 \text{ for } 1 \leq j \leq n, \\ &f(O_1(t), ..., O_n(t)) \geq f(\tilde{O}_1(t), ..., \tilde{O}_n(t)), \end{aligned} \tag{2.9}$$

if there is no j for which $\tilde{O}_j \geq \tilde{O}_j$ (for a binary system, this can be reformulated as follows: there is no such j that $\tilde{O}_j = 1$ and $O_j = 0$).

Thus, in a multi-state case, the system is coherent if and only if the structure function in each argument, is non-decreasing and all elements of the system are relevant. Mention that the largest system output performance is succeed when all of the elements' output performance rates are the largest, and the lowest system output performance is succeed when all of the elements' output performance rates are the lowest, as a result of this structure function property.

- **Homogeneity**

If all of the MSS's elements and the entire system have the same number of distinguished states, the MSS is homogeneous. All binary-state systems are homogeneous, as may be clearly seen.

2.1.2 Multi-State System Reliability and Its Measures

Acceptable and Unacceptable States. Failure Criteria

MSS behavior is characterized by its evolution in the space of states. The complete set of available system states can be separated into two distinct subsets that correspond to acceptable and unacceptable system function. A failure occurs when the system enters a subset of unacceptable states. The ability of the MSS to stay in acceptable states during the operation term is characterized as its reliability.

Therefore, the output performance $O(t)$ characterize the system operation, the state acceptability at any instant $t \geq 0$ depends on this value. In some situations, this dependency can be phrased by the acceptability function $F(O(t))$ which returns non-negative values if and only if MSS operation is acceptable. This takes place when the efficiency of the system operation is totally defined by its internal state. Such as, only states in which a network's connectivity is preserved are acceptable. In such circumstances,the consumer is interested in a certain set of MSS states. Typically, these states are defined as system failure states, implying that the system should be repaired or destroyed if they are reached.

Frequently, the system state acceptability is established by the relationship between MSS output performance and the desired level of this output performance (demand) designated outside of the system. Generally, the demand $D(t)$ is also a random process. Below we assume such a case when the demand can take discrete values from the set $d = \{d_1, ..., d_M\}$. Usually, the desired relationship between the system output performance and the demand can be phrased by the acceptability function $F(O(t), D(t))$. The acceptable system states correspond to $F(O(t), D(t)) \geq 0$ and the unacceptable states correspond to $F(O(t), D(t)) < 0$. The MSS failure criterion is determined by the previous inequality.

In many practical cases, the MSS output performance should be able to surpass the demand. In such cases the acceptability function given by the following form $F(O(t), D(t)) = O(t) - D(t)$.

The possibility of entering the subset of unacceptable states more than once throughout the operation time characterizes the system behavior. The possibility of entering the subset of unacceptable states more than once during the operation time characterizes the system behavior. When an MSS may only enter this subset once, it usually means that the system is degrading and cannot be repaired. Transitions between subsets of unacceptable and acceptable states may occur an arbitrarily times in repairable systems.

It's worth noting that dividing an MSS's state space into acceptable and unacceptable states isn't always doable. In order to determine MSS failure, only some functional linked with two stochastic processes $O(t)$ and $D(t)$ may be of interest.
Such as, the failure of MSS can be determined as an event when functional $J = \int_0^T a[O(t), D(t)]dt$ is larger than some specified value J_0 and $a(\cdot)$ is determined as some arbitrary function.

Functional J as an energy not supplied to costumers in a power system, where $O(t)$ and $D(t)$ are assessed as generating capacity and demand, where $a(\cdot)$ is identified as follows: $a(t) = D(t) - O(t)$, if $D(t) - O(t) \geq 0$, and $a(t) \equiv 0$, if $D(t) - O(t) < 0$. The functional J is called a failure criteria functional.

Relevancy and Coherency in Multi-State System Reliability Context

The MSS relevancy was taken into account as a property of the structure function that represented the system's output performance. When considering the MSS from the standpoint of reliability, the system demand must also be considered. The demand value, as well as the system output performance value, are of interest. In this case, an element is relevant if changes in its state create changes in the system's reliability without causing changes in the states of the remaining elements. The relevancy of the system is now viewed as a property related with the system's ability to accomplish a task, which is determined outside of the system. In this context element j, relevancy means that there are $O_1(t), \ldots, O_n(t)$ for which for some $o_{jk} \neq o_{jm}$

$$J\left\{a[f(O_1(t), ..., O_{j-1}(t), o_{jk}, O_{j+1}(t), ..., O_n(t))], D\right\} \leq J_0, \tag{2.10}$$

while

$$J\left\{a[f(O_1(t), ..., O_{j-1}(t), o_{jm}, O_{j+1}(t), ..., O_n(t))], D\right\} > 0.$$

Mention that this situation is more difficult than (2.8). Certainly, a relevant element according to expression (2.8) can be irrelevant according to (2.10).

One may also propose a determination of system coherency that is more closely connected to the one given for binary systems by using the acceptability function. When it comes to binary systems, coherency is determined in terms of fault and normal operation, however when it comes to MSS, all that is necessary is the structural function's monotonic behavior. In reliability, the MSS coherency means that improvements in system elements' output performance cannot lead the entire system to move from an acceptable to an unacceptable state:

$$\begin{aligned} &\text{if } F(f(\tilde{O}_1(t), ..., \tilde{O}_n(t)), D) \geq 0 \text{ and there is no } j \text{ for which } \tilde{O}_j \geq \tilde{O}_j, \\ &\text{then } F(f(O_1(t), ..., O_n(t)), D) \geq 0. \end{aligned} \tag{2.11}$$

Multi-State System Reliability Measures

In order to numerically describe the MSS behavior from a reliability standpoint, we need first to defined the MSS reliability indices. The indices of reliability can be assumed of the relevant binary-state system reliability indices.

Some indices are based on assuming the system's evolution in the time domain. Here, we can be investigated the relationship between the output performance of the system and the demand represented by the two associated stochastic processes.

The behavior of the system is described by its output performance expressed as a random variable when it is assumed in the given time instant or in a steady state, where its output performance distribution is independent of time.

Keep in mind that in a steady state the distribution of the variable demand can be represented by two vectors (d, q), where $d = \{d_1, ..., d_M\}$ is the vector of possible demand levels $d_j,\ j = 1, ..., M$ and $q = \{q_1, ..., q_M\}$ is the vector of steady-state probabilities of respective demand levels $q_j = \mathbb{P}\{D = d_j\},\ j = 1, ..., M$.

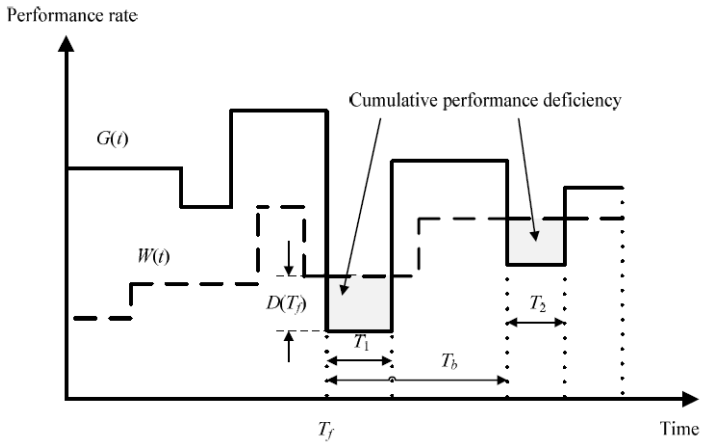

Figure 2.1: MSS behavior as stochastic process, where the $G(t)$ has been noted with $O(t)$ (Lisnianski, A. et al.(2010)).

An example of the random realization of two stochastic processes $O(t)$ and $D(t)$ can be considered in the Figure 2.1. Consider the case where the value of the system output performance exceeds the value of demand: $F(O(t), D(t)) = O(t) - D(t)$. Hence, the time to MSS failure is defined by the first time that the process $O(t)$ downcrosses the level of demand, $D(t)$.

Subsequently, we need to define the following random variables, which are important, when considering MSS evolution in the space of states during system operation period T.

T_f is called the *time to failure*, i.e., the time between the start of the system's life and the first time it reaches the subset of unacceptable states. T_b is called the *time between failures*, i.e., the amount of time between two successive transitions from one subset of acceptable states to another subset of unacceptable states. Also, N_T is *number of failures*, i.e., the number of times throughout the time interval $[0, T]$ that the system reaches the subset of unacceptable states.

For each the above times, we present the same indices. For T_f, the probability of a failure-free operation or reliability function $R(t)$, i.e, the probability that T_f is greater than or equal to the value $t(t > 0)$, where in the initial state (at instant $t = 0$) the MSS is in one of the acceptable states:

$$R(t) = \mathbb{P}\left\{T_f \geq t | F(O(0), D(0))\right\} \geq 0. \tag{2.12}$$

For T_b, the probability that the time between failures is greater than or equal to t:

$$\mathbb{P}\{T_b \geq t\}. \tag{2.13}$$

And for N_T, the probability that N_T is less than a specified n:

$$\mathbb{P}\{N_T \leq n\}. \tag{2.14}$$

Moreover, we give the mean time of indices, respectively:
Mean time to failure **(MTTF)** is the mean time up to the moment when the system enters the subset of unacceptable states for the first time:

$$E\{T_f\} = \int_0^\infty t f_{T_f}(t)dt. \tag{2.15}$$

Mean time between failures **(MTBF)**:

$$E\{T_b\} = \int_0^\infty t f_{T_b}(t)dt. \tag{2.16}$$

The expected number of system failures for the time interval $[0, T]$, is:

$$E\{N_T\} = \sum_{n=0}^{\infty} n f_{N_T}(n). \tag{2.17}$$

Also, the *Mean time to Repair* **(MTTR)** is determined as the expectation of the repair period T_r:

$$E\{T_r\} = \int_0^\infty t f_{T_r}(t)dt. \tag{2.18}$$

Multiple measures have been proposed in the literature, only a few of them will be presented here. For more complete collection of such measures, the reader is advised to look at Lisnianski, A. et al.(2010).

MSS pointwise (or instantaneous) availability $A(t, d)$ is the probability that the MSS at instant $t > 0$ is in an acceptable state:

$$A(t, d) = \mathbb{P}\{F(O(t), D(t)) \geq 0\}. \tag{2.19}$$

For the time interval $[0, T]$, the MSS average availability is the following:

$$A_T = \frac{1}{T}\int_0^T \mathbf{1}\{F[O(t), D(t)] \geq 0\} dt, \tag{2.20}$$

where

$$\mathbf{1}\left\{F[O(t),D(t)]\geq 0\right\}=\begin{cases}1, \text{ if } F[O(t),D(t)]\geq 0,\\ \\ 0, \text{ if } F[O(t),D(t)]<0.\end{cases}$$

The random variable A_T denotes the part of time during which the MSS output performance rate is acceptable.

The availability of demand DA is the following expected value of A_T

$$DA=E\left\{A_T\right\}. \tag{2.21}$$

The initial state of system has almost no effect on its availability for large $t(t\to\infty)$. So, the steady-state (stationary or long-term) MSS availability $A_\infty(d)$ for the constant demand level $D(t)=d$ can be defined on the basis of the system steady-state output performance distribution:

$$A_\infty(d)=\sum_{k=1}^{K}p_k\mathbf{1}(F(o_k,d)\geq 0), \tag{2.22}$$

where

$$\mathbf{1}\left\{F(o_i,d)\geq 0\right\}=\begin{cases}1, \text{ if } F(o_i,d)\geq 0,\\ \\ 0, \text{ if } F(o_i,d)<0,\end{cases}$$

and $p_k=\lim\limits_{t\to\infty}p_k(t)$ is the steady-state probability of the MSS state k with the corresponding output performance rate o_k.

When $F(O(t),D(t))=O(t)-D(t)$, we have $F(o_k,d)=o_k-d$ and

$$A_\infty(d)=\sum_{k=1}^{K}p_k\mathbf{1}(o_k\geq d)=\sum_{o_k\geq d}p_k. \tag{2.23}$$

The cumulative output performance curve for a MSS in a steady state is shown in Figure 2.2. The point where the cumulative output performance curve crosses the value of w corresponds to stationary availability in this diagram.

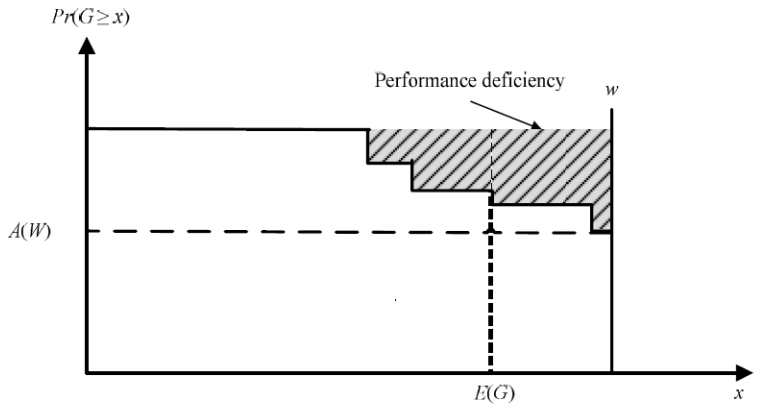

Figure 2.2: MSS steady-state cumulative output performance curve (Lisnianski, A. et al.(2010)

As previously indicated, a steady-state distribution of the variable demand can be represented by two vectors $\mathbf{d}$ and $\mathbf{q}$, where $\mathbf{d} = \{d_1, ..., d_M\}$ is the vector of possible demand levels $d_j,\ j = 1, ..., M$ and $\mathbf{q} = \{q_1, ..., q_M\}$ is the vector of steady-state probability of the respective demand levels $q_j = \mathbb{P}\{D = d_j\},\ j = 1, ..., M$.

In this situation, the steady-state availability index can be calculated as follows:

$$A_\infty(\mathbf{d},\mathbf{q}) = \sum_{m=1}^{M} A(d_m)q_m = \sum_{m=1}^{M} q_m \sum_{k=1}^{K} p_k \mathbf{1}(F(o_k, d_m) \geq 0), \tag{2.24}$$

$$q_m = \frac{T_m}{\sum\limits_{m=1}^{M} T_m} = \frac{T_m}{T},\ m = 1, ..., M. \tag{2.25}$$

The index $1 - A(d, q)$ is frequently used in power engineering and is called the loss of load probability **(LOLP)**. Here, the MSS output performance is regarded as generating capacity of the power system, while its demand is regarded as a load.

Now, we can use the output performance expectation to generate indices that characterize the average MSS output performance. At time t, the mean value of MSS instantaneous output performance is defined by the following form

$$O_{mean}(t) = E\{O(t)\}. \tag{2.26}$$

The steady-state expected output performance takes the following form, if the long-run probabilities $p_k = \lim\limits_{t\to\infty} p_k(t)$ there exist,

$$O_\infty = \sum_{k=1}^{K} p_k o_k. \tag{2.27}$$

For a certain time interval $[0, T]$, the average MSS expected output performance is determined as

$$O_T = \frac{1}{T} \int_0^T O_{mean}(t)dt. \tag{2.28}$$

Note that the mean MSS output performance is independent to demand.

A conditional expected output performance is employed in various instances. This index indicates MSS's mean output performance under the assumption that it is in an acceptable state. It takes the following form in the steady-state:

$$O_\infty^* = \frac{\sum_{k=1}^{K} o_k p_k \mathbf{1}(F(o_k, D) \geq 0)}{\sum_{k=1}^{K} p_k \mathbf{1}(F(o_k, d_m) \geq 0)}. \tag{2.29}$$

When a demand is not met, knowing the measure of system output performance deviation from that demand is useful. The instantaneous output performance deviation can be expressed as follows, in the instance where $F(O(t), D(t)) = O(t) - D(t)$

$$D(t, d) = max\{D(t) - O(t), 0\}. \tag{2.30}$$

and is named the instantaneous output performance deficiency at instant t.

Since $D(t, d)$ is a random variable at time instant t, the following measurements can be used to describe it:

• The probability that at instant t $D(t, d)$ is less than a specified level d:

$$\mathbb{P}\{D(t, d) \leq d\}, \tag{2.31}$$

▷ the mean value of the MSS output performance deficiency (deviation) at instant t:

$$D_m(t, d) = E\{D(t, d)\}. \tag{2.32}$$

The output performance deficiency is not a function of time, when the MSS is in a steady state and demand is constant $D(t) = d$. It may be calculated from the system steady-state output performance distribution as follows

$$D_\infty = \sum_{k=1}^{K} p_k \ max(d - o_k, 0). \tag{2.33}$$

For a variable demand represented by two vectors **(d,q)**, steady-state output performance deficiency D_∞ takes the following form

$$D_{\infty}(\mathbf{d},\mathbf{q}) = \sum_{m=1}^{M} \sum_{i=1}^{K} p_k q_m \ max(d_m - o_i, 0). \tag{2.34}$$

For a certain time interval $[0, T]$, the average MSS expected output performance deficiency is determined by the following form:

$$D_T = \frac{1}{T} \int_0^T D_t \ dt. \tag{2.35}$$

The cumulative output performance deficiency for certain time interval $[0, T]$ is determined as follows:

$$D_{\sum T} = \int_0^T D(t, d) \ dt. \tag{2.36}$$

When the system uses storage resources to accumulate a product, the instantaneous output performance deficiency makes no sense. The deficiency develops when the accumulated output performance in interval $[0, T]$ is less than the accumulated demand at this interval, rather that when the system output performance does not satisfy the demand. The following expression is the accumulated output performance deficiency

$$D_{\sum T} = \int_0^T (D(t) - O(t))dt = \int_0^T D(t)dt - \int_0^T O(t))dt. \tag{2.37}$$

Below, we determine characteristics when $D_{\sum T}$ is a random variable

• Throughput availability is the probability that a random $D_{\sum T}$ is less than a certain level l,

$$\mathbb{P}\{D_{\sum T} \leq l\}. \tag{2.38}$$

▷ the expected amount of the product not given to consumers during the time interval $[0, T]$:

$$D_{\sum m} = E\{D_{\sum T}\}. \tag{2.39}$$

2.2 Modern Stochastic Process Methods for MSS Reliability Assessment

Although the classical iid case simplifies analysis, it is frequently unjustified, and we are compelled to assume some level of dependence. The first-order dependence, also known as Markov dependence, is the simplest and most significant type of dependence.

Definition 2.2: (Lisnianski, A. et al.(2010)) *A stochastic process* $\{X(t)|t \geq 0\}$ *is called a Markov process if for any* $t_0 < t_1 < ... < t_{n-1} < t_n < t$ *the conditional distribution of* $X(t)$ *for given values of* $X(t_0), ..., X(t_n)$ *depends only* $X(t_n)$:

$$\mathbb{P}\{X(t) \leq x|X(t_n) = x_n,\ X(t_{n-1}) = x_{n-1}, ..., X(t_1) = x_1,\ X(t_0) = x_0\} = \mathbb{P}\{X(t) \leq x|X(t_n) = x_n\}. \tag{2.40}$$

This is a broad concept that applies to Markov processes in which the state space is continuous. Discrete-state Markov processes or Markov chains are commonly used to investigate multi-state system reliability.

The probability of the random variable at time $t > t_n$ in the Markov process is determined by the random variable's value at t_n, but not by the process;s realization of the prior to t_n. Particularly, given the current state of the process, the state probabilities at a future instant are independent of the states occupied in the past. Thus, this process is also known as "memoryless."

The conditional distribution (2.40) has invariance with regard to the time origin t_n:

$$\mathbb{P}\{X(t) \leq x|X(t_n) = x_n\} = \mathbb{P}\{X(t - t_n) \leq x|X(0) = x_n\}. \tag{2.41}$$

A Markov process of this type is considered to be homogenous.

Moreover, two necessary stochastic processes that will be employed in the future are assumed here: point and renewal processes.

A set of random points t_i on the time axis is known as a *point process*. A stochastic process $X(t)$ equal to the figure of points t_i in the interval $(0, t)$ can be associated with each point process. Point processes are commonly employed in reliability theory to characterize the occurrence of events across time.

The *Poisson process* is an important example of point processes. Poisson points are commonly used to introduce the Poisson process. These points are related with several events, and the number $N(t_1, t_2)$ of the points in an interval (t_1, t_2) of length $t = t_2 - t_1$ is a Poisson random variable with parameter λt, where λ is the average rate of occurrence of the events:

$$\mathbb{P}\{N(t_1, t_2) = k\} = \frac{e^{-\lambda t}(\lambda t)^k}{k!}. \tag{2.42}$$

The random variables $N(t_1, t_2)$ does not depend on $N(t_3, t_4)$, when the intervals (t_1, t_2) and (t_3, t_4) are not overlapping. Using the points t_i one can form the stochastic process $X(t) = N(0, t)$.

The Poisson process, like the normal distribution in probability theory, plays an necessary role in reliability theory.

The *renewal process* is well-known kind of point process. This process counts occurrences, the intervals between which are independent and identically distributed random

variables. This type of mathematical model is used to characterize the flow of failures through time, in reliability analysis.

Deem X_n is the period between the $(n-1)$st and the nth occurrence. Hence, $(X_n, n \geq 1)$ is a following of nonnegative random variables determined on a probability space $(\Omega, \mathbb{F}, \mathbb{P})$. If $X_n = 0$ then the $(n-1)st$ and the nth occurrence happen at the same time. Consider $(X_n, n \geq 1)$ to be a following of independent identically distributed random variables with the common distribution F, and to avoid trivial details we assume that $F(0) = \mathbb{P}(X_n = 0) < 1$. Let $S_0, S_1, S_2, ...$ be the random variables determined by

$$S_0 = X_0 = 0 \text{ and } S_{n+1} = S_n + X_{n+1},\ n \geq 0.$$

Under these assumptions, the sequence $S = (S_n, n \in \mathbb{N})$ is called a renewal process. The times S_n are called renewal times.

So, F is the common distribution function of X_n , and N_t is the counting process of the renewal process, especially,

$$N_t(\omega) = max\{n \in \mathbb{N} | S_n \leq t\}, t \geq 0,\ \omega \in \Omega.$$

Lemma: (Limnios, N. et al. (2001)) We have:

1. $S_n \to \infty$ (a.s.) as $n \to \infty$.

2. $N_t \to \infty$ (a.s.) as $t \to \infty$.

Limit Theorems for the Counting Process

In the sequel, we will suppose that $\mu = E(X_1) < +\infty$.

Theorem (Strong Law of Large Numbers)(Limnios, N. et al. (2001))

$$\frac{1}{t} N_t \xrightarrow{a.s.} \frac{1}{\mu}\ as\ t \to +\infty.$$

Theorem (Central Limit Theorem)(Limnios, N. et al. (2001))

If $0 < \sigma^2 = Var(X_1) < +\infty$, then,

$$\frac{N_t - t/\mu}{\sqrt{t\sigma^2/\mu^3}} \xrightarrow{d} N(0,1) \text{ as } t \to +\infty.$$

Proof: We have $\{N_t \leq n\} = \{S_n > t\}$. Set $n = [t/\mu + x\sqrt{t\sigma^2/\mu^3}]$, where $[a]$ denotes the integer part of a.

We can write,

$$\lim_{t\to\infty}\mathbb{P}(N_t < n) = \lim_{t\to\infty}\mathbb{P}\left(\frac{N_t - t/\mu}{\sqrt{t\sigma^2/\mu^3}} \leq \frac{n - t/\mu}{\sqrt{t\sigma^2/\mu^3}}\right) = \lim_{t\to\infty}\mathbb{P}\left(\frac{N_t - t/\mu}{\sqrt{t\sigma^2/\mu^3}} \leq x\right)$$

for all continuity points of the limiting distribution.

On the other hand, by the CLT for i.i.d. random variables, we get

$$\lim_{t\to\infty}\mathbb{P}(S_n < t) = \lim_{t\to\infty}\mathbb{P}\left(\frac{S_n - n/\mu}{\sqrt{n\sigma^2}} > \frac{t - n/\mu}{\sqrt{n\sigma^2}}\right) = \lim_{t\to\infty}\mathbb{P}\left(\frac{S_n - n/\mu}{\sqrt{n\sigma^2}} > -x\right) = \Phi(x).$$

2.2.1 Markov Models: Discrete-time Markov Chains

Basic Definitions and Properties

A Markov process, as previously stated, is a stochastic process whose potent behavior is such that the probability distribution for its future evolution is dependent on the present state and not on how the process entered at that state.

The Markov process is established as a Markov chain when the state space, $\mathbf{E}$, is discrete, meaning finite or countably infinite. Consider without losing generality that $\mathbf{E} = \{0, 1, 2, 3, ...\}$ when the state space is discrete and countable.

We have a discrete-time Markov chain if the parameter space, $\mathbf{T}$, is also discrete. Therefore, we let $\mathbf{T} = \{0, 1, 2, 3, ...\}$ because the parameter space is discrete. So, a following of random variables $J(0) = j_0, J(1) = j_1, J(2) = j_2, ...$, with $j_0, j_1, j_2, ...$ integer numbers, characterize a Markov chain $\{J(n), n = 0, 1, 2, ...\}$.

Now, we denote $J(n) = j$, the state of the system at time step n is j and J_0, the initial state of the system at time step 0. By using these determinations in similarly with (2.40), the Markov property can be designated as follows

$$\begin{aligned}&\mathbb{P}\{J(n) = j_n | J_0 = j_0, J_1 = j_1, ..., J_{n-1} = j_{n-1}\} = \\ &\mathbb{P}\{J(n) \leq j_n | J_{n-1} = j_{n-1}\}.\end{aligned} \tag{2.43}$$

Equation (2.43) suggests that chain behavior in the future is dependent on its present state and is independent on its past behavior, just as it is in the instance of a general Markov process. It can be expressed as follows:

$$\mathbb{P}(FUTURE|PAST, PRESENT) = \mathbb{P}(FUTURE|PRESENT)$$

Definition:(stopping time or Markov time) (Barbu, V. et al. (2009)). *A random variable T, determined on $(\Omega, \mathbb{F}, \mathbb{P})$, with values in $\bar{\mathbb{N}} = \mathbb{N} \cup \{\infty\}$, is called a stopping time with respect to the sequence $(X_n)_{n\in\mathbb{N}}$ if the occurrence of the event $\{T = n\}$ is defined by the past of the chain up to time n, $(X_k; k \leq n)$. Especially, let $\mathbb{F}_n = \sigma(X_0, ..., X_n), n \geq 0$, be the σ-algebra generated by $X_0, ..., X_n$, i.e., the information known at time n. The random variable T is called a stopping time if, for every $n \in \mathbb{N}$, $\{T = n\} \in \mathbb{F}_n$.*

Definition:(strong Markov property)(Barbu, V. et al. (2009)) *The Markov chain* $(X_n)_{n\in\mathbb{N}}$ *is said to have the strong Markov Property if, for any Markov time* T*, for any integer* $m \in \mathbb{N}$ *and state* $j \in E$ *we have*

$$\mathbb{P}(X_{m+T} = j | X_k, k \leq T) = \mathbb{P}_{X_T}(X_m = j) \ a.s.$$

Proposition: (Barbu, V. et al. (2009)) *Any Markov chain has the strong Markov property.*

Now, we need to define $p_j^{(n)}$, the probability than the chain will be in state j at step n. As a result, one can be write

$$p_j^{(n)} = \mathbb{P}\{J_n = j\}. \tag{2.44}$$

Since, we have given the above definition, we can designate the conditional probability $p_{ij}(m, n)$. This probability describe the chain's transition to state j at step n if at step m it was in state i. Thus,

$$p_{ij}^{(m,n)} = \mathbb{P}\{J(n) = j | J(m) = i\}, 0 \leq m \leq n. \tag{2.45}$$

Note that the above equation $p_{ij}^{(m,n)}$ is common as the *transition probability function* of the Markov chain.

The homogeneous Markov chains are those in which the $p_{ij}(m, n)$ is dependent only on difference $n - m$. The simpler notation is used for these chains is:

$$p_{ij}^{(n)} = \mathbb{P}\{J(m + n) = j | J(m) = i\}, 0 \leq m \leq n \tag{2.46}$$

The equation (2.46) is called *n-step transition probabilities* and it is the probability that a homogeneous Markov chain will move from state i to state j in precisely n steps.

For $n = 1$, the homogeneous Markov chains will become:

$$p_{ij}^{(1)} = \mathbb{P}\{J(m + 1) = j | J(m) = i\} = p_{ij}. \tag{2.47}$$

The probabilities p_{ij} are called one-step transition probabilities. The following transition (one-step) probability matrix $\mathbf{P}$ can be used to compact the one-step transition probabilities.

$$\mathbf{P} = [p_{ij}] = \begin{bmatrix} p_{00} & p_{01} & \cdots & p_{0M} \\ p_{10} & p_{11} & \cdots & p_{1M} \\ p_{20} & p_{21} & \cdots & p_{2M} \\ \cdots & \cdots & \cdots & \cdots \\ p_{M0} & p_{M1} & \cdots & p_{MM} \end{bmatrix}. \tag{2.48}$$

Therefore, the matrix $\mathbf{P}$ is a *stochastic matrix* and each row adds up to 1, for all $i, j \in \mathbf{E},\ 0 \leq p_{ij} \leq 1$.

The initial probability row-vector is the probability mass function of the random value $J(0)$ and produce the initial conditions of a Markov chain.

$$\mathbf{p}(0) = [p_0(0), p_1(0), ..., p_M(0)] \tag{2.49}$$

The state-transition diagram of the Markov chain is a guided graph that provides an equivalent designation of the Markov chain. The state i is stood from a node tagged i in the state diagram, and the associated one-step transition probability from state i to state j is stood from a branch tagged p_{ij} from node i to node j.

Computation of n-step Transition Probabilities and State Probabilities

The task being assumed here is to derive an expression for calculating the n-step transition probability $p_{ij}^{(n)}$ from the one-step transition probabilities $p_{ij} = p_{ij}^{(1)}$. The homogeneous Markov chain according to expression (2.46) can be rewritten as:

$$p_{ij}^{(n)} = \mathbb{P}\{J(m+n) = j | J(m) = i\}, 0 \leq m \leq n. \tag{2.50}$$

Given that the process is in state i at the 0 step, we assume the transition probability $p_{ij}^{(m+n)}$ that the process will transition to state j at the $(m+n)$ step.

To get to state j at the $(m+n)$ step the process first gets some transitional state k at step m with probability $p_{ik}^{(m)}$, later goes from k to j at step $(m+n)$ with probability $p_{kj}^{(n)}$. The Markov property implies that there are two independent occurrences. Then using the theorem of total probability we get

There are two independent occurrences, according to the Markov property. Then, using the total probability theorem, we get

$$p_{ij}^{(m+n)} = \sum_k p_{ik}^{(m)} p_{kj}^{(n)},\ i, j \in E. \tag{2.51}$$

The above equation is a variant of the well-known Chapman–Kolmogorov equation and allows active calculation of the n-step transitions probabilities.

The matrix of n-step probabilities or, especially, the matrix whose (i, j) entry is $p_{ij}^{(n)}$, is referred to as $\mathbf{P}^{(n)}$.

Thus, we define $m = 1$ and substitute n by $n - 1$ in equation (2.51), then can be rewritten in matrix form:

$$\mathbf{P}^{(n)} = \mathbf{P} \cdot \mathbf{P}^{(n-1)} = \mathbf{P}^n, \tag{2.52}$$

where $\mathbf{P}$ is the one-step probabilities of the Markov chain. Briefly, the n-step transition probability matrix is the nth power of the one-step transition probability matrix.

The unconditional state probabilities $p_j^{(n)}$ can be investigated according to the achieved results. Their values are determined by the number of steps taken since $n = 0$ and on the probabilities of the initial state at $n = 0$. It can be written by the following expression:

$$\begin{aligned} p_j^{(n)} &= \mathbb{P}\{J(n) = j\} \\ &= \sum_i \mathbb{P}(J(0) = i)\mathbb{P}(J(n) = j|J(0) = i) = \sum_i p_i^{(0)} p_{ij}^{(n)}. \end{aligned} \tag{2.53}$$

We can rewrite the matrix form expression (2.53) as follows

$$\mathbf{p}^{(n)} = \mathbf{p}^{(0)} \cdot \mathbf{P}^n, \tag{2.54}$$

where $\mathbf{p}^{(0)}$ and $\mathbf{p}^{(n)}$ are the row-vectors of the initial state probabilities and after the nth' step. This means that initial probability vector $\mathbf{p}^{(0)}$ and the one-step transition probability matrix $\mathbf{P}$ determine the unconditional state probabilities of a homogeneous Markov chain, totally.

2.2.2 Markov Models: Continuous-time Markov Chains

Main Definitions and Properties

The continuous-time Markov chain is analogous to the discrete-time Markov chain, with the exception that transitions from one state to another can occur at any time. One important case is a Markov chain with discrete-state and continuous-time. It has discrete set of values $J(t)$, $J(t) \in 1, 2, ...$, and continuous parameter t with the range of values, $t \in [0, \infty)$. In reliability applications the set $\mathbf{E}$ of states is ordinarily finite, $\mathbf{E} = \{1, 2, ..., K\}$, and so $J(t) \in \{1, 2, ..., K\}$.

For $t_0 < t_1 < ... < t_{n-1} < t_n$, the discrete-state continuous-time stochastic process $\{J(t)|t \geq 0\}$ is a Markov chain if its conditional probability mass function induges the relationship

$$\begin{aligned} &\mathbb{P}\{J(t_n) = j_n|\ J(t_{n-1}) = j_{n-1}, ..., J(t_1) = j_1,\ J(t_0) = j_0\} = \\ &\mathbb{P}\{J(t_n) = j_n|J(t_{n-1}) = j_{n-1}\}. \end{aligned} \tag{2.55}$$

The expression (2.55) is simplifies as follows, by using $t = t_{n-1}$ and $t_n = t_{n+1} + \Delta t$

$$\mathbb{P}\{J(t + \Delta t) = i|\ J(t) = j\} = \pi_{ji}(t, t + \Delta t). \tag{2.56}$$

One simple definition is frequently used is the following:

$$\pi_{ji}(t, t + \Delta t) = \pi_{ji}(t, \Delta t).$$

The above conditional probabilities are termed *transition probabilities*. The Markov process is *time-homogeneous*, if the probabilities $\pi_{ji}(t, \Delta t)$ are independent of t, but are dependent of time difference Δt. Also, $\pi_{jj}(t, \Delta t)$ is the probability that no change in the

state will happen in a time interval of length Δt if the process is in state j at the start of the interval. It should be noted that

$$\pi_{ji}(t,t) = \begin{cases} 1, \text{ if } j = i, \\ \\ 0, \text{ otherwise.} \end{cases} \tag{2.57}$$

Using (2.57), a non-negative continuous function $q_j(t)$: may be determined for each j

$$q_j(t) = \lim_{\Delta t \to 0} \frac{\pi_{jj}(t,t) - \pi_{jj}(t,t+\Delta t)}{\Delta t} = \lim_{\Delta t \to 0} \frac{1 - \pi_{jj}(t,t+\Delta t)}{\Delta t} \tag{2.58}$$

so, $\forall j$ and $i \neq j$ a non-negative continuous function $q_{ji}(t)$ given by the following equation:

$$q_{ji}(t) = \lim_{\Delta t \to 0} \frac{\pi_{ji}(t,t) - \pi_{ji}(t,t+\Delta t)}{\Delta t} = \lim_{\Delta t \to 0} \frac{\pi_{ji}(t,t+\Delta t)}{\Delta t} \tag{2.59}$$

The *transition intensity* from state i to state j at time t is defined by the function $q_j i(t)$. The transition intensities of homogeneous Markov processes are independent of t, thus are fixed.

If the process is in state j at a given time, either a transition from j to several state i occurs in the following Δt time interval, or the process stays in state j. Thus

$$\pi_{jj}(\Delta t) + \sum_{i \neq j} \pi_{ji}(\Delta t) = 1. \tag{2.60}$$

By combining (2.58) and (2.60) and declaring $q_{jj} = -q_j$ we get

$$q_{jj} = -q_j = \lim_{\Delta t \to 0} -\frac{1}{\Delta t} \sum_{i \neq j} \pi_{ji}(\Delta t) = -\sum_{i \neq j} q_{ji}. \tag{2.61}$$

The $p_i(t)$ are state probabilities of $J(t)$ at time t:

$$p_i(t) = \mathbb{P}\{J(t) = i, \} \; j = 1, ..., K, \; t \geq 0. \tag{2.62}$$

The above equation (2.62) designates the probability mass function (pmf) of $J(t)$ at time t.

Therefore, the process must be in one of K states, at any given time, $t \geq 0$.

$$\sum_{i=1}^{K} p_i(t) = 1 \tag{2.63}$$

We can phrase the probability mass function of $J(t)$ in terms of the transition probabilities $\pi_{ij}(t_1, t)$ and the probability mass function of $J(t_1)$ using the theorem of total probability, for $t > t_1$:

$$\begin{aligned} p_j(t) &= \mathbb{P}\{J(t) = j\} = \sum_{i \in S} \mathbb{P}\{J(t) = j | J(t_1) = i\} \mathbb{P}\{J(t_1) = i\} \\ &= \sum_{i \in S} \pi_{ij}(t_1, t) p_i(t_1). \end{aligned} \tag{2.64}$$

In equation (2.64) we declare $t_1 = 0$ and get this:

$$p_j(t) = \sum_{i \in S} \pi_{ij}(0, t) p_i(0). \tag{2.65}$$

The last result means that the transition probabilities $\pi_{ij}(0, t)$ and the vector of initial probability $\mathbf{p}(0) = [p_1(0), ..., p_K(0)]$ define the continuous-time Markov chain's probabilistic behavior in the future.

The transition probabilities of a Markov chain $\{J(t) | t \geq 0\}$ with continuous-time indulge the equation of Chapman–Kolmogorov $\forall i, j \in E$, which can be stated in the subsequent equation:

$$\pi_{ij}(t_1, t) = \sum_{k \in S} \pi_{ik}(t_1, t_2)\ \pi_{kj}(t_2, t), \quad 0 \leq t_1 \leq t_2 \leq t. \tag{2.66}$$

The theorem of total probability is used to prove this form:

$$\begin{aligned} &\mathbb{P}\{J(t) = j | J(t_1) = i\} \\ &= \sum_{k \in S} \mathbb{P}\{J(t) = j | J(t_2) = k, J(t_1) = i\} \mathbb{P}\{J(t_2) = k | J(t_1) = i\}. \end{aligned} \tag{2.67}$$

Then, the Markov property (2.56) is applied to expression (2.67) yielding (2.66).

At instant $t + \Delta t$, the state probabilities can be stated according to state probabilities at instant t by using the following formula:

$$p_j(t + \Delta t) = p_j(t)\big[1 - \sum_{i \neq j} q_{ij} \Delta t\big] + \sum_{i \neq j} p_i(t) q_{ij} \Delta t,\ i, j = 1, ..., K. \tag{2.68}$$

The following principles can be used to obtain Equation (2.68). There are two ways for the process to reach state j at instant $t + \Delta t$.

The first way is: at instant t, the process may already be in state j and it does not exit from this state until the moment $t + \Delta t$. The probabilities of these happenings are $p_i(t)$ and $1 - \sum_{i \neq j} q_{ji} \Delta t$, respectively.

The second way is: the process may be in one of the states $i \neq j$ at instant t and transits from state i to state j during time Δt. The probability of these events is $p_i(t)$ and $q_{ij}\Delta t$, respectively. Because the process can reach state j from any state i, these probabilities should be multiplied and summarized $\forall i \neq j$.

So, can be rewritten (2.68) as (2.65), yielding the following formula:

$$p_j(t+\Delta t) = p_j(t)\big[1 + q_{jj}\Delta t\big] + \sum_{i \neq j} p_i(t) q_{ij} \Delta t \tag{2.69}$$

or, equivalently

$$\begin{aligned} &p_j(t+\Delta t) - p_j(t) = \\ &\sum_{i=1, i\neq j}^{K} p_i(t) q_{ij}\Delta t + p_j(t) q_{jj}\Delta t = \sum_{i=1, i\neq j}^{K} p_i(t) q_{ij}\Delta t - p_j(t) \sum_{i=1, i\neq j}^{K} q_{ji}\Delta t \end{aligned} \tag{2.70}$$

The following result is the division of both sides of the form (2.70) by Δt and passing to limit $\Delta t \to 0$

$$\frac{dp_j(t)}{dt} = \sum_{i=1, i\neq j}^{K} p_i(t) q_{ij} - p_j(t) \sum_{i=1, i\neq j}^{K} q_{ji}, \;\; j = 1, 2, ..., K. \tag{2.71}$$

For the homogeneous Markov process's state probabilities $p_j(t), \;\; j = 1, ..., K$ used the (2.71) system of differential equations when the initial conditions are obtained as

$$p_j(t) = q_j, \;\; j = 1, ..., K. \tag{2.72}$$

For a state-transition diagram for continuous-time Markov chain, Equation (2.71) can be written using the following rule: The sum of the probabilities of the states that have transitions to state j multiplied by the analogous transition intensities minus the probability of state j multiplied by the sum of all transition intensities from state j, is the time derivative of $p_j(t)$ for any state j.

Producing the row-vector $\mathbf{p}(t) = [p_1(t), p_2(t), ..., p_K(t)]$ and the transition intensity matrix $\mathbf{q}$

$$\mathbf{q} = \begin{bmatrix} q_{11} & q_{12} & \cdots & q_{1K} \\ q_{21} & q_{22} & \cdots & q_{2K} \\ \cdots & \cdots & \cdots & \cdots \\ q_{K1} & q_{K2} & \cdots & q_{KK} \end{bmatrix}, \tag{2.73}$$

the diagonal elements of the above matrix are designated as $q_{jj} = -q_j$, system (2.71) can be rewritten in matrix notation:

$$\frac{dp(t)}{dt} = \mathbf{p}(t)\mathbf{q}. \tag{2.74}$$

Also, the elements of matrix of each row add up to 0: $\sum_{j=1}^{K} q_{ij} = 0, \forall i \ (1 \leq i \leq K)$.

When the system state transitions are caused by failures and repairs of its elements, the respective transition intensities are phrased by the element's failure and repair rates.

When failures and repairs of the system's elements create state transitions, the corresponding transition intensities are defined by the element's failure and repair rates.

The instantaneous conditional density of the probability of failure of an initially functional element at time t assuming that the element has not failed up to time t, is the element's failure rate $\lambda(t)$.

Briefly, $\lambda(t)$ is the time-to failure conditional probability density function. It phrases a hazard of failure in time instant t with the assumption that no failure occured before to t. The failure rate of an element at time t is designated as

$$\lambda(t) = \lim_{\Delta t \to 0} \frac{1}{\Delta t} \left[\frac{F(t + \Delta t) - F(t)}{R(t)} \right] = \frac{f(t)}{R(t)}, \tag{2.75}$$

where $F(t)$ is the Cumulative Distribution Function and $f(t)$ is probability density function of the time to failure of the element. The reliability function of the element is $R(t) = 1 - F(t)$.

The failure rate for homogeneous Markov processes is independent of t and given as

$$\lambda = MTTF^{-1}, \tag{2.76}$$

where MTTF so-called the mean time to failure, which we defined in the previous section. In the same way, $\mu(t)$ called the repair rate and is the time-to-repair conditional probability density function. The $\mu(t)$ is independent of t, for homogeneous Markov processes and given as

$$\mu = MTTR^{-1}, \tag{2.77}$$

where MTTR so-called the mean time to repair, which we defined in the previous section.

A state j is said to be *reachable* from state i if for some $t > 0$, $\pi_{ij}(t) > 0$ and is said to be an *absorbing state*, if once entered, the process is appropriated to exist in that state. If every state is reachable from every other state, a continuous-time Markov chain is called *irreducible*.

Definition: (irreducicle Markov chain) (Barbu, V. et al. (2009)) *If for any states i, j there is a positive integer n such that $p_{ij}^{(n)} > 0$, then the Markov chain is said to be irreducible.*

In many applications, the steady-state probabilities $p_i = \lim_{t\to\infty} p_i(t)$ are of interest. The following limits there are at any state $i \in E$, for an irreducible continuous-time Markov chain

$$p_i = \lim_{t\to\infty} p_i(t) = \lim_{t\to\infty} \pi_{ij}(t) = \lim_{t\to\infty} \pi_i(t) \tag{2.78}$$

and they do not depend on the initial state $j \in E$. The process is named *ergodic* when there is the steady-state probabilities. The computations are more simpler for the steady-state probabilities. The constant probabilities all time-derivatives are equal to 0, thus the set of differential equations (2.71) is reduced to a set of K algebraic linear equations, so $\frac{dp_i(t)}{dt} = 0,\ i = 1, ..., K$.

Definition: (ergodic theorem for Markov chains) (Barbu, V. et al. (2009)) *For an ergodic Markov chain we have*

$$p_{ij}^n \underset{n\to\infty}{\longrightarrow} \nu(j)$$

where ν is steady-state probability distribution, for any $i, j \in E$

For this instance in steady state $p_i = \lim_{t\to\infty} p_i(t)$, all derivatives of state probabilities on the left-hand side of (2.71) will be zeroes. The following system of algebraic linear equations should be solved in order to get the long-run probabilities:

$$0 = \sum_{i=1, i\neq j}^{K} p_i(t) q_{ij} - p_j(t) \sum_{i=1, i\neq j}^{K} q_{ji},\ j = 1, ..., K. \tag{2.79}$$

In (2.79) K equations are not linearly independent. The fact that the state probabilities add up to 1 at any moment provides an additional independent equation:

$$\sum_{i=1}^{K} p_i = 1. \tag{2.80}$$

Therefore, long-run probabilities of ergodic continuous-time Markov chains can be derived using (2.79) and (2.80).

Now, we need to assume two more significant steady state parameters of the process: one of them is the state frequency and the other is mean time of staying in state. The predicted number of arrivals into the state i per unit time determines the frequency f_i. The term frequency is connected with the process's long-run behavior. Before we going to associate the frequency, mean time of staying in state i and probability, we will assume the system evolution in the state space consist of two changable periods. These periods are: the stays in i and outside i.

Thus, we introduce again the process by two states. Define $\bar{T}_i$ as the mean duration of the stays in state i and $\bar{T}_{oi}$ as the mean duration that stays outside i. Also $\bar{T}_{ci}$ is called the mean cycle time.

$$\bar{T}_{ci} = \bar{T}_i + \bar{T}_{oi}. \tag{2.81}$$

From the designation of the state frequency it follows that, in the steady-state, f_i equals the reciprocal of the mean cycle time

The fi equals the reciprocal of the mean cycle time in the steady-state, based on the state frequency designation.

$$f_i = \frac{1}{\bar{T}_{ci}}. \tag{2.82}$$

When both sides of the form multiplied by $\bar{T}_i$, one takes

$$\bar{T}_i f_i = \frac{\bar{T}_i}{\bar{T}_{ci}} = p_i. \tag{2.83}$$

Therefore,

$$f_i = \frac{p_i}{\bar{T}_i}. \tag{2.84}$$

The last result is a basic equation, which gives the relationship between the three state parameters in the steady state.

The minimum of all random values T_{ij} is the unconditional random value T_i. The T_{ij} describe the conditional random time of staying in state i if the transition is completed from state i to any state $j \neq i$.

$$T_i = min\{T_{i1}, ..., T_{ij}\}. \tag{2.85}$$

All conditional times T_{ij} follow exponential distribution with the cdf:

$$F_{ij}(T_{ij} \leq t) = 1 - e^{-\lambda_{ij}t}. \tag{2.86}$$

All transitions from state i do not depend and then, the cdf of unconditional time T_i of staying in state i can be figured out as follows:

$$\begin{aligned} F_i(T_i \leq t) &= 1 - \mathbb{P}\{T_i > t\} = 1 - \prod_{j \neq i} \mathbb{P}\{T_{ij} > t\} \\ &= 1 - \prod_{j \neq i} [1 - \mathbb{P}\{T_{ij} \leq t\}] \\ &= 1 - \prod_{j \neq i} [1 - F_{ij}(T_{ij} < t)] = 1 - \prod_{j \neq i} e^{-\lambda_{ij}t} \\ &= 1 - e^{-\sum\limits_{j \neq i} \lambda_{ij}t}. \end{aligned} \tag{2.87}$$

As a result, the unconditional time T_i follows exponential distribution with parameter $\lambda = \sum_j \lambda_{ij}$, and the mean time of staying in state i given by:

$$\bar{T_i} = \frac{1}{\sum\limits_{j \neq i} \lambda_{ij}}. \tag{2.88}$$

Replacing $\bar{T_i}$ in expression (2.84) we finally take

$$f_i = p_i \sum_{j \neq i} \lambda_{ij}. \tag{2.89}$$

Reliability measures are commonly computed as equivalent functionals of state probabilities, p_i or $p_i(t)$.

Markov Models for Evaluating the Reliability of Multi-State Elements

Here, when we treat with a single multi-state element, we can skip index j for the definition of a set of the element's output performance rates. As a result, this set is referred as $o = \{o_1, ..., o_k\}$. We further consider that this set is ordered such that $\forall i$, $o_{i+1} \geq o_i$.

Two groupings of elements can be identified. The first group includes elements that are only seen until they fail. These elements are either unrepairable or uneconomical to repair, or just the life history up to the first failure is of relevance. The second group includes elements that are repaired after they fail and have functioning and repair periods in their life cycles. Both classes are described in the following subsections.

Non-repairable Multi-State Element

As previously stated, the lifetime of a non-repairable element endures until it enters the subset of unacceptable states for the first time. The acceptability of an element's state is dependent of the relationship between the element's output performance and the demand.

The demand is denoted by $D(t)$ and it is a random process that gets discrete values from the set $d = \{d_1, ..., d_M\}$. The acceptability function $F(O(t), D(t))$ can be used to define the desired relationship between the demand and the system output performance.

Assume a multi-state element with only minor failures determined as those that cause the element to move from state i to state i–1. Briefly, a minor failure has a small impact on element output performance. Figure 2.3 shows the state-space diagram for such an element.

The only output performance demotion specified by the stochastic process $\{O(t)|t \geq 0\}$ is element evolution in the state space. Any transition from state i to state i–1 is a transition intensity $\lambda_{i,i-1},\ i = 2, ..., k$.

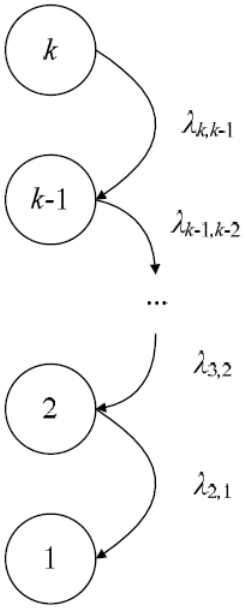

Figure 2.3: State-transition diagram for non-repairable element with minor failures (Lisnianski, A. et al.(2010)).

A process is continuous-time Markov chain when the sojourn time in any state i follows exponential distribution with parameter $\lambda_{i,i-1}$. Furthermore, it is well-known *pure death process*. Let's designate the auxiliary discrete-state continuous time stochastic process $\{J(t)|t \geq 0\}$, with $J(t) \in \{1, ..., k\}$. The stochastic process $\{O(t)|t \geq 0\}$ is closely connected with this process.

When $J(t) = i$, the corresponding output performance rate of a multi-state element is $o_i : O(t) = o_i$. The process $J(t)$ is a discrete-state stochastic process decreasing by 1 at the points t_i, $i = 1, \ldots, k$, when the corresponding transitions occur. The state probabilities of $J(t)$ are

When $J(t) = i$, a multi-state element's output performance rate is $o_i : O(t) = o_i$. When the corresponding transitions happen, the process $J(t)$ is a discrete-state stochastic process that decreases by 1 at the points t_i, $i = 1, \ldots, k$. The state probabilities of $J(t)$ are as follows:

$$p_i(t) = \mathbb{P}\{J(t) = i\}, \ i = 1, ..., k \text{ for } t \geq 0. \tag{2.90}$$

As may be seen,

$$\sum_{i=1}^{k} p_i(t) = 1. \tag{2.91}$$

for any $t \geq 0$, because the process must be in some state at all time.

In order to calculate state probabilities for the Markov process given in Figure 2.3, we can use the system (2.71) to write the following differential equations:

$$\begin{cases} \frac{dp_k(t)}{dt} = -\lambda_{k,k-1} \cdot p_k(t), \\ \\ \frac{dp_i(t)}{dt} = \lambda_{i+1,i} \cdot p_{i+1}(t) - \lambda_{i,i-1} \cdot p_i(t), \ i = 2, 3, ..., k-1, \\ \\ \frac{dp_1(t)}{dt} = \lambda_{2,1} \cdot p_2(t). \end{cases} \tag{2.92}$$

Note that there is only one transition from state k to k–1 with the intensity of $\lambda_{k,k-1}$ and no transitions to state k. In each state i, $i = 2, 3, \ldots, k-1$, there is one transition to state i from the state $i+1$ with the intensity $\lambda_{i+1,i}$ and one transition from state i to state i–1 with the intensity $\lambda_{i,i-1}$.

One can see that, from state 1, there are not transitions. That is to say, if a process reaches this state, it never leaves it. So, the state 1 is absorbing state for non repairable multi state elements.

Consider that the process starts from best state k with a maximum element output performance rate of o_k. Thus, the initial conditions are given as

$$p_k(0) = 1, p_{k-1}(0) = p_{k-2}(0) = ... = p_1(0) = 0. \tag{2.93}$$

Even for large k, one can obtain the numerical solution of the system of differential equations (2.92) under initial conditions (2.93) using readily available software tools.

The Laplace–Stieltjes transform can be used to solve the system (2.92) analytically. The following structure of linear algebraic expressions can be represented (2.92) using this transform and including the initial conditions (2.93):

$$\begin{cases} s\tilde{p}_k(s) - 1 = -\lambda_{k,k-1} \cdot \tilde{p}_k(s), \\ \\ s\tilde{p}_i(s) = \lambda_{i+1,i} \cdot \tilde{p}_{i+1}(s) - \lambda_{i,i-1} \cdot \tilde{p}_i(s), \ i = 2, 3, ..., k-1, \\ \\ s\tilde{p}_1(s) = \lambda_{2,1} \cdot \tilde{p}_2(s). \end{cases} \tag{2.94}$$

where, the Laplace-Stieltjes transform of a function $p_k(t)$ and of the derivative of a function $p_k(t)$ is $\tilde{p}_k(s) = L\{p_k(t)\} = \int_0^\infty e^{-st} p_k(t)$ and $L\{\frac{dp_k(t)}{dt}\} = s\tilde{p}_k(s) - p_k(0)$, respectively.

The system (2.94) one can rewrite in the following form:

$$\begin{cases} \tilde{p}_k(s) = \frac{1}{s+\lambda_{k,k-1}}, \\ \\ \tilde{p}_i(s) = \frac{\lambda_{i+1,i}}{s+\lambda_{i,k-1}} \cdot \tilde{p}_{i+1}(s), \ i = 2, 3, ..., k-1, \\ \\ \tilde{p}_1(s) = \frac{\lambda_{2,1}}{s} \cdot \tilde{p}_2(s). \end{cases} \tag{2.95}$$

The states $i+1, ..., k$, where the element output performance is up of level o_i are acceptable commonly, where $F(o_i, d) = o_i - d$ for the fixed demand level $o_{i+1} \geq d > o_i$, $i = 1, ..., k-1$.

The probability of the state with the smallest output performance $p_1(t)$ defines the unreliability function of the multi-state element for the constant demand level $o_2 \geq d \geq o_1$. Therefore, the reliability function determined as the probability that the element is not in its worst state (total failure) is

The unreliability function of the multi-state element for the fixed level demand $o_2 \geq d \geq o_1$ is determined by the state probability with the smallest output performance $p_1(t)$. As a result, the reliability function, which is determined as the probability of the first element has complete failure, is

$$R_1(t) = 1 - p_1(t). \tag{2.96}$$

Generally, if the fixed demand is $o_{i+1} \geq o_i,\ i = 1, ..., k-1$, the unreliability function is a sum of the probabilities of the unacceptable states $1, 2, \ldots, i$. Hence, the reliability function for the ith element given by

$$R_i(t) = 1 - \sum_{j=1}^{i} p_j(t). \tag{2.97}$$

For this fixed demand level, the mean time up to multi-state element failure can be defined as the mean time up to the process entrance into state i.

Bellow, we compute the sum of the time periods when the process exist in each state $j > i$. Therefore the process starts from the best state k with the highest element output performance rate o_k.

$$MTTF_i = \sum_{j=i+1}^{k} \frac{1}{\lambda_{j,j-1}},\ i = 1, ..., k. \tag{2.98}$$

According to (2.27), the element mean instantaneous output performance at time t can be calculated as

$$E_t = \sum_{i=1}^{k} o_i p_i(t). \tag{2.99}$$

According to (2.33), the element mean instantaneous output performance deficiency for the fixed demand d can be calculated as

$$D_t = \sum_{i=1}^{k} p_i(t) max(d - o_i, 0). \tag{2.100}$$

Assume a non-repairable multi-state element that can have both major and minor failures. Meaning a major failure is defined as one that leads an element to move from state i to state $j : j < i - 1$. The Figure 2.4 shows the state-space diagram for such an element, which represents transitions corresponding to both minor and major failures.

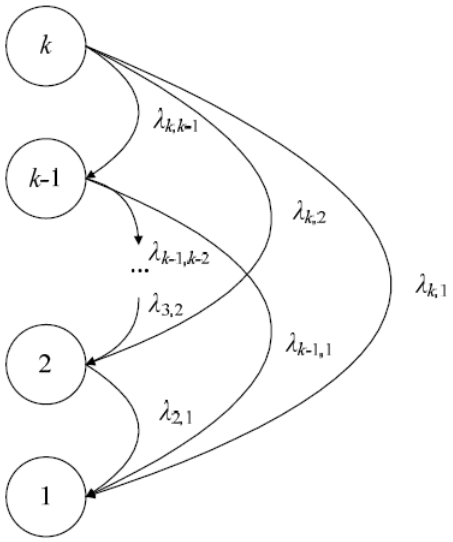

Figure 2.4: State-transition diagram for non-repairable element with minor and major failures (Lisnianski, A. et al. (2010)).

According to Equations (2.71), we can write the differential equations for state probabilities for the continuous-time Markov chain represented by previous state-space diagram with the initial conditions (2.93):

$$\begin{cases} \frac{dp_k(t)}{dt} = -p_k(t) \cdot \sum\limits_{e=1}^{k-1} \lambda_{k,e}, \\ \frac{dp_i(t)}{dt} = \sum\limits_{e=i+1}^{k} \lambda_{e,i} \cdot p_e(t) - p_i(t) \cdot \sum\limits_{e=1}^{i-1} \lambda_{i,e}, \ i = 2, 3, ..., k-1, \\ \frac{dp_1(t)}{dt} = \sum\limits_{e=2}^{k} \lambda_{e,1} \cdot p_e(t). \end{cases} \tag{2.101}$$

The mean instantaneous output performance and the mean instantaneous output performance deficiency may be defined using (2.99) and (2.100), after solving previous system and finding the state probabilities $p_i(t),\ i = 1, ..., k$.

The unavailability of the element with both minor and major failures is defined by the sum of the unacceptable states probabilities. Using the expression (2.97) can be defined the element reliability function, given that for the fixed demand $d(o_i < d \leq o_{i+1})$.

Repairable Multi-State Element

The more general model of a multi-state element is the model with repair. The repairs can also be both minor and major. The repairable model is a more generic model of a multi-state element. It has minor and major repairs. The minor repairs come back an element from state j to state $j + 1$ and the major repairs come back it from state j to state i, where $i > j + 1$.

An element with only minor failures and repairs is a specific instance of the repairable multi-state element. The birth and death process is the stochastic process that based on

this element. The Figure 2.5 (a) shows the state-space diagram of this process and (b) shows the repairable multi-state element with minor and major failures and repairs.

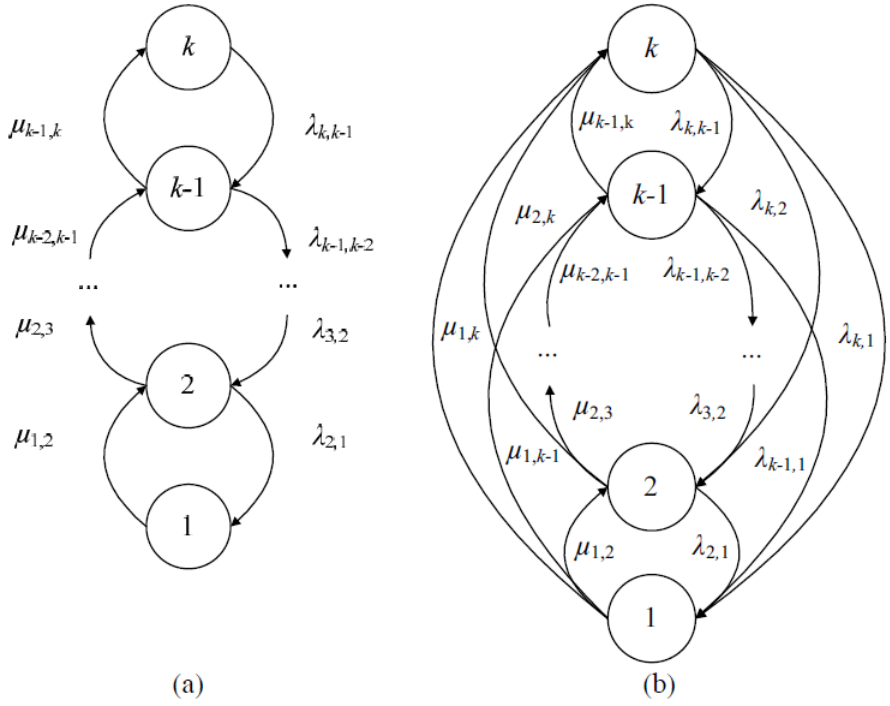

Figure 2.5: State-transition diagrams for repairable element with minor failures and repairs (a) and for repairable element with minor and major failures and repairs (b) (Lisnianski, A. et al.(2010)).

One can write the differential equations' system for the state probabilities of such elements using the initial conditions (2.93). The solve of this system give the state probabilities $p_i(t),\ i = 1, ..., k$..

$$\begin{cases} \frac{dp_k(t)}{dt} = \sum\limits_{e=1}^{k-1} \mu_{e,k} \cdot p_e(t) - p_k(t) \cdot \sum\limits_{e=1}^{k-1} \lambda_{k,e}, \\ \frac{dp_i(t)}{dt} = \sum\limits_{e=i+1}^{k} \lambda_{e,i} \cdot p_e(t) + \sum\limits_{e=1}^{i-1} \mu_{e,i} \cdot p_e(t) - p_i(t) \left(\sum\limits_{e=1}^{i-1} \lambda_{i,e} + \sum\limits_{e=i+1}^{k} \mu_{i,e} \right), \\ i = 2, 3, ..., k-1, \\ \frac{dp_1(t)}{dt} = \sum\limits_{e=2}^{k} \lambda_{e,1} \cdot p_e(t) - p_1(t) \sum\limits_{e=2}^{k} \mu_{1,e}, \end{cases} \tag{2.102}$$

The states $i+1, ..., k$, where the element output performance is up of level o_i are acceptable commonly, where $F(o_i, d) = o_i - d$ for the fixed demand level $o_{i+1} \geq d > o_i,\ i = 1, ..., k-1$. Hence, the instantaneous availability given by

$$A_i(t) = \sum_{e=i+1}^{k} p_e(t). \tag{2.103}$$

Using (2.99) and (2.100) can be defined the element mean instantaneous output performance and output performance deficiency.

For the steady-state probabilities the computations are more simpler. The set of differential equations (2.102) is reduced to a set of k algebraic linear equations because for the fixed probabilities all time-derivatives are equal to 0, hence, $\frac{dp_i(t)}{dt} = 0,\ i = 1, ..., k$.

Let the long-run probabilities $p_i = \lim\limits_{t\to\infty} p_i(t)$. The following system of algebraic linear equations should be solved in order to get the probabilities:

$$\begin{cases} 0 = \sum\limits_{e=1}^{k-1} \mu_{e,k} \cdot p_e - p_k \sum\limits_{e=1}^{k-1} \lambda_{k,e}, \\ \\ 0 = \sum\limits_{e=i+1}^{k} \lambda_{e,i} \cdot p_e + \sum\limits_{e=1}^{i-1} \mu_{e,i} \cdot p_e - p_i \left(\sum\limits_{e=1}^{i-1} \lambda_{i,e} + \sum\limits_{e=i+1}^{k} \mu_{i,e} \right), \\ i = 2, 3, ..., k-1, \\ \\ 0 = \sum\limits_{e=2}^{k} \lambda_{e,1} p_e - p_1 \cdot \sum\limits_{e=2}^{k} \mu_{1,e}, \end{cases} \tag{2.104}$$

In (2.104) k equations are not linearly independent. The fact that the state probabilities add up to 1 at all time provides an other independent equation:

$$\sum_{i=1}^{k} p_i = 1. \tag{2.105}$$

The probability of the event when the element enters the set of unacceptable states for the first time is used to define the reliability function for repairable multi-state elements. It doesn't really matter which one of the unacceptable states is visited initially and how the element operates after the first time it enters the set of unacceptable states.

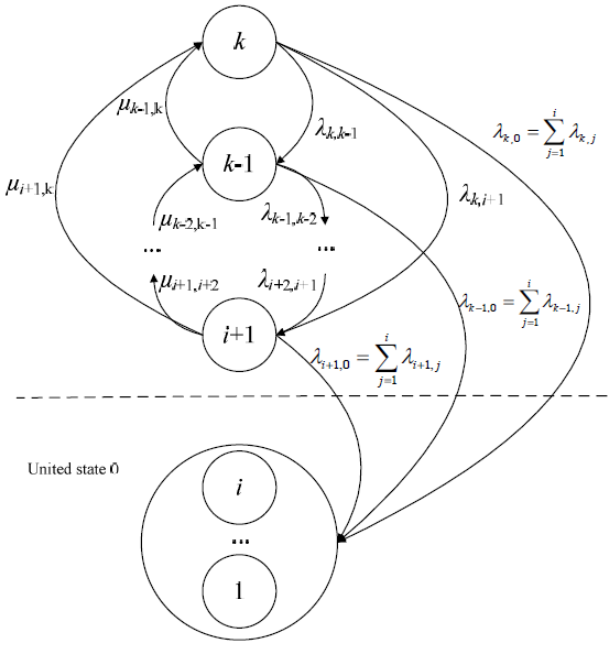

Figure 2.6: State-transition diagram for definition of reliability function $R_i(t)$ for repairable element. (Lisnianski, A. et al.(2010))

In order to we derive the element reliability function $R_i(t)$, for the fixed demand $d(o_i < d \leq o_{i+1})$, we should make a new Markov model.

All states $1, 2, \ldots, i$ of the element corresponding to the output performance rates that are smaller than the demand d should be combined into the same absorbing state. This absorbing state can be assumed as state 0 and all repairs that come back the element to the set of acceptable states should be prohibited.

This corresponds to zeroing all the transition intensities $\mu_{0,m}$ for $m = i + 1, ..., k$.

The sum of the transition rates from state m to all the unacceptable states is defined the transition rate $\lambda_{m,0}$ from all acceptable states $m(m > i)$ to the absorbing state 0:

$$\lambda_{m,0} = \sum_{j=1}^{i} \lambda_{m,j}, \;\; m = k, k-1, ..., i+1. \tag{2.106}$$

The state-transition graph for computation of the reliability function is presented in Figure 2.6. For this graph, the state probability $p_0(t)$ describes the reliability function of the element because after the first entrance into the absorbing state 0 the element never leaves it: $R_i(t) = 1 - p_0(t)$.

Figure 2.6 shows the state-transition graph used to compute the reliability function. In this graph, the state probability $p_0(t)$ describes the reliability function of the element because after the first enter the absorbing state 0 the element cannot leaves: $R_i(t) = 1 - p_0(t)$.

The definition of reliability function of the element introduced by the following system:

$$\begin{cases} \frac{dp_k(t)}{dt} = \sum\limits_{e=i+1}^{k-1} \mu_{e,k} \cdot p_e(t) - p_k(t) \cdot \left(\sum\limits_{e=1}^{k-1} \lambda_{k,e} + \lambda_{k,0} \right), \\ \\ \frac{dp_j(t)}{dt} = \sum\limits_{e=i+1}^{k} \lambda_{e,j} \cdot p_e(t) + \sum\limits_{e=1}^{j-1} \mu_{e,j} \cdot p_e(t) - p_j(t) \left(\sum\limits_{e=i+1}^{j-1} \lambda_{j,e} + \lambda_{j,0} + \sum\limits_{e=i+1}^{k} \mu_{j,e} \right), \\ \text{for } i < j < k \\ \\ \frac{dp_0(t)}{dt} = \sum\limits_{e=i+1}^{k} \lambda_{e,0} \cdot p_e(t). \end{cases} \tag{2.107}$$

In order to solve this system we use initial conditions

$$p_k(0) = 1, p_{k-1}(0) = ... = p_i(0) = p_0(0) = 0. \tag{2.108}$$

after the solving we will get the reliability function as follows

$$R_i(t) = 1 - p_0(t) = \sum_{j=i+1}^{k} p_j(t). \tag{2.109}$$

The ultimate state probabilities for system (2.107) are:

$$p_k = p_{k-1} = ... = p_{i+1} = 0, \ p_0 = 1, \tag{2.110}$$

since the element always entrances into the absorbing state 0 when $t \to \infty$.

The mean time to first failure, when the element output performance degrades from level demand d for the first time, where $o_i < d \leq o_{i+1}$, can be calculated using the computed reliability function.

$$MTTF_i = \int_0^{\infty} R_i(t)dt. \tag{2.111}$$

Reliability measures are commonly computed using of state probabilities p_i or $p_i(t)$.

Markov Models for Evaluating the Reliability of Multi-State Systems

Assume a system comprise various multi-state elements. A singular system state is created by combining the states of these elements. Any system element j can have k_j various states, each corresponding to a different output performance rate, as represented by the set $o_i = \{o_{j1}, ..., o_{jk_j}\}$. At each given time t, the present state of element j and, as a result, the present value of the element output performance rate $O_j(t)$ are random variables. The element output performance rate $O_j(t)$ takes values from $o_j : O_j(t) \in o_j$. For

any element j, the output performance rate is determined as continuous-state Markov process in the time interval $[0, T]$, where T is the MSS functional period.

Based on the general MSS model, we consider that

$$L^n = \{o_{11}, ..., o_{1k_1}\} \times \{o_{21}, ..., o_{2k_2}\} \times ... \times \{o_{n1}, ..., o_{nk_n}\}$$

is a space of achievable combinations of output performance rates for all n system elements and $o = \{o_1, ..., o_K\}$ is a space of available values of these output performance rate for the total system.

The system structure function is determined by the transform $\phi(O_1(t), ..., O_n(t)) : L^n \to o$, which is the space of element output performance rates into the space of system output performance rates at any t. Since, by using the structure function, the total MSS output performance rate can be calculated for each combination of output performance rates of system elements.

The present state of the total MSS and, as a result, the present value of the system output performance rate $O(t)$ at any t are random variables. A continuous-time Markov chain $O(t)$ takes values from $o : O(t) \in o = \{o_1, ..., o_K\}$.

Consider that Markov processes for various elements are independent and that there are no state transitions of any various elements occur at the same time. Especially, a system may be only one failure or one repair at any given time t.

The Markov method to MSS reliability evaluation has two ways: first way is development of the state-space diagram for the total system and second way is the evaluation of the system's reliability based on solving a system of differential equations corresponding to the diagram.

In Markov analysis, particularly for the MSS, the suitable design of the state-transition diagram is crucial. When the modeled system is great enough, the explosion of the number of states remains a major issue. In such instances, a graphical representation of a state space diagram is frequently difficult, so a typical description of the system is one of the possible options. Using this description, we can investigate the state-space graph systematically because we know the rules that control MSS evolution. It's also necessary to keep in mind that the state-space diagram serves just as a guide. The transition intensity matrix $\mathbf{q}$, which specifies the system of differential equations and consequently the related Markov model is the focus.

Hence, we're talking about the formalized generation of the transition intensity matrix (2.74) and, consequently, the Markov model generation. Based on this concept, we present effective algorithms for the reliability evaluation. The following is an achievable algorithm for Markov model generation for the MSS.

Algorithm for the generation of the Markov model

1. Adjustment of the failure and repair rate sets

The given element failure and repair rates should be placed in the following ordered set of failure rates, for each element j of the MSS

$$\left\{\lambda^{(j)}_{k_j,k_j-1}, \lambda^{(j)}_{k_j,k_j-2}..., \lambda^{(j)}_{k_j,1}, \lambda^{(j)}_{k_j-1,k_j-2}, \lambda^{(j)}_{k_j-1,k_j-3}, ..., \lambda^{(j)}_{k_j-1,1}, ..., \lambda^{(j)}_{3,2}, \lambda^{(j)}_{3,1}, \lambda^{(j)}_{2,1}\right\}$$

and the ordered set of repair rates

$$\left\{\mu_{1,2}^{(j)},...,\mu_{1,k_j-1}^{(j)},\mu_{1,k_j}^{(j)},\mu_{2,3}^{(j)},...,\mu_{2,k_j-1}^{(j)},\mu_{2,k_j}^{(1)},...,\mu_{k_j-2,k_j-1}^{(j)},\mu_{k_j-2,k_j}^{(j)},\mu_{k_j-1,k_j}^{(j)}\right\}.$$

If there is no failure that causes a reduction in element output performance from level o_{jm} to level $o_{jm-\Delta m}$ for element j, the associated failure rate $\lambda_{m,m-\Delta m}^{(j)}$ is zero in the failure rate set. Similarly, if there is no repair that rebounds the output performance of j from level $o_{jm-\Delta m}$ to level o_{jm}, the associated repair rate $\mu_{m-\Delta m,n}^{(j)}$ is zero in the repair rate set.

2. Generation of Multi State System states

All the $K = k_1k_2\ldots k_n$ available MSS states are created as various combinations of all the achievable output performance levels of the system's elements. A set $\{o_{1i},...,o_{n1}\},\ i \in [1,k_1],...,l \in [1,k_n]$, of analogous states of the system elements should be allocated to every system state.

3. Enumeration of the system states and the calculation of the Multi State System output performance

We should enumerate all states of system. The enumeration order is not significant for algorithms that based on computer. What is really significant is the relation between the number of states $n_s(n_s \in [1,K])$, the set of output performance rates of elements in this state $\{o_{1i},...,o_{n1}\}$ and the MSS output performance rate o_{n_s} in this state that is defined by the MSS structure function

$$g_{n_s} = \phi(o_{1i},...,o_{n1}),\ n_s = 1,...,K.$$

4. State-transition analysis and producing the transition matrix

The connections between any system state n_s and other states must be defined at this step. Failures and repairs of the system elements define these connections.

Based on the hypothesis that there are no transitions in any various elements in the same time, the transition from an arbitrary system state described by the set of element output performances $\{o_{1i},\ldots,o_{jm},\ldots,o_{nh}\}$ is feasible only to one of the states in which just one of the elements changes its output performance:

$$\{o_{1i},\ldots,o_{jm},\ldots,o_{nh}\} \rightarrow \{o_{1i},\ldots,o_{jf},\ldots,o_{nh}\}$$

where $m \neq f,\ 1 \leq j \leq n$. The element failure represented of the transition in which $f < m$, with transition intensity $\lambda_{m,f}^{(j)}$ and the element repair represented of the transition in which $f > m$, with transition intensity $\mu_{m,f}^{(j)}$.

For the definition of all the transitions in the MSS state-space diagram, all the pairs of system states that differ by the state of a single element must be chosen. The associated transition intensities should be selected from the associated ordered sets, for every pair.

If the MSS transits from state n_1 to state n_2 due to a failure with the intensity of the arbitrary element j, then the element $q_{n_1n_2}$ of transition matrix $\mathbf{q}$ placed in the intersection of row n_1 and column n_2 is

$$q_{n_1n_2} = \lambda_{m,f}^{(j)}. \tag{2.112}$$

If the MSS transits from state n_1 to state n_2 due to a repair with intensity $\mu_{m,f}^{(j)}(f > m)$ of an arbitrary element j, then the element $q_{n_1n_2}$ of transition matrix $\mathbf{q}$ placed in the intersection of row n_1 and column n_2 is

$$q_{n_1n_2} = \mu_{m,f}^{(j)}. \tag{2.113}$$

However, if there is no transition from state n_1 to state n_2, then the element $a_{n_1n_2}$ of transition matrix $\mathbf{q}$ placed in the intersection between row n_1 and column n_2 is 0:

$$q_{n_1n_2} = 0. \tag{2.114}$$

5. Designation of diagonal elements in the transition intensity matrix

The definition of the diagonal elements of the transition intensity matrix $\mathbf{q}$ is the final stage in its creation. The sum of elements in each row of matrix $\mathbf{q}$ is zero. Thus, the diagonal elements of the matrix $\mathbf{q}$ determined as follows:

$$q_{ii} = -\sum_{n=1,n\neq i}^{K} q_{in},\ i = 1, ..., K. \tag{2.115}$$

A transition intensity matrix for MSS can be obtained by using the five-step algorithm. The system of differential equations (2.74) that describes the system behavior can be given from the matrix.

The above algorithm is generic and can create a Markov model for even the most complex MSS, lowering the risk of errors.

Instantaneous availability, instantaneous expected output performance, and instantaneous output performance deficiency are all MSS reliability indices that can be found in the same manner as was proved for multi-state element.

First of all, we should solve the system of differential equations and should find the probabilities $p_i(t)$ must for all system states $i = 1, \ldots, K$.

The MSS instantaneous availability for a fixed demand level d can be calculated as the sum of probabilities of all acceptable states, states where MSS output performance is larger than or equal to d. As a result, the following form can be used to determine MSS instantaneous availability:

$$A(t) = \sum_{i=1}^{K} p_i(t) \cdot 1(o_i \geq d), \tag{2.116}$$

MSS mean instantaneous output performance is designated by the following

$$E_t = \sum_{i=1}^{K} o_i p_i(t), \tag{2.117}$$

and also MSS mean instantaneous output performance deficiency is determined as follows

$$D_t = \sum_{i=1}^{K} p_i(t) max(d - o_i, 0). \tag{2.118}$$

The Markov model must be changed, in order to derive the MSS reliability function $R_i(t)$ for the fixed demand d, $o_i < d \leq o_{i+1}$. All the system unacceptable states where the output performance rate is less than demand d should be merged into a single absorbing state with the number 0.

Transitions from state 0 to any other state that is acceptable should be prohibited. The sum of the transition rates from state j to all unacceptable states should be used to calculate the transition rate from any acceptable state j to the absorbing state.

As a result of these adjustments, we have a new transition intensity matrix.The probability of state 0 $p_0(t)$ is obtained by solving the differential equation (2.74) with this matrix, and the system reliability function is determined as $R(t) = 1 - p_0(t)$.

2.2.3 Markov Reward Models

Basic Definition and Model Description

The Markov approach was used to find several significant reliability indices of MSS in the previous subsections. Assume other indices like state frequencies and the mean number of system failures throughout an functional period. The fact which is really significant is the Markov reward models assumed here are quite useful for MSS life cycle cost analysis and reliability-related cost computation.

The continuous-time Markov chain with a set of states $\{1, \ldots, K\}$ and transition intensity matrix $\mathbf{q}=[q_{ij}],\ i, j = 1, ..., K$ is considered in this model. If the process remains in any state i for the period of the time unit, a particular amount of money r_{ii} should be salaried.

It is recommended that a particular amount of money r_{ii} be salaried each time that the process transits from state i to state j. These amounts of money r_{ii} and r_{ij} are referred to as *rewards*. Rewards can be interpreted in a variety of ways, not only as money. It could be the energy of a power generation system or the productivity of a production line.

The Markov process with rewards is designated a Markov process with rewards related with its states and transitions. A new matrix $\mathbf{r}=[r_{ij}],\ i, j = 1, ..., K$ of rewards is defined, for these. The process reduces to a standard continuous-time discrete-state Markov process if all rewards are zero.

Remark that the rewards r_{ii} and r_{ij} have different dimensions. Thus, reward r_{ii} is measured in cost units per time unit, if r_{ij} is measured in cost units. The value that is

important is the total expected reward accumulated up to the moment t under specialized initial conditions.

The total expected reward accumulated up to time t is defined by $V_i(t)$, given the initial state of the process at the moment $t = 0$ is state i.

In order to find the total expected rewards, we should solve the following system of differential equations under specialized initial conditions.

$$\frac{dV_i(t)}{dt} = r_{ii} + \sum_{j=1, j\neq i}^{K} q_{ij} r_{ij} + \sum_{j=1}^{K} q_{ij} V_j(t),\ i = 1, ..., K. \tag{2.119}$$

The following steps can be taken to obtain System (2.119).

Consider that at the moment $t = 0$ the process is in state i. The process can either stay in current state or transit to another state j, within the time increment Δt. If it is in state i throughout time Δt, the expected reward accumulated during this time is $r_{ii}\Delta t$. Because the process is still in state i at the start of the time interval $[\Delta t, \Delta t + t]$, the expected reward for this interval is $V_i(t)$ and for the total interval $[0, \Delta t + t]$ is $V_i(\Delta t + t) = r_{ii}\Delta t + V_i(t)$. The probability that the process will remain in state i for the time interval Δt equals 1 minus the probability that it will move to any other state $j \neq i$ for this interval:

$$\pi_{ii}(0, \Delta t) = 1 - \sum_{j=1, j\neq i}^{K} q_{ij}\Delta t = 1 + q_{ii}\Delta t. \tag{2.120}$$

Further, throughout time Δt the process can transit to another state $j \neq i$ with the probability $\pi_{ii}(0, \Delta t) = 1 + q_{ii}\Delta t$. In this instance, the expected reward accumulated throughout the time interval $[0, \Delta t]$ is r_{ij}. At the start of the time interval $[\Delta t, \Delta t + t]$ the process is in state j. Since, the expected reward for this interval is $V_j(t)$ and the expected reward for the interval $[0, \Delta t + t]$ is $V_i(\Delta t + t) = r_{ij} + V_j(t)$.

To calculate the total expected reward we should summarize the products of rewards and associated probabilities for all of the states. Hence, for small Δt we have

$$V_i(\Delta t + t) \approx (1 + q_{ii}\Delta t)[r_{ii}\Delta t + V_i(t)] + \sum_{j=1, j\neq i}^{K} q_{ij}\Delta t[r_{ij}\Delta t + V_j(t)],\ i = 1, ..., K. \tag{2.121}$$

Ignoring the terms with an order larger than Δt the last expression can be rewritten by the following form:

$$\frac{V_i(\Delta t + t) - V_i(t)}{\Delta t} = r_{ii} + \sum_{j=1, j\neq i}^{K} q_{ij} r_{ij} + \sum_{j=1}^{K} q_{ij} V_j(t),\ i = 1, ..., K. \tag{2.122}$$

Since, we determine the vector column of total expected rewards $V(t)$ with components $V_1(t), ..., V_K(t)$ and the vector column u with components

$$u_i = r_{ii} + \sum_{j=1, j\neq i}^{K} q_{ij} r_{ij},\ i = 1, ..., K, \tag{2.123}$$

we will get Equation (2.119) in matrix notation:

$$\frac{d}{dt}\mathbf{V}(t) = \mathbf{u} + \mathbf{q}\mathbf{V}(t). \tag{2.124}$$

The system (2.119) can be solved under initial conditions $V_i(0) = 0,\ i = 1, \ldots, K$.

For the designating of the steady-state solution of (2.119), we can solve the following system of algebraic equations

$$0 = \mathbf{u} + \mathbf{q}\mathbf{V}(t). \tag{2.125}$$

where 0 is a vector column with zero elements.

2.2.4 Semi-Markov Models

As previously stated, when the transition time between any states is distributed exponentially, a discrete-state continuous-time stochastic process can only be described as a continuous-time Markov chain. This aspect severely limits the Markov chain model's application to problems in the real world. A semi-Markov process model is one technique to examine processes with arbitrarily distributed sojourn periods. A semi-Markov model's key advantage is that it permits non-exponential distributions for state transitions and generalizes certain types of stochastic processes. Therefore, in real world the lifetime and repair times are not exponential, this is very significant. Semi-Markov processes have a complicated general theory, but here we consider certain sections of reliability evaluation utilizing semi-Markov processes that don't require a lot of calculations. In the real-world, the engineers can examine the reliability of MSSs with arbitrary transition periods without Monte-Carlo simulation, using relatively simple calculation approaches. This particularly associates to MSS steady-state behavior.

Embedded Markov Chain and Definition of Semi-Markov Process

For designating a semi-Markov process, assume a system that at any moment $t \geq 0$ can be in one of various achievable states $o_1, ..., o_K$.

The system behavior is determined by the discrete-state continuous-time stochastic output performance process $O(t) \in \{o_1, ..., o_K\}$. The following equation gives the initial state i of the system and one-step transition probabilities:

$$\begin{aligned} &O(0) = o_i,\ i \in \{1, ..., K\}, \\ &\pi_{jk} = \mathbb{P}\{O(t_n) = o_k | O(t_{n-1}) = o_j\},\ j, k \in \{1, ..., K\}. \end{aligned} \tag{2.126}$$

where π_{jk} is the probability that the system move from state j with output performance rate o_j to state k with output performance rate o_k. The probabilities $\pi_{jk},\ j, k \in \{1, ..., K\}$

designate the one-step transition probability matrix $\pi = [\pi_{jk}]$ for the discrete-time chain $O(t_n)$, where transitions from one to another state may happen only at discrete time moments $t_1, t_2, ..., t_{n-1}, t_n, ...$ So, a Markov chain $O(t_n)$ is named Markov chain embedded in stochastic process $O(t)$, or *embedded Markov chain.*

For every $\pi_{jk} \neq 0$ a random variable corresponds T^*_{jk} with the cdf

$$W_{jk}(t) = W_{jk}(T^*_{jk} \leq t) \tag{2.127}$$

and pdf $w_{jk}(t)$. The random variable T^*_{jk} is a conditional sojourn time in state j and describes the time that the system stays in the j under the assumption that the system transits from state j to state k.

Figure 2.7 depicts a graphical representation of possible realization of the assumed process. At the initial time instant $O(0) = o_i$, the process transits from the initial state i to state j, with output performance rate o_j, with probability π_{ij}. The process is in state i throughout random time T^*_{ij} with cdf W_{ij}, if the next state is state j. The probability of the transition from state j to other state k is π_{jk}, since the process transits from state j. If the system transits from state j to state k, there is in state j throughout random time T^*_{jk} with cdf W_{jk} up to the transition to state k.

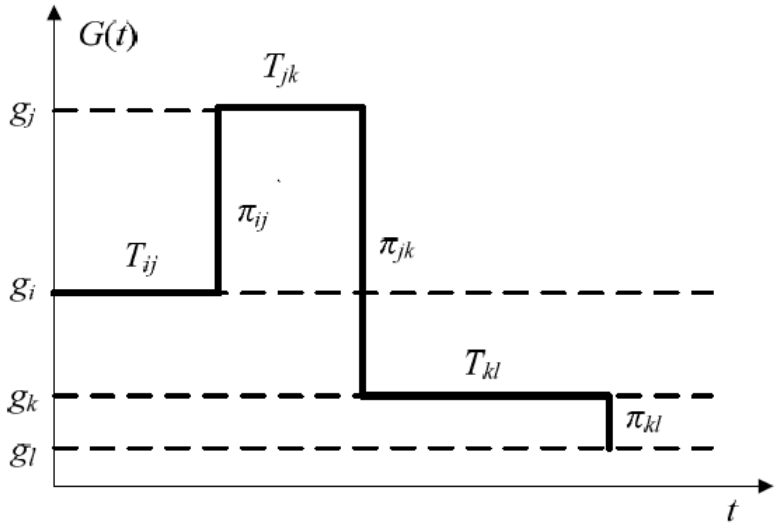

Figure 2.7: Semi Markov stochastic process, where the $G(t)$ has been noted with $O(t)$ (Lisnianski, A. et al.(2010)).

This process can be continued over an arbitrary period T.

If the future state and the associated sojourn time in the present state must be determined independently of the process's previous history, the stochastic process $O(t)$ is called a *semi-Markov process.* In this instance, the chain $O(t_n)$ will be a Markov chain with one-step transition probabilities $\pi_{jk},\ j, k \in \{1, ..., K\}$ and be called an *embedded Markov chain.*

Hence, in order to determine the Semi Markov process one has to determine the initial state of the process and the matrices $\pi = [\pi_{jk}]$ and $W(t) = [W_{ij}(t)]$, $\forall\ j, k \in \{1, ..., K\}$.

Notice that in the process in which we skip the arbitrarily distributed times between transitions and we interest the time instants of transitions is called homogeneous discrete time Markov chain. Nevertheless, the process lose Markov properties, if we take into account the sojourn times in several states. Thus, Markov process is considered the process only at time instants of transitions. This explanation for the name of the process semi-Markov.

The kernel matrix $\mathbf{Q}$(t) is the most fundamental determination of the semi-Markov process. The matrix element $Q_{ij}(t)$ defines the probability that a *one-step transition* from state i to state j occurs throughout the time interval $[0, T]$. *One-step transition probabilities for embedded Markov chain* can be calculated as follows:

$$\pi_{ij} = \lim_{t\to\infty} Q_{ij}(t) \tag{2.128}$$

using kernel matrix and the cdf $W_{ij}(t)$ of the conditional sojourn time in state i given by:

$$W_{ij}(t) = \frac{1}{\pi_{ij}} Q_{ij}(t). \tag{2.129}$$

According to the kernel matrix, the cumulative distribution function $W_i(t)$ of unconditional sojourn time T_i in every state i can be designated as

$$W_i(t) = \sum_{j=1}^{K} Q_{ij}(t) = \sum_{j=1}^{K} \pi_{ij} W_{ij}(t). \tag{2.130}$$

Thus, the probability density function of the unconditional sojourn time in state i with output performance rate o_i can be written

$$w_i(t) = \frac{d}{dt} W_i(t) = \sum_{j=1}^{K} \pi_{ij} w_{ij}(t). \tag{2.131}$$

Based on (2.131), the mean unconditional sojourn time in state i can be obtained as

$$\bar{T_i} = \int_0^\infty t w_i(t) dt = \sum_{j=1}^{K} \pi_{ij} \bar{T}^*_{ij}, \tag{2.132}$$

where $\bar{T}^*_{ij}$ is the mean conditional sojourn time in state i given that the system transits from state i to j.

The stochastic behavior of a semi-Markov process is totally defined by the kernel matrix $\mathbf{Q}$(t) and the initial state. When studying MSS reliability, the following principles can be used to find the kernel matrix for a semi-Markov process. Transitions between states are frequently carried out as a result of occurrences such as failures and repairs. The cumulative distribution function of time between each occurrence is known. The transition is made based on which occurrence comes first in a competition between the occurrences.

Figure 2.8 presents state-transition diagram for the semi-Markov process, which has 3 possible transitions from initial state 0. When events of various types 1,2, and 3 occur, the process will transition from state 0 to states 1, 2, and 3. The time between events of type 1 is random variable $T_{0,1}$ with cumulative distribution function $W_{0,1}(t)$. The process transits from state 0 to 1, if an event of type 1 happens first. The random variable $T_{0,2}$ determines the time between events of type 2 with cumulative distribution function $W_{0,2}(t)$. The process transits from state 0 to 2, if an event of type 2 happens before the others.

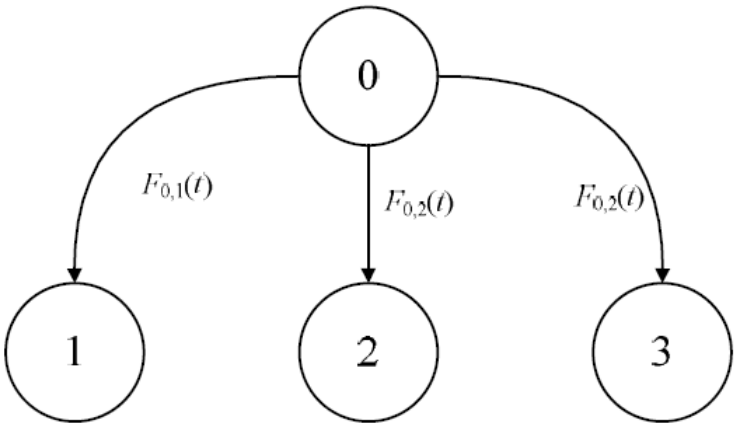

Figure 2.8: State Transition Diagram of simplest semi-Markov Process, where the $F_{ij}(t)$ has been noted with $W_{ij}(t)$ (Lisnianski, A. et al.(2010)).

The time between events of type 3 is random variable $T_{0,3}$ with cdf $W_{0,3}(t)$ and the process transits from state 0 to 3, if an event of type 3 happens first.

The probability that the random variable $T_{0,1}$ that is lower than $T_{0,2}$ and $T_{0,3}$, under the assumption $T_{0,1} \leq t$, is the probability $Q_{01}(t)$ in which the process will transit from state 0 to 1 up to time t.

Thus,

$$\begin{aligned} Q_{01}(t) &= \mathbb{P}\{(T_{0,1} \leq t)\&(T_{0,1} > t)\&(T_{0,3} > t)\} \\ &= \int_0^t dW_{0,1}(s) \int_t^\infty dW_{0,2}(s) \int_t^\infty dW_{0,3}(s) \\ &= \int_0^t [1 - W_{0,2}(s)][1 - W_{0,3}(s)]\, dW_{0,1}(s). \end{aligned} \tag{2.133}$$

Similarly, we receive

$$Q_{02}(t) = \int_0^t [1 - W_{0,1}(s)][1 - W_{0,3}(s)]\, dW_{0,2}(s), \tag{2.134}$$

$$Q_{03}(t) = \int_0^t [1 - W_{0,1}(s)][1 - W_{0,2}(s)]\, dW_{0,3}(s). \tag{2.135}$$

The kernel matrix for a semi-Markov process with the state-transition diagram shown in Figure 2.8, given by the following form

$$Q(t) = \begin{bmatrix} 0 & Q_{01}(t) & Q_{02}(t) & Q_{03}(t) \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}. \tag{2.136}$$

Expressions (2.134) – (2.136) can be easily generalized to any number of transitions from initial state 0.

Evaluation of Reliability Indices Based on Semi-Markov Processes

For finding the MSS reliability indices, should be made the system state-space diagram. The difference in the semi-Markov model is the times of transition may be distributed arbitrarily. According to the transition time distributions $W_{ij}(t)$, kernel matrix $\mathbf{Q}$(t) should be determined based on the method introduced in the previous section.

In analysis of semi-Markov process, a prime problem is the finding the state probabilities. Let $\theta_{ij}(t)$ be the probability that the process that begins in initial state i at time $t = 0$ will be in state j at time t. It was shown that probabilities $\theta_{ij}(t),\ i, j \in 1, \ldots, K$, can be derived from the solution of the following system of integral equations:

$$\theta_{ij}(t) = \delta_{ij}[1 - W_i(t)] + \sum_{k=1}^{K} \int_0^t q_{ik}(\tau)\theta_{kj}(t - \tau)d\tau, \tag{2.137}$$

where

$$q_{ik}(\tau) = \frac{dQ_{ik}(\tau)}{d\tau}, \tag{2.138}$$

$$W_i(t) = \sum_{j=1}^{K} Q_{ij}(t), \tag{2.139}$$

$$\delta_{ij} = \begin{cases} 1, \text{ if } i = j, \\ 0, \text{ if } i \neq j. \end{cases} \tag{2.140}$$

The system of linear integral equations (2.137) is the primary system in semi-Markov processes theory. With the system's solving, we can find all the probabilities $\theta_{ij}(t),\ i, j \in \{1, ..., K\}$, for a semi-Markov process with a given kernel matrix $[Q_{ij}(t)]$ and initial state.

According to the probabilities $\theta_{ij}(t),\ i, j \in \{1, ..., K\}$, we can easily find significant reliability indices. Consider that system states are ordered based on their output performance rates $o_K \geq o_{K-1} \geq ... \geq o_2 \geq o_1$ and demand $o_n \geq d \geq o_{n-1}$ is fixed. State K with output performance rate g_K is the initial state. In this instance, system instantaneous

availability is used as the probability that a system starting at moment $t = 0$ from state K will be at moment $t \geq 0$ in every state $o_K, ..., o_n$. Thus,

$$A(t, d) = \sum_{j=n}^{K} \theta_{Ki}(t). \tag{2.141}$$

The mean system instantaneous output performance and the mean instantaneous output performance deficiency can be calculated as

$$E_t = \sum_{i=1}^{K} o_i \theta_{Ki}(t) \tag{2.142}$$

and

$$D_t(d) = \sum_{i=1}^{n-1} (d - o_i)\theta_{Ki}(t) \cdot \mathbf{1}(d > o_i). \tag{2.143}$$

In the general instance, the system of integral equations (2.137) can be solved only numerically. The method of the Laplace–Stieltjes transform can be used to determine an analytical solution of the system. As was done for Markov models, we define a Laplace–Stieltjes transform of function $w(x)$ as

$$\tilde{w}(s) = L\{w(x)\} = \int_0^t e^{-sx} w(x) dx. \tag{2.144}$$

Applying the Laplace–Stieltjes transform to both sides of (2.137) we get

$$\tilde{\theta}_{ij}(s) = \delta_{ij}\tilde{\Psi}_i(s) + \sum_{k=1}^{K} \pi_{ik}\tilde{w}_{ik}(s)\tilde{\theta}_{kj}(s), \ 1 \leq i, j \leq K, \tag{2.145}$$

where $\tilde{\Psi}_i(s)$ is the Laplace–Stieltjes transform of the function

$$\Psi_i(t) = 1 - W_i(t) = \int_t^{\infty} w_i(t) dt = \mathbb{P}\{T_i > t\} \tag{2.146}$$

and, thus,

$$\tilde{\Psi}_i(s) = \frac{1}{s}[1 - \tilde{w}_i(s)]. \tag{2.147}$$

The system of algebraic equations (2.145) determines Laplace–Stieltjes transform of probabilities $\theta_{ij}(t)$, $i, j \in \{1, ..., K\}$, as a function of the main parameters of a semi-Markov process.

It is also possible to get the steady-state probabilities, if we solve this system. The formulas for computation of steady-state probabilities are given below. Steady-state

probabilities $\theta_{ij} = \lim\limits_{t\to\infty}\theta_{ij}(t)$ are independent of the initial state of process I, and for their definition, we can use the index θ_j. It is demonstrated that

$$\theta_j = \frac{p_j\bar{T}_j}{\sum\limits_{j=1}^{K} p_j\bar{T}_j}, \tag{2.148}$$

where $p_j,\ j = 1,\ldots,K$ are steady-state probabilities of the embedded Markov chain and these are the solutions of the following system:

$$\begin{cases} p_j = \sum\limits_{i=1}^{K} p_i\pi_{ij}\ j = 1,...K, \\ \\ \sum\limits_{i=1}^{K} p_i = 1. \end{cases} \tag{2.149}$$

Keep in mind, the first K equations in (2.149) are linearly dependant and we cannot solve the system without the last equation $\sum\limits_{i=1}^{K} p_i = 1$.

For finding the reliability function, a new semi-Markov model should be constructed in the same way as the respective Markov models: all states corresponding to output performance rates less than fixed demand d should be merged into a single absorbing state 0. All transitions that come back the system from this absorbing state should be prohibited. The reliability function is derived from this new model as $R(d,t) = \theta_{K0}(t)$.

2.3 Statistical Analysis of Reliability Data for MSS

2.3.1 Basic Concepts of Statistical Estimation Theory

In this section, we'll go through the fundamentals of using statistical approaches to MSSs reliability assessment. We'll remain to the Markov model concept and use contemporary techniques for estimation of transition intensity rates. But first, we'll need explain the fundamentals of statistical estimating theory.

Assume we conduct an experiment with a random outcome $\mathbf{X}$, $X \in \mathbb{R}$, where $\mathbb{R}$ is defined the sample space, or the collection of all possible outcomes of our experiment. Now, we consider the circumstance $X = \{x_1,...,x_n\}$, in which x_i are independent observations of n items, selected at random from a population. The set $\{x_1,...,x_n\}$ is determined as a random sample of size n and every x_i is an observation.

A sample is typically consisted of independent and identically distributed random observations, in statistics. This means that observations of a given sample are obtained independently and under the same conditions.

Let $X_1,...,X_n$ be a random sample from a distribution F_θ that is stated up to a vector of unknown parameters θ. For example, the sample could come from a Poisson distribution with unknown mean value or a Normal distribution with unknown mean and variance. While it is common in probability theory to assume that all of a distribution's

parameters are known, this is not the case in statistics, where one of the major problems is to utilize observed data to make conclusions about unknown parameters.

Consider that the probability law of $\mathbf{X}$ which has a well-known mathematical type, say, with pdf $f(x;\theta_1,...,\theta_k)$, where $\theta_1,...,\theta_k$ are *unknown* parameters of the population. Consider for instance the normal distribution

$$f(x;\mu,\sigma)=\frac{1}{\sigma\sqrt{2\pi}}exp\{-(x-\mu)^2/2\sigma^2\},\ -\infty<x<+\infty,$$

where μ and σ are unknown parameters of the population, so

$$\{\theta_1,\theta_2\}=\{\mu,\sigma\}.$$

There will always be an infinite number of sample values functions, known as statistic, which can be offered to estimate one or more of the parameters. Typically, a statistic $S=S(\mathbf{X})$ is each function of $\mathbf{X}$. Any statistic used to estimate the value of an unknown parameter θ is called an *estimator* of θ. While the *estimate* is the observed value of the estimator.

Obviously, the best estimate is the one that is closest to the true value of the parameter to be estimated. Especially, the best estimate may be defined as the statistic whose distribution is as concentrated as possible to the true value of the parameter. Thus, the primary estimate problem in the previous case can be phrased as follows:
Define the sample observations' functions

$$\hat{\theta}_1(x_1,...,x_n),...,\hat{\theta}_k(x_1,...,x_n)$$

so that their distribution is as concentrated as possible to the parameters' true value. As estimators, the estimating functions are established.

Properties of Estimators

Certain estimators' properties are the consistency, unbiasedness, efficiency, and sufficiency.

Consistency: An estimator $\hat{\theta}$ is consistent if the probability that $\hat{\theta}$ will deviate from parameter θ more than every constant amount $\varepsilon>0$, approaches zero as the sample size n grows up. Typically, let $\hat{\theta}_n$ be an estimator of parameter θ according to sample of size n. Thus, $\{\hat{\theta}_n\}$ is a consistent following of estimators of θ if $\forall\varepsilon>0$

$$\lim_{n\to\infty}\mathbb{P}\{|\hat{\theta}_n-\theta|>\varepsilon\}=0. \tag{2.150}$$

Apparently, consistency is a property regarding the behavior of an estimator for great values of sample size n, i.e., as $n\to\infty$. Its behavior, when n is finite, is neglected.

Unbiasedness: This is a property related with finite n. A statistic $\hat{\theta}_n=\hat{\theta}_n(x,...,x_n)$ is characterized as an unbiased estimate of parameter θ if

$$E\{\hat{\theta}_n\}=\theta, \tag{2.151}$$

The estimator's bias is given by

$$b_{\hat{\theta}} = [E\{\hat{\theta}_n\} - \theta]. \tag{2.152}$$

the bias bereaves a statistic result of representativeness by systematically distorting it. It is not the same as a random error that can deform at every chance but balances out on the average. It is worth mentioning that, the bias is a systematic error. The bias is zero for an unbiased estimator.

It was proved that unbiased estimator is always consistent estimator. A consistent estimator, on the other hand, is not always unbiased.

Efficiency: This is an significant criterion for evaluating an estimator's quality. It is preferable to get a close estimate to the true value. The variance is a measure of closeness, hence an estimator's efficiency is inversely related to its variance. A consistent estimator, $\hat{\theta}_1$, will be more efficient than another estimator, $\hat{\theta}_2$, if $Var\{\hat{\theta}_1\} < Var\{\hat{\theta}_2\}$, where $Var\{\}$ is the variance. The most efficient estimator is one that has sampling variance lower than that of any other estimate, in a category of consistent estimators for a parameter. When there is such an estimator, it gives a criterion for the measure of efficiency of the other estimators.

If we have a $\hat{\theta}_1$ which is the most efficient estimator with variance V_1, and another estimator $\hat{\theta}_2$ with variance V_2, afterwards the efficiency E of $\hat{\theta}_2$ is determined as

$$E = \frac{\hat{\theta}_1}{\hat{\theta}_2}. \tag{2.153}$$

Obviously E cannot be grater than 1.

Sufficiency: If an estimator θ contains all the information in the sample about a parameter θ, it is said to be sufficient for that parameter. No other estimator generated from the same sample can give further information about the parameter, if it is sufficient.

Main Estimation Methods

In this subsection, we will present the two most common ways of estimation obtains considered in statistics are Point and Interval estimations. The estimates are called point estimates, can define a separate quantity as an estimate of θ. Such are, the maximum likelihood method and the method of moments, for designating estimators of unknown parameters. Also, in the interval estimation, rather than expressing a special value as our estimate of θ, we give an interval within which we believe that lies. Moreover, we assess how much confidence we can place in such an interval estimate. We demonstrate how to get an interval estimate of the unknown mean of a normal distribution with a known variance.

Point Estimation

The Point Estimation is an estimation technique which suggest that the unknown parameters should be estimated by matching population moments with the appropriate sample moments.

The method of moments is an estimation strategy which propose that estimating unknown parameters by matching population moments with the relevant sample moments.

We consider that the sample $\{x_1, ..., x_n\}$ was derived using n observations of the continuous random variable X. Of course, one can determine the sample mean and sample variance, which are the first and the second moments, as the corresponding sample's expected values of size n by the following forms:

$$\bar{x} = \frac{1}{n}\sum_{i=1}^{n} x_i \tag{2.154}$$

and

$$S^2 = \frac{1}{n}\sum_{i=1}^{n}(x_i - \bar{x})^2. \tag{2.155}$$

Thereafter, $\bar{x}$ and S^2 can be utilized as point estimates of the distribution mean μ and variance σ^2, respectively. It's worth noting that estimator of variance (2.155) is biased because $\bar{x}$ is estimated from the same sample. Nevertheless, this bias can be removed away by multiplying it by $n/(n-1)$:

$$S^2 = \frac{1}{n-1}\sum_{i=1}^{n}(x_i - \bar{x})^2. \tag{2.156}$$

The sample moments are equated to the relevant distribution moments using the method of moments. The estimators of the distribution parameters are determined from the solutions of the equations. The method of moments estimates are usually consistent, although they may not be efficient.

In statistics, a kind of estimator known as *the maximum likelihood estimator* is commonly used. This technique is based on the concept of computing parameter values that maximize the probability of obtaining a specific sample. The following logic is used to get at it.

Let $f(x_1, ..., x_n|\theta)$ as the joint probability mass function of the discrete random variables $X_1, X_2, ..., X_n$ and let it be their joint probability density function when they are jointly continuous random variables. Since, θ is considered unknown, we express f as a function of θ.

Since $f(x_1, ..., x_n|\theta)$ reflects the likelihood that the values $x_1, x_2, ..., x_n$ would be observed when θ is the real value of the parameter, a sensible estimate of θ would appear to be the value offering the greatest likelihood of the observed values.

Alternatively, the maximum likelihood estimate $\hat{\theta}$ is the value of θ which maximize $f(x_1, ..., x_n|\theta)$ where $x_1, x_2, ..., x_n$ are the observed values. The function $f(x_1, ..., x_n|\theta)$ is usually stated to as the likelihood function of θ.

In defining the maximizing value of θ, it is commonly useful to use the fact that $f(x_1, ..., x_n|\theta)$ and $\log[f(x_1, ..., x_n|\theta)]$ have their maximum at the same value of θ. As a result, we may get $\hat{\theta}$ by maximizing $\log[f(x_1, ..., x_n|\theta)]$.

Interval Estimation

Assume that $X_1, ..., X_n$ is a sample from a normal population with unknown mean μ and known variance σ^2. It has been shown that $\bar{X} = \frac{1}{n}\sum_{i=1}^{n} X_i$ is the estimator for μ. We don't expect that the sample mean $\bar{X}$ to exactly equal μ, but we expect it to be close. Thus, rather than a point estimate, it can be useful to be able to define an interval within which we have a some degree of confidence that μ lies within. To get such an interval estimator, we use the probability distribution of the point estimator. Let's have a look at how it functions.

Since, the point estimator $\bar{X}$ is normal with mean μ and variance σ^2/n, it follows that

$$\frac{\bar{X} - \mu}{\sigma/\sqrt{n}} = \sqrt{n}\frac{\bar{X} - \mu}{\sigma}$$

has a standard normal distribution. Hence,

$$\mathbb{P}\left\{\bar{X} - 1.96\frac{\sigma}{\sqrt{n}} < \mu < \bar{X} + 1.96\frac{\sigma}{\sqrt{n}}\right\} = 0.95$$

Meaning, 95% of the time, the sample average $\bar{X}$ will be such that the distance between it and the mean μ will be less than $1.96\frac{\sigma}{\sqrt{n}}$. If we observe the sample and $\bar{X} = \bar{x}$, then we say that "with 95 percent confidence"

$$\bar{x} - 1.96\frac{\sigma}{\sqrt{n}} < \mu < \bar{x} + 1.96\frac{\sigma}{\sqrt{n}}.$$

2.3.2 Classical Parametric Estimation for Binary-State System

Basic Considerations

For binary-state systems, we assume statistical methods for estimating the reliability model parameter, for example the λ of the Exponential distribution. The propose is to determine the parameter's point estimate and its confidence interval.

In general, estimation of parameters can be based on both field data and data received through a particular reliability or life test. On the test of reliability, a sample of items is put to the test in the same environment in which they are expected to work. All failure times are registered. There are two different kinds of testing. The first is testing with failed item substitution, in which each item should be substituted with a new one after it fails, and the second is testing without substitution. For further analysis, a following of recorded times to failure is termed a given sample. An complete sample is one in which all objects failed throughout a test and all failure timings are known for a specific observation period.

Taking an complete sample of observations is commonly impractical in the actual world. Often we pause the test after a certain amount of time has passed or since a certain number of failed items has been observed. Else, the test gets too time consuming or too costly. So, for certain items the lifetime is censored, i.e., the information about it has the form "the lifetime exceeds some value t". Contemporary products are ordinarily reliable enough so that a complete sample is a rare. Broadly, reliability data are not complete and we are using censored samples.

Define N, the number of elements on the test, and we suppose that all elements tested at the same time.

If just r items fail throughout the test period, T, the failure times are known, and the failed items are not replaced, the sample is referred to as singly *censored on the right at* T. In this instance, we only know that the failure times of $N-r$ unfailed items are longer than the test period T. *Right censored at* T means that the exact value of an observation is unknown, but it is known that it is larger than or equal to T.

The failure time for an component is called *left censored* when the failure time is unknown but known to be less than a specific value. This form of censoring is never utilized in reliability.

The *time-terminated test* is a test that ends at a predefined nonrandom time T. Throughout the test time, the number of failures observed, r, will be a random variable. Type I right censoring refers to a situation like this. In the time-terminated life test, n components are placed on a test and the number of them that failed throughout the test period, as well as the time to failure for each them, is recorded.

The *failure-terminated test* is a test that ends at a predefined nonrandom number of failures in advance have been observed. In this instance, the period of the test is a random variable. Type II right censoring refers to a situation like this. Both time-terminated and failure-terminated reliability tests can be run with or without replacement of every failed item.

Exponential Distribution Point Estimation

Time-terminated Test with Replacement

Assume that N identical items are used on test with replacement. The test ends after a stated time T_s. Thereafter, the total time T on test, which includes both failed and unfailed items, is calculated as

$$T = NT_k. \tag{2.157}$$

If r failures have been observed up to time T_s, then we can find the component failure rate using the maximum likelihood point estimate:

$$\hat{\lambda} = \frac{r}{T}. \tag{2.158}$$

So, we can take the respective estimate of the items' mean time to failure:

$$\hat{MTTF} = \frac{T}{r}. \tag{2.159}$$

Note that, the number of units tested throughout the test, n_{test}, is

$$n_{test} = N + r. \tag{2.160}$$

If we use only one component on test ($N = 1$) and record r failures throughout the total time T, then we can be obtained

$$\hat{\lambda} = \frac{r}{T} = \frac{r}{T_s}. \tag{2.161}$$

Remember that expressions (2.158)–(2.161) are true if replacement times are negligibly small. If it is not so, thereafter the entire accumulated replacement time, T_R, should be determined and the failure rate can be estimated by the following equation:

$$\hat{\lambda} = \frac{r}{T_s - T_R}. \tag{2.162}$$

Time-terminated Test Without Replacement

Assume that N identical items are used on test without replacement. As in the previous case, the test ends after a stated time, T_s, throughout which r failures were recorded. Thereafter, the total time T on test, which includes by both failed and unfailed items is

$$T = (N - r)T_s + \sum_{i=1}^{r} t_i, \tag{2.163}$$

where,

t_i : time of record up to failure for failed component i,

$\sum_{i=1}^{r} t_i$: accumulated time on test of the r failed components,

$(N - r)T_s$: accumulated time on test of the unfailed components.

Failure-terminated Test with Replacement

Assume that N identical components are used on test with replacement and an component is used as soon as it fails. The test ends after a time, T_r, when the rth failure has happened.

The total time T, related with failed and unfailed components is

$$T = NT_r. \tag{2.164}$$

When the distribution of the time to failure is Exponential, then λ can be estimated as

$$\hat{\lambda} = \frac{r}{T} = \frac{r}{NT_r}. \tag{2.165}$$

The corresponding estimate of mean time to failure is obtained by

$$\hat{MTTF} = \frac{T}{r}. \tag{2.166}$$

Now, the total number of components tested is

$$n_{test} = N + r - 1, \tag{2.167}$$

because the test ends when the last failed component fails, and so the last failed component is not replaced.

Failure-terminated Test Without Replacement

Assume that N identical components are used on test without replacement – when a failure happens, the failed component is not replaced by a new one. The test ends after a time, T_r, when the rth failure has happened.

The total time T, is given by

$$T = (N - r)T_r + \sum_{i=1}^{r} t_i, \tag{2.168}$$

where,

t_i : time of record up to failure for failed component i,

$\sum_{i=1}^{r} t_i$: accumulated time on test of the r failed components,

$(N - r)T_r$: accumulated time on test of the unfailed (surviving) components.

When the distribution of the time to failure is Exponential, then λ can be estimated as

$$\hat{\lambda} = \frac{r}{T} = \frac{r}{(N - r)T_r + \sum_{i=1}^{r} t_i}. \tag{2.169}$$

The estimate of mean time to failure is obtained by

$$\hat{MTTF} = \frac{T}{r} = \frac{(N - r)T_r + \sum_{i=1}^{r} t_i}{r}. \tag{2.170}$$

In this instance the total number of components tested is

$$n_{test} = N \tag{2.171}$$

since no components are being replaced.

Interval Estimation for Exponential Distribution

Here assumed the *failure-terminated test* and presented that if the the distribution of the time to failure is Exponential with parameter λ, the variable $\frac{2r\lambda}{\hat{\lambda}} = 2\lambda T$ has the χ^2 distribution with $2r$ degrees of freedom. Hence, we can write the following expression:

$$\mathbb{P}\left\{\chi^2_{a/2;2r} \leq \frac{2r\lambda}{\hat{\lambda}} \leq \chi^2_{1-a/2;2r}\right\} = 1-\alpha. \tag{2.172}$$

Considering $\hat{\lambda} = \frac{r}{T}$ after resetting we will have a two-sided confidence interval for the true value of λ:

$$\mathbb{P}\left\{\frac{1}{2T}\,\chi^2_{a/2;2r} \leq \lambda \leq \frac{1}{2T}\,\chi^2_{1-a/2;2r}\right\} = 1-\alpha. \tag{2.173}$$

Thus, we can get the one-sided confidence interval or the upper confidence bound:

$$\mathbb{P}\left\{\lambda \leq \frac{1}{2T}\,\chi^2_{1-a/2;2r}\right\} = 1-\alpha. \tag{2.174}$$

For the *time-terminated test* the specific confidence bounds are not provided. The approximate two-sided confidence interval for the failure rate, λ, was given as

$$\mathbb{P}\left\{\frac{1}{2T}\,\chi^2_{a/2;2r} \leq \lambda \leq \frac{1}{2T}\,\chi^2_{1-a/2;2r+2}\right\} = 1-\alpha. \tag{2.175}$$

The respective the one-sided confidence interval or the upper confidence bound is calculated by

$$\mathbb{P}\left\{\lambda \leq \frac{1}{2T}\,\chi^2_{1-a/2;2r+2}\right\} = 1-\alpha. \tag{2.176}$$

2.3.3 Estimation of Transition Intensities via Output Performance Observations

Multi-State Markov Model and Observed Reliability Data. Problem Formulation

Absolutely, a binary-state system is the most basic example of a MSS with two distinctive states. In the previous sections, we were mentioned in the Point estimation for transition intensities of binary Markov models. Despite the fact that this is an real practical problem, there have been essentially no investigations into it in multi-state environment until recently.

For example, in the ground of power system reliability assessment, it has been known that using basic two-state models to describe big generating units in generating capacity adequacy assessments might result in negative assessments. Many utilities use multi-state models instead of two-state representations to evaluate unit reliability more accurately.

These models use the steady-state probabilities of a unit remaining at various generating capacity levels. A long-run probability of a unit remaining at a specific capacity level is usually simply described as the part of the function time when the unit is at this capacity level. The investigation of the short-term behavior of MSSs cannot be relied on steady-state probabilities. The research should be conducted using a broad MSS model with known transition intensities between any of the model's states. The goal is to estimate these transition intensities based on actual MSS failures and repair statistics, which is represented by the observed realization of an output performance stochastic process.

The appropriate method for point and interval estimate of transition intensities via output performance observation will be presented below. The moethod was primarily presented in Lisnianski (2008).

Figure 2.9 shows a general Markov model of an Multi State system with minor and major failures and repairs.

In the model, there are N states, with every state $i \in [1, ..., N]$ has its own output performance level o_i. In most cases, state N is related with the nominal output performance level and state 1 is related with total system failure, and all subsequent states $i \in [2, ..., N-1]$ are related with the respective limited output performance levels o_i. The transition intensity from state i to state j is determined as q_{ij}.

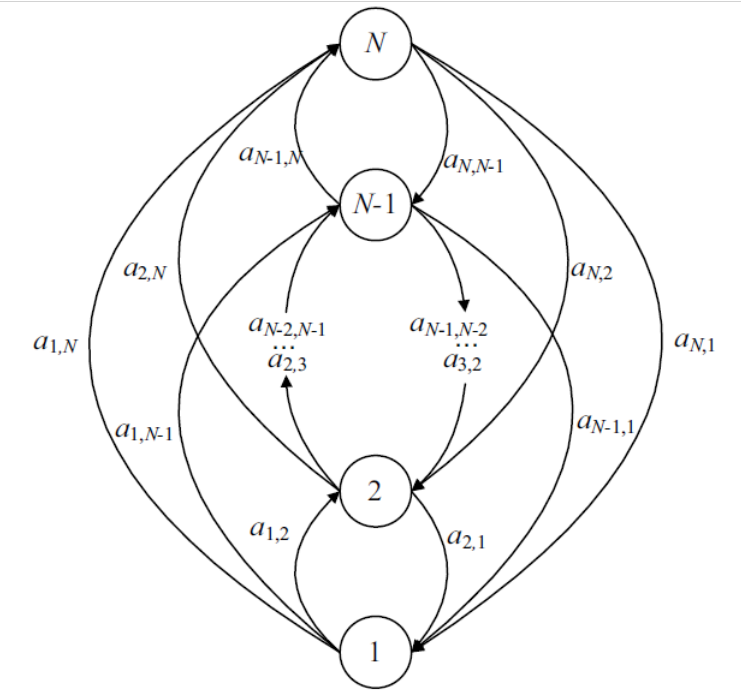

Figure 2.9: General Markov model for a Multi State System, where the a_{ij} has been noted with q_{ij} (Lisnianski and Levitin 2003).

Thus, MSS output performance is known for each time $t \in [0, T]$, where T is the complete observation time, as well as the respective times of MSS transitions from each output performance level o_i to level o_i, $i, j \in [1, ..., N]$.

Figure 2.10 shows an example of a single realization of such a stochastic process.

The stochastic process $O_A(t)$ is a discrete-state continuous-time process. The following definitions are provided for this stochastic process.

$T_i^{(m)}$: sojourn time of the system of mth remaining in state i throughout observation time T,

k_i: accumulated number of system enters state i or accumulated number of system exits from state i to another state, throughout observation time T,

k_{ij} : accumulated number of system transitions from state i to another state $j \neq i$, throughout observation time T.

Figure 2.10 shows the realization throughout the MSS observation time T remained in state $N(k_N = 4)$ four times, once it transited from state N to state $N-1(k_{N,N-1} = 1)$, from state N to state $3(k_{N,3} = 1)$, and from state N to state $1(k_{N,1} = 1)$.

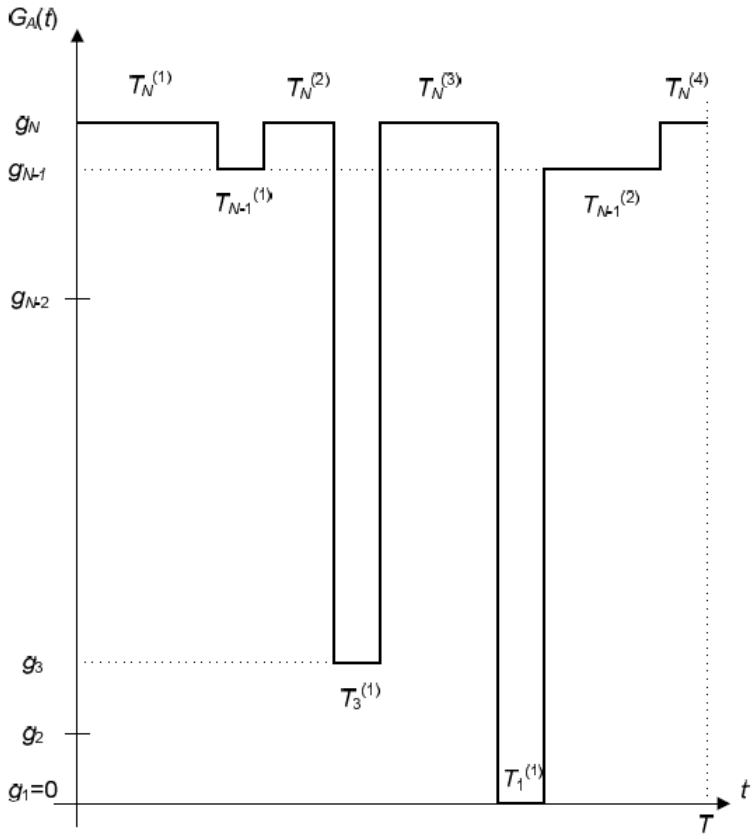

Figure 2.10: Multi State System output performance $O_A(t)$ as a stochastic process, where the $G_A(t)$ has been denoted with $O_A(t)$ (Lisnianski, A. et al., 2010).

Hence, the reliability data for Multi State System that can be given from the observation of the output performance stochastic process throughout time T are the following.

For every state i are known:

1. $\{T_i^{(1)}, ..., T_i^{(k_i)}\}$ sample of system sojourn times in state i throughout T,

2. k_{ij} number of system transitions from state i to another state j throughout T and

3. k_i number of system remains in state i or number of system exits from state i to another state, throughout T.

The goal is to estimate transition intensities $q_{ij},\ i,j \in [1,...,N]$ according to a single realization of discrete-state continuous-time stochastic process $O_A(t)$ that was observed throughout time T.

Method Description

As previously stated, the stochastic process $O_A(t)$ is a discrete-state, continuous-time Markov process. We present a new stochastic process that is associated to the $O_A(t)$ process .

If we avoid random times between transitions from state i to state $j \neq i$ in process $O_A(t)$ and we interest only time moments of transitions, then the process will be a discrete-state, discrete-time Markov chain.

A Markov chain $O_{Am}(n),\ n = 0,1,2,...$, embedded in process $O_A(t)$, is a chain that can only be examined at time instants of transitions in the underlying process $O_A(t)$. The embedded Markov chain $O_{Am}(t)$ is absolutely determined by its initial state's probability distribution and one-step transition probabilities $\pi_{ij},\ i,j = [1,...,N]$.

Figure 2.9 shows the transitions between several states of the model as a result of events such as failures and repairs. Since, a Markov model describes a Multi State System is described, the cdf $W_{ij}(t)$ of time between transition from state i to another state $j \neq i$ is determined by the respective transition intensity such as

$$W_{ij}(t) = 1 - e^{-q_{ij}(t)}, \tag{2.177}$$

where q_{ij} is the transition intensity from state i to state j.

The distribution of the so-called conditional sojourn time T_{ij} in state i, which describes the system sojourn time in state i under the premise that the unit transits from state i to state j, is designated by the function $W_{ij}(t)$. A one-step transition probability of discrete-state, continuous-time process $O_A(t)$ is defined as the probability $Q_ij(t)$ that the unit will transit from state i to state j up to time t, if it is in state i at initial time instant $t = 0$. These probabilities $Q_{ij}(t),\ i,j = 1,...,N$ determine kernel matrix $\mathbf{Q}(t)$ for stochastic process $O_A(t)$:

$$Q(t) = |Q_{ij}(t)|. \tag{2.178}$$

The following way can be used to calculate these one-step probabilities for a kernel matrix. Each probability $Q_{ik}(t)$ determines the probability that random variable T_{ik} will be minimal among all other random variables $T_{ij},\ j \neq i,\ j \neq k,\ j = 1,...,N$, which designates all achievable transitions from state i to another state. So, $\forall\ k \neq i$ we will have

$$T_{ik} = min\left\{T_{i1},...,T_{i,k-1},T_{i,k+1},...,T_{iN}\right\}. \tag{2.179}$$

Relied on (2.179) we get the one-step probability $Q_{ik}(t)$ as the probability that under the assumption $T_{ik} \leq t$ the random variable T_{ik} will be lowest than the others T_{ij} $j \neq i,\ j \neq k,\ j = 1, ..., N$.

Thus, $\forall\ i = 1, 2, ..., N$ and $k \neq i$ can be derived the following form

$$\begin{aligned} Q_{ik}(t) &= \mathbb{P}\left\{(T_{ik} \leq t)\&(T_{i1} > t)\&...\&(T_{i,k-1} > t)\&(T_{i,k+1} > t)\&...\&(T_{iN} > t)\right\} \\ &= \int_0^t dW_{ik}(s) \int_t^\infty dW_{i1}(s)... \int_t^\infty dW_{i,k-1}(s) \int_t^\infty dW_{i,k+1}(s)... \int_t^\infty dW_{iN}(s) \\ &= \int_0^t [1 - W_{i1}(s)]...[1 - W_{i,k-1}(s)][1 - W_{i,k+1}(s)]...[1 - W_{iN}(u)]dW_{ik}(s). \end{aligned} \tag{2.180}$$

Using (2.180) and considering in equation (2.177), one obtains

$$Q_{ik}(t) = \frac{q_{ik}}{\sum_{j=1}^N q_{ij}} \left(1 - e^{-\sum_{j=1}^N q_{ij} t}\right). \tag{2.181}$$

The cumulative distribution function $W_i(t)$ and the probability density function $w_i(t)$ of unconditional sojourn time T_i in any state i, according to one-step probabilities $Q_{ij}(t),\ i, j = 1, ..., N$ can be given as:

$$\begin{aligned} W_i(t) &= \sum_{k=1}^N Q_{ik}(t) = 1 - e^{-\sum_{j=1}^N q_{ij} t} \\ w_i(t) &= \sum_{j=1}^N q_{ij} \left(1 - e^{-\sum_{j=1}^N q_{ij} t}\right) \end{aligned} \tag{2.182}$$

The unconditional sojourn time T_i is an exponentially distributed random variable with mean

$$T_{i_{mean}} = \frac{1}{\sum_{j=1}^N q_{ij}} = \frac{1}{A}, \tag{2.183}$$

where $A = \sum_{j=1}^N q_{ij}$.

If we use the sample $\{T_i^{(1)}, T_i^{(2)}, ..., T_i^{(k_i)}\}$, we can take an estimation $\hat{T}_{i_{mean}}$ of the mean unconditional sojourn time according to time-terminated test

$$\hat{T}_{i_{mean}} = \frac{\sum_{j=1}^{k_i} T_i^{(j)}}{k_i}. \tag{2.184}$$

The following expression can be used to estimate the sum A of intensities of all transitions that leave state i based on (2.183) and (2.184):

$$\hat{A} = \frac{1}{\hat{T}_{i_{mean}}} = \frac{k_i}{\sum_{j=1}^{k_i} T_i^{(j)}}. \tag{2.185}$$

The sum of intensities for all transitions that depart from any state i may be estimated using formula (2.185). An extra formula can be obtained in the following approach to estimate individual transition intensities.

In order to obtain, the one step transition probabilities for embedded Markov chain $O_{Am}(t)$, we are based on kernel matrix $\mathbf{Q}$(t):

$$\pi_{ij} = \lim_{t\to\infty} Q_{ij}(t). \tag{2.186}$$

Considering expression (2.181) we will have

$$\pi_{ik} = \lim_{t\to\infty} Q_{ik}(t) = \lim_{t\to\infty} \left\{ \frac{q_{ik}}{\sum_{j=1}^{N} q_{ij}} \left(1 - e^{-\sum_{j=1}^{N} q_{ij} t}\right) \right\} = \frac{q_{ik}}{\sum_{j=1}^{N} q_{ij}} \tag{2.187}$$

or equivalently

$$q_{ik} = \pi_{ik} \sum_{j=1}^{N} q_{ij}. \tag{2.188}$$

The probabilities of one-step transition π_{ik} of an embedded Markov chain can be approximated as a ratio of respective numbers of transitions based on an observed single realization of the output performance stochastic process.

$$\hat{\pi}_{ik} = \frac{k_{ik}}{k_i}. \tag{2.189}$$

When estimates (2.185) and (2.189) are substituted into expression (2.188), the transition intensity is estimated as follows:

$$\hat{q}_{ik} = \hat{\pi_{ik}}\hat{A} = \frac{k_{ik}}{k_i}\frac{1}{\hat{T}_{i_{mean}}} = \frac{k_{ik}}{T_{\sum i}},\ i,k = [1,...,N],\ i \neq k, \tag{2.190}$$

where $T_{\sum i}$ is the system's accumulated time residence in state i throughout total observation time T.

For a Markov MSS with N states the sum $\sum_{j=1}^{N} q_{ij} = 0$, since

$$\hat{q_{ii}} = -\sum_{j=1, i\neq j}^{N} \hat{q_{ij}}. \tag{2.191}$$

Algorithm for Point Estimation of Transition Intensities for Multi-State System

For a Markov multi-state systems with N available states, the following data processing algorithm is recommended.

1. Compute accumulated time of the system's stay in state i throughout total observation time T:

$$T_{\sum i} = \sum_{m=1}^{k_i} T_i^{(m)}.$$

2. Estimate transition intensity $\hat{q_{ij}}$ from state i to state $j \neq i$ using the following form:

$$\hat{q_{ij}} = \frac{k_{ij}}{T_{\sum i}}.$$

3. For $j = i$, estimate the transition intensities using the following equation:

$$\hat{q_{ii}} = -\sum_{j=1, i \neq j}^{N} \hat{q_{ij}}.$$

Interval Estimation of Transitions Intensities for Multi-State System

The output performance of the Multi State system was observed throughout time T; thus, in this instance we are dealing with a time-terminated test. Hence, according to expression (2.175), one can write the following two-sided confidence interval for the true value of q_{ij}:

$$\mathbb{P}\left\{\frac{1}{2T_{\sum i}}\ \chi^2_{a/2;2k_{ij}} \leq q_{ij} \leq \frac{1}{2T_{\sum i}}\ \chi^2_{1-a/2;2k_{ij}+2}\right\} = 1 - \alpha. \tag{2.192}$$

According to expression (2.176), the respective upper confidence bound or the one-side confidence interval for the true value of q_{ij} can be given as

$$\mathbb{P}\left\{q_{ij} \leq \frac{1}{2T_{\sum i}}\ \chi^2_{1-a;2k_{ij}+2}\right\} = 1 - \alpha. \tag{2.193}$$

Chapter 3

The H-class of Distributions

If $\mathbf{A}$ is a $n \times n$ matrix, then the permanent of $\mathbf{A}$, stated by per $\mathbf{A}$, is determined as

$$per\mathbf{A} = \sum_{\sigma \in S_n} \prod_{i=1}^{n} a_{i\sigma(i)}$$

where S_n is the set of permutations of $1, 2, ..., n$. So, The permanent is defined similarly to the determinant, except that all terms in the expansion are given a positive sign.

If $a_1, a_2, ...$ are column vectors, thereafter

$$\begin{matrix}[a_1 & a_2 & ...] \\ i_1 & i_2 & \end{matrix}$$

will state the matrix obtained by taking i_1 copies of a_1, i_2 copies of a_2 and so on.

Suppose $X_1, ..., X_n$ independent random variables with distribution functions $F_1, ..., F_n$ and densities $f_1, ..., f_n$ and $X_{1:n} \leq ... \leq X_{n:n}$ the respective order statistics.

Below we present that the density of any order statistic is easily written in terms of a permanent. Thus, the density of $X_{r:n} (1 \leq r \leq n)$ is obtained by

$$h_{r:n}(x) = \frac{1}{(r-1)!(n-r)!}\ per \begin{bmatrix} F_1(x) & 1 - F_1(x) & f_1(x) & \\ . & . & . & . \\ . & . & . & . \\ F_n(x) & 1 - F_n(x) & f_n(x) & \end{bmatrix},\ -\infty < x < \infty$$

In the same way, the distribution function of $X_{r:n} (1 \leq r \leq n)$ can be written in terms of permanents. Thus, the distribution function of $X_{r:n} (1 \leq r \leq n)$ is obtained by

$$H_{r:n}(x) = \sum_{i=r}^{n} \frac{1}{i!(n-i)!}\ per \begin{bmatrix} F_1(x) & 1 - F_1(x) \\ . & . \\ . & . \\ F_n(x) & 1 - F_n(x) \end{bmatrix},\ -\infty < x < \infty$$

If $S \subset N = \{1, 2, ..., n\}$ thereafter S' will be the complement of S in N and $|S|$ will be the cardinality of S. The $X_{r:S}$ designated the $r-th$ order statistic for $\{X_i | i \in S\}$ and $H_{r:S}(x)$, the distribution of $X_{r:S}$. We will replace S with its cardinality, when there is no confusion. For constant x, $\mathbf{F}$ will be the column vector $(F_1(x), ..., F_n(X))'$ and $\mathbf{1}$ the column of all ones. At last, $\mathbf{A}[S|.)$ is denoted the matrix given from $\mathbf{A}$ by taking all the rows whose indices are in S.

Theorem 3.1: (Balasubramanian, K. et al. (1991)) For arbitrary $F_1, F_2, ..., F_n$ and $n \geq 2$,

$$a)\ H_{r:n}(x) = \sum_{j=0}^{n-r} (-1)^{n-r-j} \binom{n-j-1}{n-r-j} \sum_{|S|=n-j} H_{|S|:S}(x)$$

$$b)\ H_{r:n}(x) = \sum_{j=0}^{r-1} (-1)^{r-j-1} \binom{n-j-1}{n-r} \sum_{|S|=n-j} H_{1:S}(x).$$

Assume the random variable X has an arbitrary distribution function $F(x)$. Two families of distribution functions with a positive parameter λ are defined as follows:

$$\textbf{Family I}\text{: } F^{\lambda}(x) = [F(x)]^{\lambda},\ \lambda > 0$$

and

$$\textbf{Family II}\text{: } F_{\lambda}(x) = 1 - [1 - F(x)]^{\lambda},\ \lambda > 0.$$

If $X^{(\lambda)}$ with distribution function $F^{\lambda}(x)$ and $X_1, ..., X_n$ are distributed independently as $X^{(\lambda_1)}, ..., X^{(\lambda_n)}$ respectively. Then

$$\begin{aligned} H_{|S|:S}(x) &= \prod_{i \in S} F^{\lambda_i}(x) = \prod_{i \in S} [F(x)]^{\lambda_i} \\ &= [F(x)]^{\lambda_S} = F^{\lambda_S}(x), \quad \lambda_S = \sum_{i \in S} \lambda_i \end{aligned}$$

and from (a) of Theorem 3.1, can be obtained

$$H_{r:n}(x) = \sum_{j=0}^{n-r} (-1)^{n-r-j} \binom{n-j-1}{n-r-j} \sum_{|S|=n-j} F^{\lambda_S}(x).$$

Also, if $X_{(\lambda)}$ with distribution function $F_{\lambda}(x)$ and $X_1, ..., X_n$ are independent distributed as $X_{(\lambda_1)}, ..., X_{(\lambda_n)}$ respectively. Then

$$H_{1:S}(x) = 1 - \prod_{i \in S}[1 - F_{\lambda_i}(x)] = 1 - \prod_{i \in S}[1 - F(x)]^{\lambda_i}$$
$$= 1 - [1 - F(x)]^{\lambda_S} = F_{\lambda_S}(x), \quad \lambda_S = \sum_{i \in S} \lambda_i$$

and from (b) of Theorem 3.1, can be obtained

$$H_{r:n}(x) = \sum_{j=0}^{r-1} (-1)^{r-j-1} \binom{n-j-1}{n-r} \sum_{|S|=n-j} F_{\lambda_S}(x).$$

It has been established (Makrides A., (2022)) that any distribution of the following structure

$$F(x; \lambda) = 1 - e^{-\lambda(H(x))}, \quad x \geq 0 \tag{3.1}$$

is closed under minimum, where $\lambda > 0$ is the shape parameter. Then, for an increasing function H(x) with $H(0) = 0$, (3.1) represents a class of distributions called the "H-class". The main feature of the family (3.1) is that the cdf of the minimum ordered statistic of a random sample $X_1, X_2, ..., X_n$ from (3.1) falls into the same family (see Barbu, V. et al. (2017)). Assuming that $h(\cdot)$ is the derivative of $H(\cdot)$, it is easy to see that the density function of a typical member of the H-class (3.1) is

$$f(x; \lambda) = \lambda h(x) e^{-\lambda(H(x))} \tag{3.2}$$

while the reliability function is given by

$$R(x; \lambda) = 1 - F(x; \lambda) = e^{-\lambda(H(x))}. \tag{3.3}$$

Then, for all members of the H-class the following equations hold

$$F(x; \lambda) = 1 - (1 - F(x; 1))^{\lambda} \tag{3.4}$$

and

$$R(x; \lambda) = (R(x; 1))^{\lambda} \tag{3.5}$$

where $F(x; \lambda)$ & $R(x; \lambda)$ represent the cumulative and reliability functions of the typical members of the H-class with shape parameter $\lambda = 1$. Representative members of the family are classical distributions like the Exponential and Weibull. Note that additional distributional parameters associated with $H(\cdot)$, may also be involved in (3.1). Note further that $H(\cdot)$ may not necessarily be a distribution.

3.1 H-class: A class closed under minima

In this section we establish that the H-class in (3.1) is closed under minima which is a significant property which is vital in the statistical inference of the multi-state setting of the next section.

Theorem *If $X_1, X_2, ..., X_n$ are independent identically distributed random variables' from (3.1), then the cdf F_{min} and the reliability function R_{min} of $X_{(1)}$ satisfy properties (3.4) & (3.5).*

Proof: For the required cdf we can easily see that

$$\begin{aligned} F_{min}(x;\lambda) = P(X_{(1)} \leq x) &= 1 - \prod_{i=1}^{n}(P(X_i \geq x)) \\ &= 1 - [R(x;\lambda)]^n = 1 - [e^{-\lambda H(x)}]^n \\ &= 1 - e^{-n\lambda H(x)} \end{aligned}$$

and

$$R_{min}(x;\lambda) = 1 - F_{min}(x;\lambda) = e^{-n\lambda H(x)}$$

such that

$$f_{min}(x;\lambda) = n[1 - F(x;\lambda)]^{n-1} f(x) = n[R(x;\lambda)]^{n-1} f(x)$$

$$= n[e^{-\lambda H(x)}]^{n-1} f(x)$$

which belongs to H-class in (3.1) with shape parameter $n\lambda$.

Remark The results of this section can be generalized by dropping the assumption of identically distributed random variables. Indeed, if one considers the case of independent random variables which though are not necessarily identically distributed (inid) and assumes a random sample $X_1, X_2, ..., X_n$ with the cumulative distribution function of X_i, $i = 1, ..., n$ denoted by

$$F(x;\lambda_i) = 1 - e^{-\lambda_i H(x)} \tag{3.6}$$

then, Theorem still holds with F_{min} belonging to the H-class (3.1) with parameter $\sum_{i=1}^{n}\lambda_i$, that is

$$F_{min}(x;\underset{\sim}{\lambda}) = 1 - e^{-\sum_{i=1}^{n}\lambda_i H(x)} \tag{3.7}$$

where $\underset{\sim}{\lambda} = (\lambda_1, ..., \lambda_n)'$ and also

$$R(x;\underset{\sim}{\lambda}) = e^{-\sum_{i=1}^{n}\lambda_i H(x)}. \tag{3.8}$$

3.2 Distributions of H-class

Below we present classical distributions that belong in the H-class.

Exponential distribution: The simplest family of distributions that is a subset of H-Class, is non other than the Exponential, where the H(·) function is linear.

$$f(x) = \lambda\, e^{-\lambda x} \quad \text{and} \quad F(x) = 1 - e^{-\lambda x},\ x\epsilon[0,\infty),\ \lambda > 0$$

Hence, $H(x) = x$ and $h(x) = 1$, with the parameter λ of H-class being the same as the λ of the Exponential distribution.

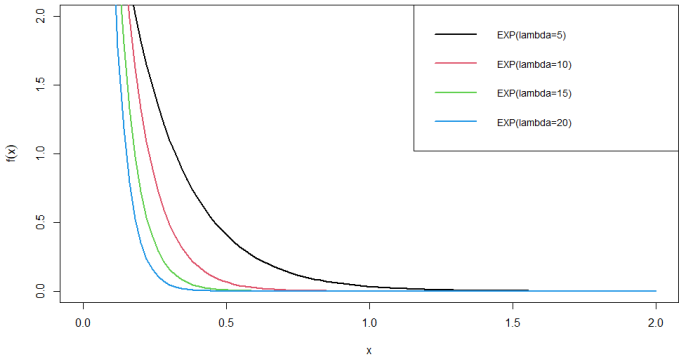

Figure 3.1: Plot of multiple pdfs of Exponential where the black line is for $\lambda = 5$, red is for $\lambda = 10$, green is for $\lambda = 15$ and blue is for $\lambda = 20$.

Rayleigh distribution: The next simplest family that belongs to H-Class, is Rayleigh, where the H(·) function is quadratic.

$$f(x) = \frac{x}{\sigma^2}\, e^{-(\frac{x}{\sqrt{2}\sigma})^2} \text{ and } F(x) = 1 - e^{-(\frac{x}{\sqrt{2}\sigma})^2},\ x\epsilon[0,\infty),\ \sigma > 0$$

Hence, $H(x) = x^2$ and $h(x) = 2x$, with parameter $\lambda = \frac{1}{2\sigma^2}$.

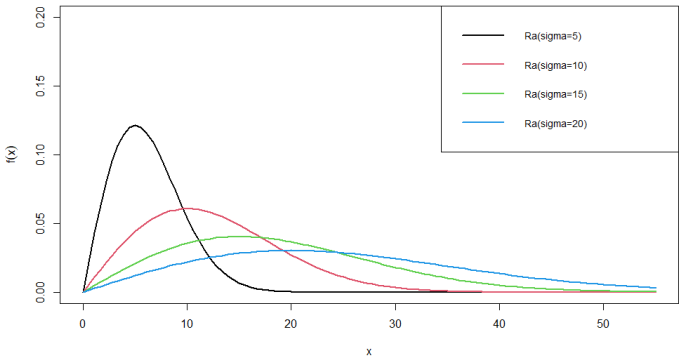

Figure 3.2: Plot of multiple pdfs of Rayleigh where the black line is for $\sigma = 5$, red is for $\sigma = 10$, green is for $\sigma = 15$ and blue is for $\sigma = 20$.

Weibull distribution: The next members of H-Class, is Weibull, where the H(·) function is a polynomial of order k.

$$f(x) = \frac{k}{a}\left(\frac{x}{a}\right)^{k-1} e^{-(t/a)^k} \text{ and } F(x) = 1 - e^{-(x/a)^k},\ x \geq 0,\ a\epsilon(0,+\infty),\ k\epsilon(0,+\infty)$$

Hence, $H(x) = \left(\frac{x}{a}\right)^k$ and $h(x) = \frac{k}{a}\left(\frac{x}{a}\right)^{k-1}$, with parameter $\lambda = 1$.

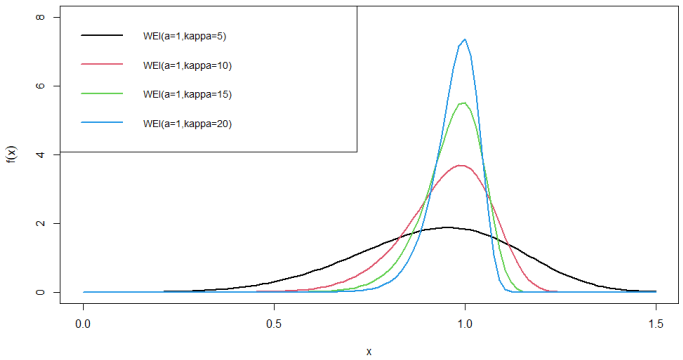

Figure 3.3: Plot of multiple pdfs of Weibull where the black line is for $a = 1$ and $k = 5$, red is for $a = 1$ and $k = 10$, green is for $a = 1$ and $k = 15$ and blue is for $a = 1$ and $k = 20$.

Gompertz distribution: Then we have Gompertz distribution, with the H(·) function being of exponential form.

$$f(x) = a\ e^{bx - \frac{a}{b}(e^{bx}-1)} \text{ and } F(x) = 1 - e^{-(\frac{a}{b})(e^{bx}-1)},\ x\epsilon[0,\infty),\ a,b > 0$$

Hence, $H(x) = e^{bx} - 1$ and $h(x) = be^{bx}$, with parameter $\lambda = \frac{a}{b}$.

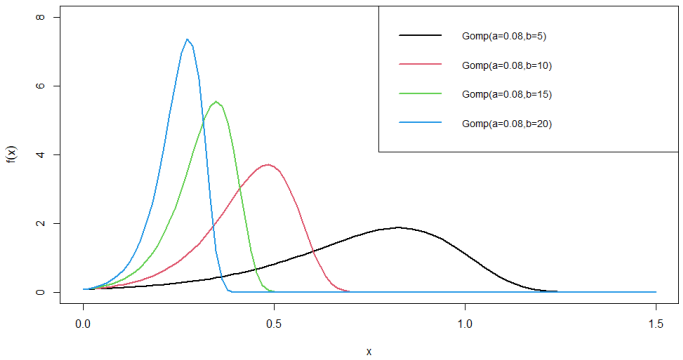

Figure 3.4: Plot of multiple pdfs of Gompertz where the black line is for $a = 0.08$ and $b = 5$, red is for $a = 0.08$ and $b = 10$, green is for $a = 0.08$ and $b = 15$ and blue is for $a = 0.08$ and $b = 20$.

Pareto distribution: Last but not least, we have Pareto distribution, with the H(·) function being of logarithmic form.

$$f(x) = \frac{ab^a}{x^{a+1}} = a\left(\frac{b}{x}\right)^a \ \frac{1}{x} = a\left(e^{ln(\frac{b}{x})^a}\right) \ \frac{1}{x} \quad \text{and}$$

$$F(x) = 1 - e^{ln(\frac{b}{x})^a} = 1 - e^{-a \ ln(\frac{x}{b})}, \ x\epsilon[b, \infty), \ a, b > 0$$

Hence, $H(x) = ln(\frac{x}{b})$ and $h(x) = \frac{1}{x}$, with parameter $\lambda = a$.

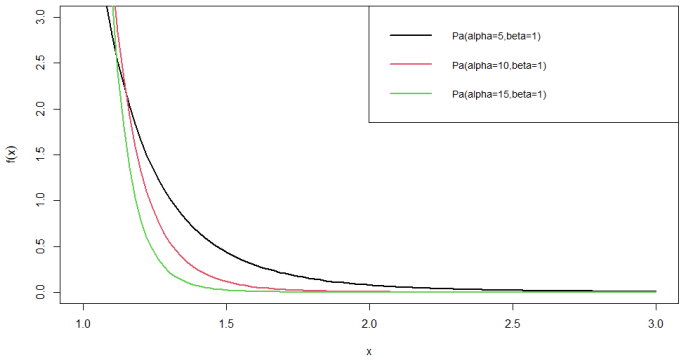

Figure 3.5: Plot of multiple pdfs of Pareto where the black line is for $a = 5$ and $b = 1$, red is for $a = 10$ and $b = 1$ and green is for $a = 15$ and $b = 1$.

3.3 Approximation of non H-class distributions

Consider the Modified Weibull distribution introduced in Delgarm, L. et al. (2015) defined by

$$G(z) = 1 - e^{-az^\gamma e^{\beta z}}, \quad z > 0, \ a, \gamma > 0, \ \beta \geq 0 \tag{3.9}$$

and

$$g(z) = az^{\gamma-1}(\gamma + \beta z)e^{\beta z - az^\gamma e^{\beta z}}, \quad z > 0, \ a, \gamma > 0, \ \beta \geq 0. \tag{3.10}$$

If N is a random variable with zero-truncated Poisson mass distribution with parameter λ then the conditional distribution of the minimum ordered statistic $X = Z_{(1)}$ of a random sample from (3.9) given N, is obtained by

$$f_{X|N}(x|n) = ane^{-anx^\gamma e^{\beta x}} x^{\gamma-1}(\gamma + \beta x)e^{\beta x}, \quad x > 0, \ a, \gamma > 0, \ \beta \geq 0 \tag{3.11}$$

and

$$F_{X|N}(x|n) = 1 - e^{-anx^\gamma e^{\beta x}} \tag{3.12}$$

Summing over all values of N we obtain the marginal distribution given below

$$f_X(x) = a\lambda x^{\gamma-1}(1-e^{-\lambda})^{-1}(\gamma+\beta x)\; e^{\beta x - ax^{\gamma}e^{\beta x} - \lambda(1-e^{-ax^{\gamma}e^{\beta x}})}. \tag{3.13}$$

The above distribution is known as the **Modified Weibull Poisson (MWP)** distribution with cumulative distribution function

$$F(x) = \frac{e^{\lambda}}{e^{\lambda}-1}\left(1-e^{-\lambda(1-e^{-ax^{\gamma}e^{\beta x}})}\right), x>0,\; a,\gamma>0,\; \beta>0. \tag{3.14}$$

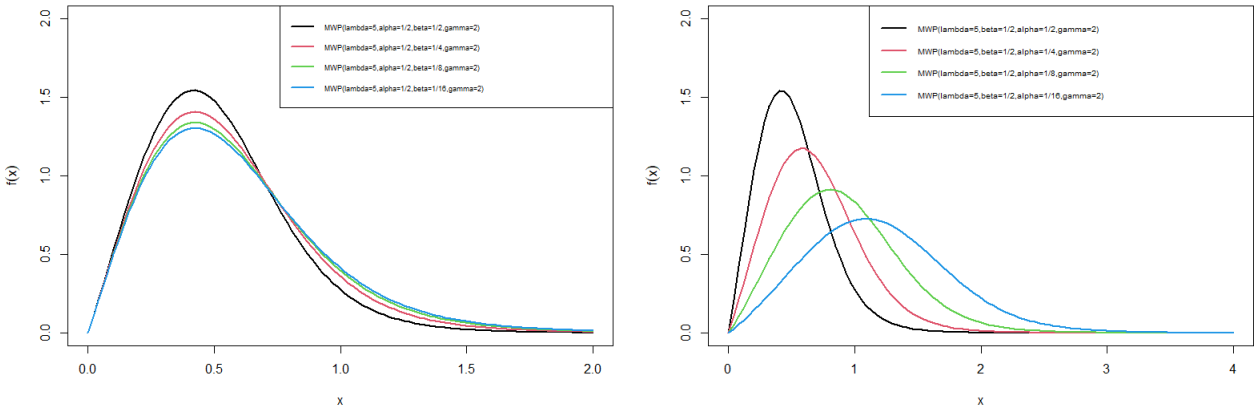

Figure 3.6: The left plot shows the multiple pdfs of MWP where the black is for $\lambda=5$, $a=1/2$, $\beta=1/2$ and $\gamma=2$, red is for $\lambda=5$, $a=1/2$, $\beta=1/4$ and $\gamma=2$, green is for $\lambda=5$, $a=1/2$, $\beta=1/8$ and $\gamma=2$ and blue $\lambda=5$, $a=1/2$, $\beta=1/16$ and $\gamma=2$. The right plot shows the multiple pdfs of MWP where the black is for $\lambda=5$, $a=1/2$, $\beta=1/2$ and $\gamma=2$, red is for $\lambda=5$, $a=1/4$, $\beta=1/2$ and $\gamma=2$, green is for $\lambda=5$, $a=1/8$, $\beta=1/2$ and $\gamma=2$ and blue $\lambda=5$, $a=1/16$, $\beta=1/2$ and $\gamma=2$.

The **Modified Weibull Poisson (MWP)** is of special interest in Reliability theory since it combines zero-truncated Poisson distribution (which give us the number of parts of the machine that will fail) and Weibull distribution (which is an appropriate distribution for modelling time until failure). Thus, Poisson will give us how many parts have failed, Weibull will give us at what time each one of them failed and the minimum between them, will answer us what is the minimum time until the first failure of the machine.
However the MWP is that it does not belong to the H-class. This is because of the constant $\frac{e^{\lambda}}{e^{\lambda}-1}$ that appears in equation (3.14). Hence, we have to search which distributions of H-class best approximate it. The criterion based on which we will deem two distributions close to one another will be the ratio of their densities to be close to one.

In order to showcase the ability to approximate MWP by H-class distributions we present below multiple examples.

We start to find the best approximation of MWP with parameters $\lambda = 5, \beta = 4, a = 1$ and $\gamma = 2$. Which are chosen arbitrarily. By trial and error, we find that the ratio is closed to one for Gompertz with parameters $a = 0.8$ and $b = 10$ in the interval (0.1,0.4). In this interval, MWP has an overall probability of $\mathbb{P}(0.1 < X < 0.4) = 0.7645428$.

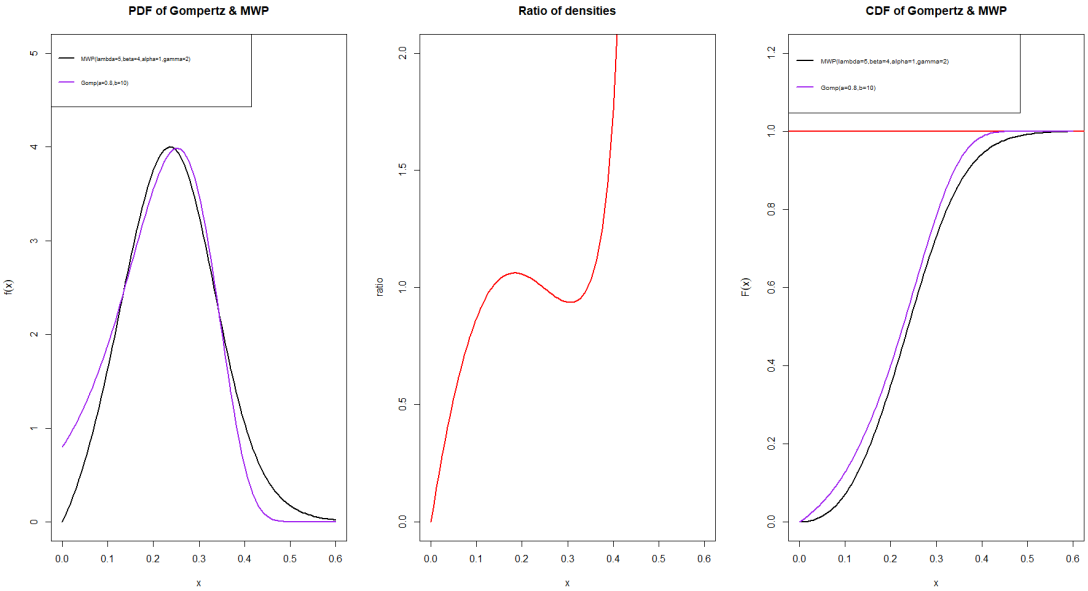

Figure 3.7: The left plot shows the pdf of MWP($\lambda = 5, \beta = 4, a = 1, \gamma = 2$) with black line and the pdf of Gompertz($a = 0.8, b = 10$) with purple line, while the right plot shows the cdf of distributions. The central plot shows the ratio of the densities of MWP & Gompetz distribution.

Moreover, the best approximation of MWP with parameters $\lambda = 10, \beta = 1/16, a = 1/16$ and $\gamma = 2$, is Gompertz with parameters $a = 1/2$ and $b = 1$ in the interval (0.5,2.5). In this interval, MWP has an overall probability of $\mathbb{P}(0.5 < X < 2.5) = 0.8266786$.

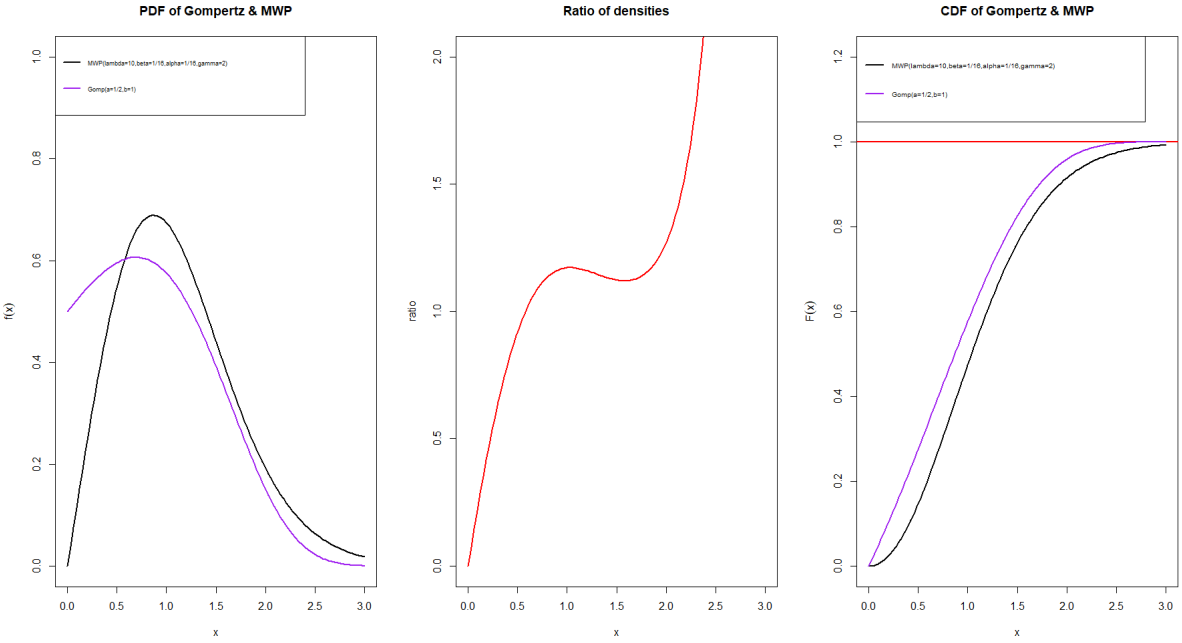

Figure 3.8: The left plot shows the pdf of MWP($\lambda = 10, \beta = 1/16, a = 1/16, \gamma = 2$) with black line and the pdf of Gompertz($a = 1/2, b = 1$) with purple line and the right plot shows the cdf of these. The central plot shows the ratio of densities.

Another example is the approximation of MWP with parameters $\lambda = 5, \beta = 1/16, a = 1/16$ and $\gamma = 2$, from Weibull with parameters $a = 1.8$ and $k = 2$ in the interval (0,4). In this interval, MWP has an overall probability of $\mathbb{P}(0 < X < 4) = 0.9796949$.

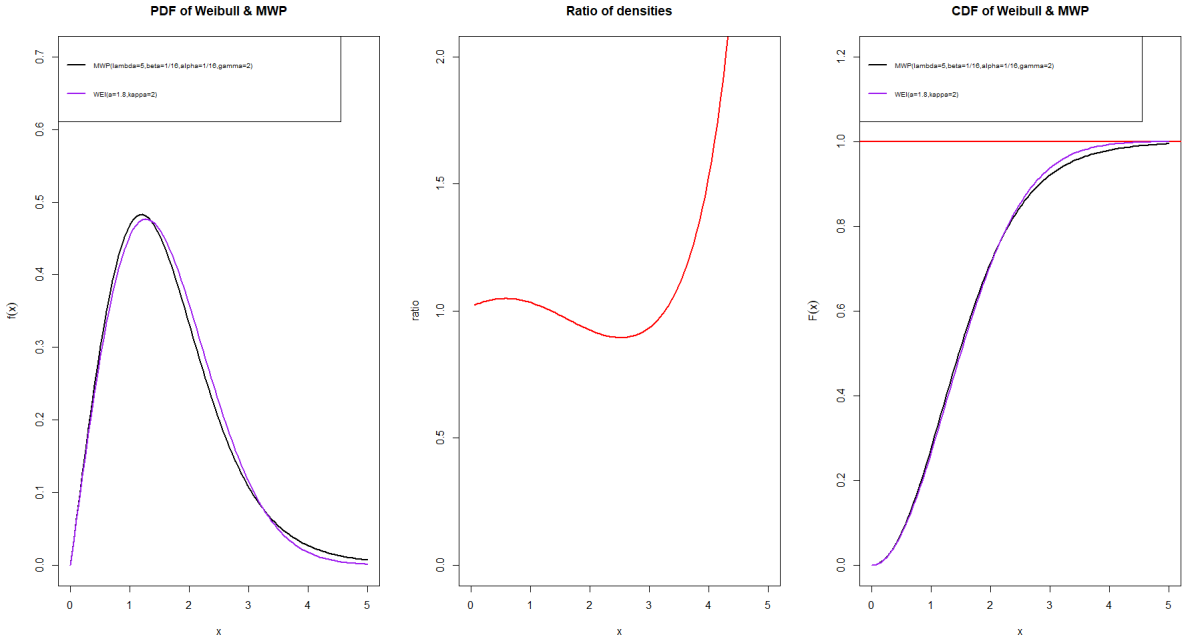

Figure 3.9: The left and right plots show the pdf and cdf of MWP($\lambda = 5, \beta = 1/16, a = 1/16, \gamma = 2$) with black line and Weibull($a = 1.8, k = 2$) with purple line respectively. The central plot shows the ratio of densities.

Also, the approximation of MWP with parameters $\lambda = 5, \beta = 1/2, a = 1$ and $\gamma = 2$ is Weibull with parameters $a = 1/2$ and $k = 2$ in the interval (0.3,1.3). In this interval, MWP has an overall probability of $\mathbb{P}(0.3 < X < 1.3) = 0.7413275$.

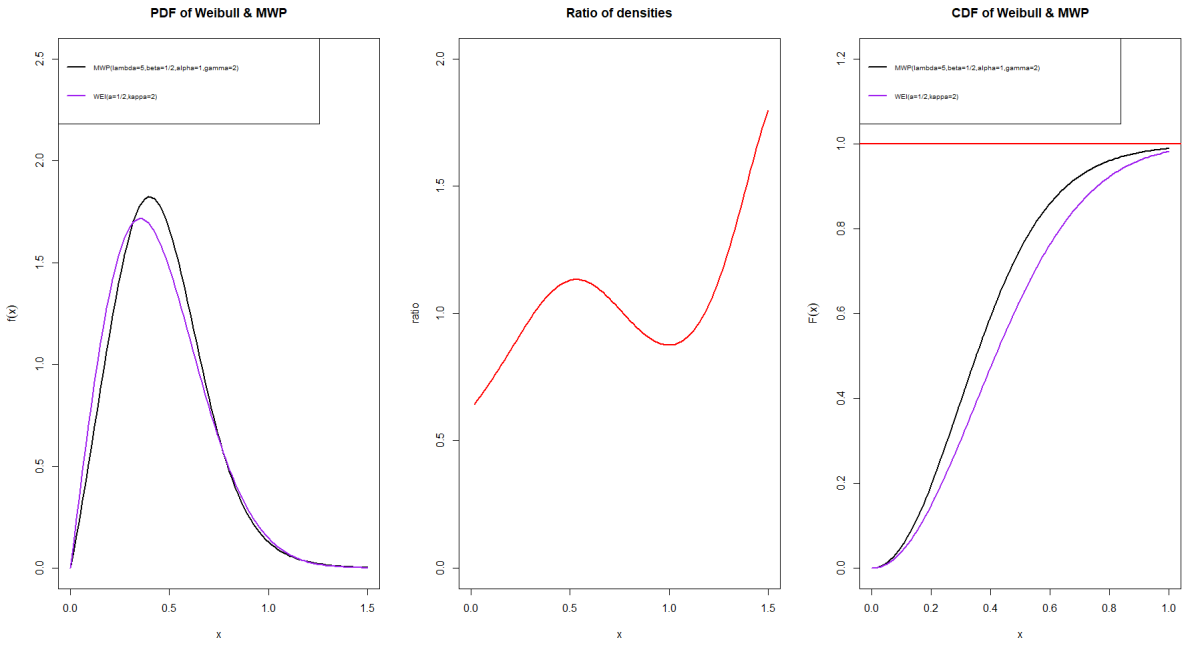

Figure 3.10: The left and right plot show the pdf and the cdf of MWP($\lambda = 5, \beta = 1/2, a = 1, \gamma = 2$) with black line and Weibull($a = 1/2, k = 2$) with purple line, respectively. The central plot shows the ratio of densities.

Furthermore, the best approximation of MWP with parameters $\lambda = 5, \beta = 1/200, a = 1/200$ and $\gamma = 2$, we find that is closed to one for Rayleigh with parameter $\sigma = 5$ in the interval (1,16). In this interval, MWP has an overall probability of $\mathbb{P}(1 < X < 16) = 0.9581969$.

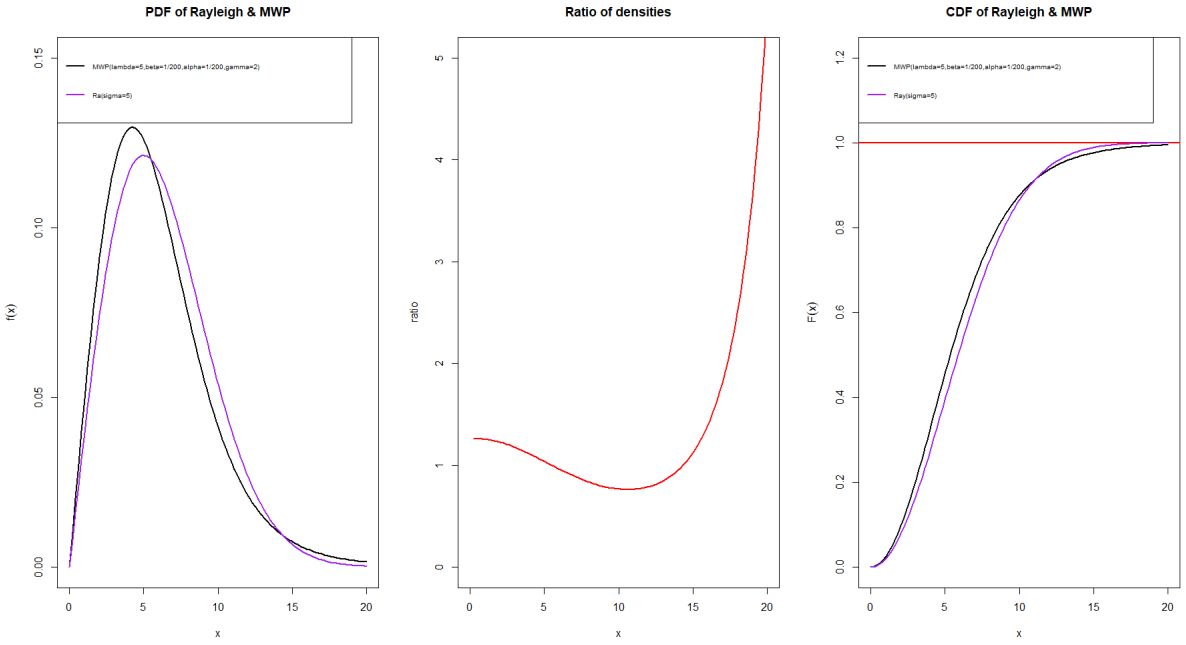

Figure 3.11: The left and right plots show the pdf and the cdf of MWP($\lambda = 5, \beta = 1/200, a = 1/200, \gamma = 2$) with black line and Rayleigh($\sigma = 5$) with purple line, respectively. The central plot shows the ratio of densities.

At last, the approximation of MWP with parameters $\lambda = 1/100, \beta = 1/100, a = 1/800$ and $\gamma = 2$, is Rayleigh with parameter $\sigma = 17$ in the interval (15,40). In this interval, MWP has an overall probability of $\mathbb{P}(15 < X < 40) = 0.1005515$.

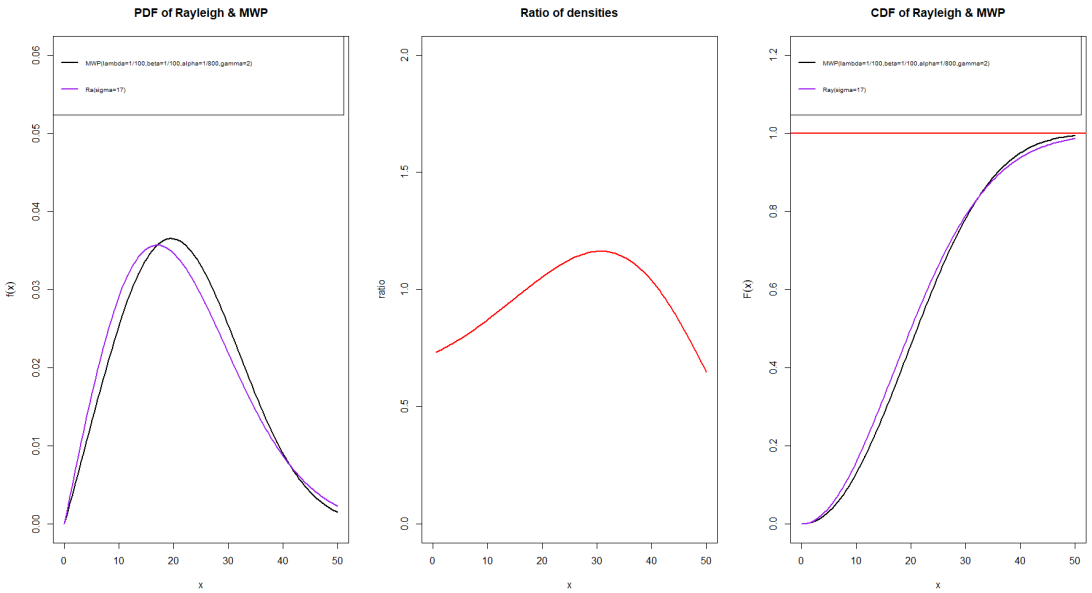

Figure 3.12: The left and right plots show the pdf and the cdf of MWP($\lambda = 1/100, \beta = 1/100, a = 1/800, \gamma = 2$) with black line and Ra($\sigma = 17$) with purple line, respectively. The central plot shows the ratio of the densities of MWP & Rayleigh distribution.

3.4 The Semi-Markov Model under the H-class

A specific system with N states is considered in this work which has the property that the next state to be visited after state i is the one for which T_{ik} is the minimum, where $T_{i1}, T_{i2}, ..., T_{iN}$ are the waiting times at state i before moving to state $1, 2, 3, ..., N$. Under these settings the following equations hold true for the class of distributions (3.1).

$$Q_{ik}(t) = \frac{\lambda_{ik}}{\sum_{j=1}^{N} \lambda_{ij}} \left(1 - e^{-\sum_{j=1}^{N} \lambda_{ij} H(t)} \right)$$

$$p_{ik} = \frac{\lambda_{ik}}{\sum_{j=1}^{N} \lambda_{ij}}$$

$$W_i(t) = 1 - e^{-\sum_{j=1}^{N} \lambda_{ij} H(t)}$$

$$w_i(t) = \sum_{j=1}^{N} \lambda_{ij} h(t) \left(1 - e^{-\sum_{j=1}^{N} \lambda_{ij} H(t)} \right)$$

where $W_i(t)$ coincide with the distribution $W_{ik}(t)$ that does not depend on the visited state k thus the subscript "k" is omitted. Similarly, the pdf is denoted by $w_i(t)$.

3.5 Parameter Estimators of the H-class distributions

Consider the following counting processes: First, for any states $i, k \epsilon E$ and $t \epsilon \mathbb{R}_+$, let us introduce the counting processes $N_i(t) =$ the number of visits to state i of process $(J_n)_{n\epsilon\mathbb{N}}$ up to time t, where $(J_n)_{n\epsilon\mathbb{N}}$ describe the states at time instant S_n and $(S_n)_{n\epsilon\mathbb{N}}$ are the successive time points. Also, $N_{ik}(t) =$ the number of jumps of $(J_n)_{n\epsilon\mathbb{N}}$ from state i to state j up to time t :

$$N_i(t) = \sum_{n=0}^{N(t)-1} \mathbf{1}_{\{J_n=i\}} = \sum_{n=0}^{\infty} \mathbf{1}_{\{J_n=i,\ S_{n+1}\leq t\}}$$

$$N_{ik}(t) = \sum_{n=1}^{N(t)} \mathbf{1}_{\{J_{n-1}=i, J_n=k\}} = \sum_{n=1}^{\infty} \mathbf{1}_{\{J_{n-1}=i, J_n=k, S_n\leq t\}}$$

where $i, k \epsilon E$, $t \epsilon \mathbb{R}_+$ and $N(t)$ is defined by $N(t) = max\{n \in \mathbb{N} | S_n \leq t\}$, as the amount of transitions up to time t.

Let M be the total observation time. We proceed now with the general form of the likelihood formula for L trajectories.

The evolution of the system, taking into account several sample paths $l = 1, ..., L$ of a semi-Markov process, $\{j_0^{(l)}, x_1^{(l)}, j_1^{(l)}, x_2^{(l)}, ..., j_{N^{(l)}(M)}^{(l)}\}$, can be expressed as

$$\mathcal{L} = \prod_{l=1}^{L} \lambda_{j_0}^{(l)} \; p_{j_0^{(l)} j_1^{(l)}} \; f_{j_0^{(l)}}(x_1^{(l)}) \; ... \; f_{j_{N^{(l)}(M)-1}^{(l)}}(x_{N^{(l)}(M)}^{(l)}) \tag{3.15}$$

$$\tag{3.16}$$

$$= \left(\prod_{i \epsilon E} \lambda_i^{N_{i,0}^{(L)}} \right) \left(\prod_{i,j \epsilon E} p_{ij}^{\sum_{l=1}^{L} N_{ik}^{(l)}(M)} \right) \left(\prod_{l=1}^{L} \prod_{i \epsilon E} \prod_{j=1}^{N_i^{(l)}(M)} f_i(x_i^{(l,r)}) \right) \tag{3.17}$$

where,

- $N_{i,0}^{(L)} = \sum_{l=1}^{L} \mathbf{1}_{\{J_0^{(l)} = i\}}$
- $N_i^{(l)}(M)$: the number of exits from state i up to time M of the l^{th} trajectory, l=1,...,L.
- $N_{ik}(L, M) = \sum_{l=1}^{L} N_{ik}^{(l)}(M)$
- $N_{ik}^{(l)}(M)$: the number of transitions from state i to state j up to time M of the l^{th} trajectory, l=1,...,L.
- $x_i^{(l,r)}$: the sojourn time in state i during the r^{th} visit, $r = 1, ..., N_i^{(l)}(M)$ of the l^{th} trajectory, l=1,...,L.
- $N_{i,M}(L) = \sum_{l=1}^{L} \mathbf{1}_{\{J_{N^{(l)}(M)}^{(l)} = i\}}$ is the number of visits of state i, as last visited state, over the L trajectories; observe that $\sum_{i \epsilon E} N_{i,M}(L) = L$

The above likelihood under the class (3.4) can be rewritten as

$$\mathcal{L} = \left(\prod_{i \epsilon E} \lambda_i^{N_{i,0}(L)} \right) \left(\prod_{l=1}^{L} \prod_{i,k \epsilon E} \lambda_{ik}^{N_{ik}^{(l)}(M)} \right) \times$$

$$\times \prod_{l,i,j} \left\{ \left(1 - F(x_i^{(l,r)}) \right)^{\sum_{k \epsilon E} \lambda_{ik}} \frac{f(x_i^{(l,r)})}{1 - F(x_i^{(l,r)})} \right\}$$

and we get the estimator of λ_{ik} given by

$$\hat{\lambda_{ik}}(L, M) = -\frac{N_{ik}(L, M)}{\sum_{l=1}^{L} B_{(i)}^{(l)}(M)},$$

where

$$B_{(i)}^{(l)}(M) = \sum_{k=1}^{N_i^{(l)}(M)} log\left(1 - F(X_i^{(l,r)}\right).$$

The initial probabilities can be estimated using the following expression

$$\hat{\lambda_i}(L, M) = \frac{N_{i,0}^{(L)}}{L}.$$

Using the proper expression among the previous ones, for the parameter estimates, the following estimators can be easily obtained:

$$\hat{p_{ik}}(M) = \frac{\hat{\lambda_{ik}}(L, M)}{\sum_{j=1}^{N} \hat{\lambda_{ij}}(L, M)} = \frac{N_{ik}(M)}{N_i(M)}, \tag{3.18}$$

$$\hat{W}_i(t, M) = \left(1 - e^{-H(t) \sum_{j\epsilon E} \hat{\lambda_{ij}}(L,M)}\right) \tag{3.19}$$

and

$$\hat{Q_{ik}}(t, M) = \frac{\hat{\lambda_{ik}}(L, M)}{\sum_{j\epsilon E} \hat{\lambda_{ij}}(L, M)} \left(1 - e^{-H(t) \sum_{j\epsilon E} \hat{\lambda_{ij}}(L,M)}\right). \tag{3.20}$$

3.5.1 The case of general H-class

In the general case of H-class of distributions, the estimator of the parameter λ_{ik} becomes

$$\hat{\lambda_{ik}}(L, M) = \frac{N_{ik}(L, M)}{\sum_{l=1}^{L} \sum_{j=1}^{N_i^{(l)}(M)} H\left(X_i^{(l,r)}\right)} \tag{3.21}$$

where for $H(\cdot)$ one can consider any distribution function.

3.5.2 The case of Gompertz distribution

The estimator of the parameter λ_{ik} for Gompertz can be modified as

$$\hat{\lambda_{ik}}(L, M) = \frac{N_{ik}(L, M)}{\sum_{l=1}^{L} \sum_{k=1}^{N_i^{(l)}(M)} \left(e^{b(x_i^{(l,r)})} - 1\right)} \tag{3.22}$$

3.5.3 The case of Weibull distribution

The estimator of the parameter λ_{ik} for Weibull can be modified as

$$\hat{\lambda_{ik}}(L,M) = \frac{N_{ik}(L,M)}{\sum_{l=1}^{L}\sum_{k=1}^{N_i^{(l)}(M)}\left((x_i^{(l,r)})^k\right)} \tag{3.23}$$

3.5.4 The case of Rayleigh distribution

The estimator of the parameter λ_{ik} for Rayleigh can be modified as

$$\hat{\lambda_{ik}}(L,M) = \frac{N_{ik}(L,M)}{\sum_{l=1}^{L}\sum_{k=1}^{N_i^{(l)}(M)}\left((x_i^{(l,r)})^2\right)} \tag{3.24}$$

Conclusion

The main purpose of this thesis was the study of semi-Markov Processes and Reliability Analysis, via theoretical issues and applications. After the presentation of the necessary theory on Order Statistics and multi-state systems, we introduced the H-class of distributions which is closed under minimum. The H-class is a very significant family of distributions, as it contains various distributions with the great variety of distributional shapes. However, some distributions, that we are interested in, do not belong to the H-class, like the MWP distribution.

In this work, we achieved two important goals. The first goal was the approximation of distributions which are not part of the H-class, i.e. they are not closed under minimum, by members of the H-class. More specifically, we focus on the Modified Weibull Poisson which is of great interest in Reliability theory since it combines zero-truncated Poisson distribution and the Weibull distribution for the simultaneous inference for the number of failures and the times to failure. The second goal was, using the multi-state systems and semi-Markov methodology, the construction of maximum likelihood estimators for the parameters of the distribution of H-class. These results are significant because they are providing a framework for statistical inference as well as the transition probabilities among the system's states for the members of H-class, but also for any distribution that can be approximated by members of H-class.

Last but not least, the flexibility of the approach can be guaranteed by the variety of distributions that belong in the H-class and whose distributional shapes cover a wide range of functional forms.

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Appendix A

Lemma: (Limnios, N. et al. (2001)) We have:

1. $S_n \to \infty$ (a.s.) as $n \to \infty$.

2. $N_t \to \infty$ (a.s.) as $t \to \infty$.

Proof: (1) For any fixed $a > 0$, by the Chebychev inequality, we get

$$\mathbb{P}(S_n \leq a) = \mathbb{P}(e^{-S_n} \geq e^{-a}) \leq \frac{E(e^{-S_n})}{e^{-a}} = e^a (Ee^{(X_1)})^n.$$

Therefore $F(0) < 1$ we have $Ee^{(X_1)} < 1$, which yields

$$\mathbb{P}(\lim_{n\to\infty} S_n \leq a) = \lim_{n\to\infty} \mathbb{P}(S_n \leq a) = 0.$$

(2) Due to the equality $\{S_{n-1} > t\} = \{N_t < n\}$ we can write, for any fixed $n \in \mathbb{N}^*$,

$$\mathbb{P}(\lim_{t\to\infty} N_t < n) = \lim_{t\to\infty} \mathbb{P}(N_t < n) = \lim_{t\to\infty} \mathbb{P}(S_{n-1} > t) = 0.$$

Limit Theorems for the Counting Process

In the sequel, we will suppose that $\mu = E(X_1) < +\infty$.

Theorem (Strong Law of Large Numbers)(Limnios, N. et al. (2001))

$$\frac{1}{t} N_t \xrightarrow{a.s.} \frac{1}{\mu} \; as \; t \to +\infty.$$

Proof: We have $N_t \to \infty$ as $t \to \infty$ (a.s.). On the other hand, from the inequalities $S_{N_t} \leq t < S_{N_t+1}$, we deduce that

$$\frac{X_1 + ... + X_{N_t}}{N_t} \leq \frac{t}{N_t} < \frac{X_1 + ... + X_{N_t} + X_{N_t+1}}{N_t}.$$

Now, since $X_{N_t+1} < \infty$ (a.s.), it follows that

$$X_{N_t+1}/N_t \to 0 \text{ (a.s.)} \quad \text{as } t \to +\infty.$$

We see now that the left and right terms of the above inequalities tend to the same limit (as a consequence of the strong law of large numbers for i.i.d. random variable). Since $E(X_1) = \mu$, the proof is achieved.

Appendix B

R code

```
####------ pdf MWP ------####

f <- function(x, lambda, alpha, beta, gamma) {
  y <- alpha*lambda*(x^(gamma-1))*((1-exp(-lambda))^(-1))*(gamma+beta*x)*exp(beta*x-alpha*(x^gamma)
  *exp(beta*x)-lambda*(1-exp(-alpha*(x^gamma) *exp(beta*x))))
}


###------lambda=5,alpha=1/2,beta=c(1/2,1/4,1/8,1/16),gamma=2
xs <-seq(0,2,length=100)
ys2 <-f(xs,lambda=5,alpha=1/2,beta=1/2,gamma=2)
ys3 <-f(xs,lambda=5,alpha=1/2,beta=1/4,gamma=2)
ys4 <-f(xs,lambda=5,alpha=1/2,beta=1/8,gamma=2)
ys5 <-f(xs,lambda=5,alpha=1/2,beta=1/16,gamma=2)


plot(xs,ys2,type="l",lwd=2,xlab="x",ylim = c(0,2),ylab=expression(f(x)))
lines(xs,ys3,type="l",col=2,lwd=2)
lines(xs,ys4,type="l",col=3,lwd=2)
lines(xs,ys5,type="l",col=4,lwd=2)
legend("topright",cex=0.55,c("MWP(lambda=5,alpha=1/2,beta=1/2,gamma=2)",
"MWP(lambda=5,alpha=1/2,beta=1/4,gamma=2)","MWP(lambda=5,alpha=1/2,beta=1/8,gamma=2)",
"MWP(lambda=5,alpha=1/2,beta=1/16,gamma=2)"),col=1:4,lty=rep(1,4),lwd=rep(2,4))


###------lambda=5,beta=1/2,alpha=c(1/2,1/4,1/8,1/16),gamma=2
xs <-seq(0,4,length=100)
ys2 <-f(xs,lambda=5,beta=1/2,alpha=1/2,gamma=2)
ys3 <-f(xs,lambda=5,beta=1/2,alpha=1/4,gamma=2)
ys4 <-f(xs,lambda=5,beta=1/2,alpha=1/8,gamma=2)
ys5 <-f(xs,lambda=5,beta=1/2,alpha=1/16,gamma=2)


plot(xs,ys2,type="l",lwd=2,xlab="x",ylim = c(0,2),ylab=expression(f(x)))
lines(xs,ys3,type="l",col=2,lwd=2)
lines(xs,ys4,type="l",col=3,lwd=2)
lines(xs,ys5,type="l",col=4,lwd=2)
legend("topright",cex=0.6,c("MWP(lambda=5,beta=1/2,alpha=1/2,gamma=2)",
"MWP(lambda=5,beta=1/2,alpha=1/4,gamma=2)","MWP(lambda=5,beta=1/2,alpha=1/8,gamma=2)",
"MWP(lambda=5,beta=1/2,alpha=1/16,gamma=2)"),col=1:4,lty=rep(1,4),lwd=rep(2,4))



####------pdf Exp------####

f <- function(x, lambda) {
  y <- lambda*exp(-lambda*x)
}

###------lambda=c(5,10,15,20)
xs <-seq(0,1.5,length=100)
ys2 <-f(xs,lambda=5)
ys3 <-f(xs,lambda=10)
ys4 <-f(xs,lambda=15)
ys5 <-f(xs,lambda=20)
```

```
plot(xs,ys2,type="l",lwd=2,xlab="x",ylim = c(0,1.5),ylab=expression(f(x)))
lines(xs,ys3,type="l",col=2,lwd=2)
lines(xs,ys4,type="l",col=3,lwd=2)
lines(xs,ys5,type="l",col=4,lwd=2)
legend("topright",cex=0.9,c("EXP(lambda=5)","EXP(lambda=10)","EXP(lambda=15)",
"EXP(lambda=20)"),col=1:4,lty=rep(1,4),lwd=rep(2,4))


####------pdf Rayleigh----####

f <- function(x, sigma) {
  y <- (x/(sigma^2))*exp((-x^2)/(2*sigma^2))
}

###------lambda=c(5,10,15,20)
xs <-seq(0,55,length=100)
ys2 <-f(xs,sigma=5)
ys3 <-f(xs,sigma=10)
ys4 <-f(xs,sigma=15)
ys5 <-f(xs,sigma=20)


plot(xs,ys2,type="l",lwd=2,xlab="x",ylim = c(0,0.2),ylab=expression(f(x)))
lines(xs,ys3,type="l",col=2,lwd=2)
lines(xs,ys4,type="l",col=3,lwd=2)
lines(xs,ys5,type="l",col=4,lwd=2)
legend("topright",cex=0.9,c("Ra(sigma=5)","Ra(sigma=10)","Ra(sigma=15)",
"Ra(sigma=20)"),col=1:4,lty=rep(1,4),lwd=rep(2,4))


####------pdf Weibull----####

f <- function(x, a, kappa) {
  y <- (kappa/a)*((x/a)^(kappa-1))*(exp(-(x/a)^kappa))
}


###------a=1,kappa=c(5,10,15,20)
xs <-seq(0,1.5,length=100)
ys2 <-f(xs,a=1,kappa=5)
ys3 <-f(xs,a=1,kappa=10)
ys4 <-f(xs,a=1,kappa=15)
ys5 <-f(xs,a=1,kappa=20)


plot(xs,ys2,type="l",lwd=2,xlab="x",ylim = c(0,8),ylab=expression(f(x)))
lines(xs,ys3,type="l",col=2,lwd=2)
lines(xs,ys4,type="l",col=3,lwd=2)
lines(xs,ys5,type="l",col=4,lwd=2)
legend("topleft",cex=0.9,c("WEI(a=1,kappa=5)","WEI(a=1,kappa=10)",
"WEI(a=1,kappa=15)","WEI(a=1,kappa=20)"),col=1:4,lty=rep(1,4),lwd=rep(2,4))



####------pdf Gompertz----####

f <- function(x, a, b) {
  y <- a*exp(b*x-((a/b)*(exp(b*x)-1)))
}


###------alpha=0.08, beta=c(5,10,15,20)
xs <-seq(0,1.5,length=100)
ys2 <-f(xs,a=0.08,b=5)
ys3 <-f(xs,a=0.08,b=10)
ys4 <-f(xs,a=0.08,b=15)
ys5 <-f(xs,a=0.08,b=20)

plot(xs,ys2,type="l",lwd=2,xlab="x",ylim = c(0,8),ylab=expression(f(x)))
lines(xs,ys3,type="l",col=2,lwd=2)
lines(xs,ys4,type="l",col=3,lwd=2)
```

```
lines(xs,ys5,type="l",col=4,lwd=2)
legend("topright",cex=0.9,c("Gomp(a=0.08,b=5)","Gomp(a=0.08,b=10)", "Gomp(a=0.08,b=15)",
"Gomp(a=0.08,b=20)"),col=1:4,lty=rep(1,4),lwd=rep(2,4))


####-----pdf Pareto----####

f <- function(x, alpha, beta) {
  y <- (alpha*(beta^alpha))/(x^(alpha+1))
}


###-----,alpha=c(5,10,15),beta=1
xs <-seq(1,3,length=100)
ys2 <-f(xs,alpha=5,beta=1)
ys3 <-f(xs,alpha=10,beta=1)
ys4 <-f(xs,alpha=15,beta=1)



plot(xs,ys2,type="l",lwd=2,xlab="x",ylim = c(0,3),ylab=expression(f(x)))
lines(xs,ys3,type="l",col=2,lwd=2)
lines(xs,ys4,type="l",col=3,lwd=2)
legend("topright",cex=0.9,c("Pa(alpha=5,beta=1)","Pa(alpha=10,beta=1)",
"Pa(alpha=15,beta=1)"),col=1:3,lty=rep(1,3),lwd=rep(2,3))



####-----ratio mwp & gompertz----####

#No1

#gompertz
f1 <- function(x, a, b) {
  y1 <- a*exp(b*x-((a/b)*(exp(b*x)-1)))
}

#mwp
f2 <- function(x, lambda, alpha, beta, gamma) {
  y <- alpha*lambda*(x^(gamma-1))*((1-exp(-lambda))^(-1))*(gamma+beta*x)
  *exp(beta*x-alpha*(x^gamma)*exp(beta*x)-lambda*( 1-exp(-alpha*(x^gamma) *exp(beta*x))))
}


xs <-seq(0,0.6,length=100)
ys1 <-f2(xs,lambda=5,beta=4,alpha=1,gamma=2)
ys2 <-f1(xs,a=0.8,b=10)


plot(xs,ys1,type="l",main="PDF of Gompertz & MWP",lwd=2,xlab="x",ylim = c(0,5),ylab=expression(f(x)))
lines(xs,ys2,type="l",col="purple",lwd=2)
legend("topleft",cex=0.55,c("MWP(lambda=5,beta=4,alpha=1,gamma=2)",
"Gomp(a=0.8,b=10)"),col=c(1,"purple"),lty=rep(1,5),lwd=rep(2,5))


#ratio
ys3 <-ys1/ys2
plot(xs,ys3,type="l",col="red",main="Ratio of densities",lwd=2,xlab="x",
ylim = c(0,2),ylab=expression(ratio))


#No2

#gompertz
f1 <- function(x, a, b) {
  y1 <- a*exp(b*x-((a/b)*(exp(b*x)-1)))
}

#mwp
f2 <- function(x, lambda, alpha, beta, gamma) {
  y <- alpha*lambda*(x^(gamma-1))*((1-exp(-lambda))^(-1))*(gamma+beta*x)
  *exp(beta*x-alpha*(x^gamma)*exp(beta*x)-lambda*( 1-exp(-alpha*(x^gamma) *exp(beta*x))))
}
```

```
xs <-seq(0,3,length=100)
ys1 <-f2(xs,lambda=10,beta=1/16,alpha=1/16,gamma=2)
ys2 <-f1(xs,a=1/2,b=1)


plot(xs,ys1,type="l",main="PDF of Gompertz & MWP",lwd=2,xlab="x",ylim = c(0,1),ylab=expression(f(x)))
lines(xs,ys2,type="l",col="purple",lwd=2)
legend("topleft",cex=0.55,c("MWP(lambda=10,beta=1/16,alpha=1/16,gamma=2)",
"Gomp(a=1/2,b=1)"),col=c(1,"purple"),lty=rep(1,5),lwd=rep(2,5))


#ratio
ys3<-ys1/ys2
plot(xs,ys3,type="l",col="red",main="Ratio of densities",lwd=2,xlab="x",
ylim = c(0,2),ylab=expression(ratio))




####-----ratio mwp & weibull---

#No1

#weibull
f1 <- function(x, a, kappa) {
  y <- (kappa/a)*((x/a)^(kappa-1))*(exp(-(x/a)^kappa))
}

#mwp
f2 <- function(x, lambda, alpha, beta, gamma) {
  y <- alpha*lambda*(x^(gamma-1))*((1-exp(-lambda))^(-1))*(gamma+beta*x)
  *exp(beta*x-alpha*(x^gamma)*exp(beta*x)-lambda
  *( 1-exp(-alpha*(x^gamma) *exp(beta*x))))
}


xs <-seq(0,5,length=100)
ys1 <-f2(xs,lambda=5,alpha=1/16,beta=1/16,gamma=2)
ys2 <-f1(xs,a=1.8,kappa=2)


plot(xs,ys1,type="l",main="PDF of Weibull & MWP",lwd=2,xlab="x",ylim = c(0,0.7),ylab=expression(f(x)))
lines(xs,ys2,type="l",col="purple",lwd=2)
legend("topleft",cex=0.6,c("MWP(lambda=5,beta=1/16,alpha=1/16,gamma=2)",
"WEI(a=1.8,kappa=2)"),col=c(1,"purple"),lty=rep(1,5),lwd=rep(2,5))


#ratio
ys3 <-ys1/ys2
plot(xs,ys3,type="l",col="red",main="Ratio of densities",lwd=2,xlab="x",
ylim = c(0,2),ylab=expression(ratio))


#No2

#weibull
f1 <- function(x, a, kappa) {
  y <- (kappa/a)*((x/a)^(kappa-1))*(exp(-(x/a)^kappa))
}

#mwp
f2 <- function(x, lambda, alpha, beta, gamma) {
  y <- alpha*lambda*(x^(gamma-1))*((1-exp(-lambda))^(-1))*(gamma+beta*x)
  *exp(beta*x-alpha*(x^gamma)*exp(beta*x)-lambda*( 1-exp(-alpha*(x^gamma) *exp(beta*x))))
}


xs <-seq(0,1.5,length=100)
ys1 <-f2(xs,lambda=5,alpha=1/2,beta=1,gamma=2)
ys2 <-f1(xs,a=1/2,kappa=2)
```

```
plot(xs,ys1,type="l",main="PDF of Weibull & MWP",lwd=2,xlab="x",ylim = c(0,2.5),ylab=expression(f(x)))
lines(xs,ys2,type="l",col="purple",lwd=2)
legend("topleft",cex=0.6,c("MWP(lambda=5,beta=1/2,alpha=1,gamma=2)",
"WEI(a=1/2,kappa=2)"),col=c(1,"purple"),lty=rep(1,5),lwd=rep(2,5))


#ratio
ys3 <-ys1/ys2
plot(xs,ys3,type="l",col="red",main="Ratio of densities",lwd=2,xlab="x"
,ylim = c(0,2),ylab=expression(ratio))



####-----ratio rayleigh-MWP-----

#No1

#rayleigh
f1 <- function(x, sigma) {
  y <- (x/(sigma^2))*exp((-x^2)/(2*sigma^2))
}

#mwp
f2 <- function(x, lambda, alpha, beta, gamma) {
  y2 <- alpha*lambda*(x^(gamma-1))*((1-exp(-lambda))^(-1))*(gamma+beta*x)
  *exp(beta*x-alpha*(x^gamma)*exp(beta*x)-lambda*( 1-exp(-alpha*(x^gamma) *exp(beta*x))))
}


xs <-seq(0,20,length=100)
ys1 <-f2(xs,lambda=5,beta=1/200,alpha=1/200,gamma=2)
ys2 <-f1(xs,sigma=5)


plot(xs,ys1,type="l",main="PDF of Rayleigh & MWP",lwd=2,xlab="x",ylim = c(0,0.15),ylab=expression(f(x)))
lines(xs,ys2,type="l",col="purple",lwd=2)
legend("topleft",cex=0.6,c("MWP(lambda=5,beta=1/200,alpha=1/200,gamma=2)",
"Ra(sigma=5)"),col=c(1,"purple"),lty=rep(1,5),lwd=rep(2,5))


#ratio
ys3 <-ys1/ys2
plot(xs,ys3,type="l",col="red",main="Ratio of densities",lwd=2,xlab="x",
ylim = c(0,5),ylab=expression(ratio))


#No2

#rayleigh
f1 <- function(x, sigma) {
  y <- (x/(sigma^2))*exp((-x^2)/(2*sigma^2))
}

#mwp
f2 <- function(x, lambda, alpha, beta, gamma) {
  y2 <- alpha*lambda*(x^(gamma-1))*((1-exp(-lambda))^(-1))*(gamma+beta*x)
  *exp(beta*x-alpha*(x^gamma)*exp(beta*x)-lambda*( 1-exp(-alpha*(x^gamma) *exp(beta*x))))
}


xs <-seq(0,50,length=100)
ys1 <-f2(xs,lambda=1/100,beta=1/100,alpha=1/800,gamma=2)
ys2 <-f1(xs,sigma=17)


plot(xs,ys1,type="l",main="PDF of Rayleigh & MWP",lwd=2,xlab="x",ylim = c(0,0.06),ylab=expression(f(x)))
lines(xs,ys2,type="l",col="purple",lwd=2)
legend("topleft",cex=0.6,c("MWP(lambda=1/100,beta=1/100,alpha=1/800,gamma=2)",
"Ra(sigma=17)"),col=c(1,"purple"),lty=rep(1,5),lwd=rep(2,5))
```

```
#ratio
ys3 <-ys1/ys2
plot(xs,ys3,type="l",col="red",main="Ratio of densities",lwd=2,xlab="x",
ylim = c(0,2),ylab=expression(ratio))



####-----cdf MWP & Gompertz----


#gompertz
F1 <- function(x, a, b) {
  y1 <- (1-exp(-(-(a/b)*(1-exp(b*x)))))
}

#mwp
F2 <- function(x, lambda, alpha, beta, gamma) {
  y2 <- (exp(lambda)/(exp(lambda)-1))*(1-exp(-lambda*( 1-exp(-alpha*(x^gamma)*exp(beta*x) ) ) ))
}


#No1

xs <-seq(0,0.6,length=100)
ys1 <-F2(xs,lambda=5,beta=4,alpha=1,gamma=2)
ys2 <-F1(xs,a=0.8,b=10)


plot(xs,ys1,type="l",main="CDF of Gompertz & MWP",lwd=2,xlab="x",ylim = c(0,1.2),ylab=expression(F(x)))
abline(h=1, col="red",lwd=2)
lines(xs,ys2,type="l",col="purple",lwd=2)
legend("topleft",cex=0.6,c("MWP(lambda=5,beta=4,alpha=1,gamma=2)",
"Gomp(a=0.8,b=10)"),col=c(1,"purple"),lty=rep(1,5),lwd=rep(2,5))


#No2

xs <-seq(0,3,length=100)
ys1 <-F2(xs,lambda=10,beta=1/16,alpha=1/16,gamma=2)
ys2 <-F1(xs,a=1/2,b=1)


plot(xs,ys1,type="l",main="CDF of Gompertz & MWP",lwd=2,xlab="x",ylim = c(0,1.2),ylab=expression(F(x)))
abline(h=1, col="red",lwd=2)
lines(xs,ys2,type="l",col="purple",lwd=2)
legend("topleft",cex=0.6,c("MWP(lambda=10,beta=1/16,alpha=1/16,gamma=2)",
"Gomp(a=1/2,b=1)"),col=c(1,"purple"),lty=rep(1,5),lwd=rep(2,5))


####-----cdf MWP & Weibull----


#weibull
F1 <- function(x, a, kappa) {
  y1 <- 1-exp(-(x/a)^kappa)
}

#mwp
F2 <- function(x, lambda, alpha, beta, gamma) {
  y2 <- (exp(lambda)/(exp(lambda)-1))*(1-exp(-lambda*( 1-exp(-alpha*(x^gamma)*exp(beta*x) ) ) ))
}


#No1
xs <-seq(0,5,length=100)
ys1 <-F2(xs,lambda=5,beta=1/16,alpha=1/16,gamma=2)
ys2 <-F1(xs,a=1.8,kappa=2)


plot(xs,ys1,type="l",main="CDF of Weibull & MWP",lwd=2,xlab="x",ylim = c(0,1.2),ylab=expression(F(x)))
abline(h=1, col="red",lwd=2)
```

```
lines(xs,ys2,type="l",col="purple",lwd=2)
legend("topleft",cex=0.6,c("MWP(lambda=5,beta=1/16,alpha=1/16,gamma=2)",
"WEI(a=1.8,kappa=2)"),col=c(1,"purple"),lty=rep(1,5),lwd=rep(2,5))


#No2
xs <-seq(0,1,length=100)
ys1 <-F2(xs,lambda=5,beta=1/2,alpha=1,gamma=2)
ys2 <-F1(xs,a=1/2,kappa=2)


plot(xs,ys1,type="l",main="CDF of Weibull & MWP",lwd=2,xlab="x",ylim = c(0,1.2),ylab=expression(F(x)))
abline(h=1, col="red",lwd=2)
lines(xs,ys2,type="l",col="purple",lwd=2)
legend("topleft",cex=0.6,c("MWP(lambda=5,beta=1/2,alpha=1,gamma=2)",
"WEI(a=1/2,kappa=2)"),col=c(1,"purple"),lty=rep(1,5),lwd=rep(2,5))



####-----cdf MWP & Rayleigh----

#ray
F1 <- function(x, sigma) {
  y1 <- 1-exp((-x^2)/(2*sigma^2))
}

#mwp
F2 <- function(x, lambda, alpha, beta, gamma) {
  y2 <- (exp(lambda)/(exp(lambda)-1))*(1-exp(-lambda*( 1-exp(-alpha*(x^gamma)*exp(beta*x) ) ) ))
}

#No1
xs <-seq(0,20,length=100)
ys1 <-F2(xs,lambda=5,beta=1/200,alpha=1/200,gamma=2)
ys2 <-F1(xs,sigma=5)


plot(xs,ys1,type="l",main="CDF of Rayleigh & MWP",lwd=2,xlab="x",ylim = c(0,1.2),ylab=expression(F(x)))
abline(h=1, col="red",lwd=2)
lines(xs,ys2,type="l",col="purple",lwd=2)
legend("topleft",cex=0.6,c("MWP(lambda=5,beta=1/200,alpha=1/200,gamma=2)",
"Ray(sigma=5)"),col=c(1,"purple"),lty=rep(1,5),lwd=rep(2,5))


#No2
xs <-seq(0,50,length=100)
ys1 <-F2(xs,lambda=1/100,beta=1/100,alpha=1/800,gamma=2)
ys2 <-F1(xs,sigma=17)


plot(xs,ys1,type="l",main="CDF of Rayleigh & MWP",lwd=2,xlab="x",ylim = c(0,1.2),ylab=expression(F(x)))
abline(h=1, col="red",lwd=2)
lines(xs,ys2,type="l",col="purple",lwd=2)
legend("topleft",cex=0.6,c("MWP(lambda=1/100,beta=1/100,alpha=1/800,gamma=2)",
"Ray(sigma=17)"),col=c(1,"purple"),lty=rep(1,5),lwd=rep(2,5))



####-----posostiaia----

#ray
F4 <- function(x, sigma) {
  y1 <- 1-exp((-x^2)/(2*sigma^2))
}

#weibull
F3 <- function(x, a, kappa) {
  y1 <- 1-exp(-(x/a)^kappa)
}

#gompertz
F1 <- function(x, a, b) {
  y1 <- (1-exp(-(-(a/b)*(1-exp(b*x)))))
```

```
}

#MWP
F2 <- function(x, lambda, alpha, beta, gamma) {
  y2 <- (exp(lambda)/(exp(lambda)-1))*(1-exp(-lambda*( 1-exp(-alpha*(x^gamma)*exp(beta*x) ) ) ))
}


#ratio 1
myfun1 <-F2(0.4,5,4,1,2)-F2(0.1,5,4,1,2)
my_fun1 <-F1(0.4,0.8,10)-F1(0.1,0.8,10)


#ratio 2
myfun2 <-F2(2.5,10,1/16,1/16,2)-F2(0.5,10,1/16,1/16,2)
my_fun2 <-F1(2.5,1/2,1)-F1(0.5,1/2,1)


#ratio 3
myfun3 <-F2(4,5,1/16,1/16,2)-F2(0,5,1/16,1/16,2)
my_fun3 <-F3(4,1.8,2)-F3(0,1.8,2)

#ratio 4
myfun4 <-F2(1.3,5,1/2,1,2)-F2(0.3,5,1/2,1,2)
my_fun4 <-F3(1.3,1/2,2)-F3(0.3,1/2,2)

#ratio 5
myfun5 <-F2(16,5,1/200,1/200,2)-F2(1,5,1/200,1/200,2)
my_fun5 <-F4(16,5)-F4(1,5)

#ratio 6
myfun6 <-F2(40,1/100,1/100,1/800,2)-F2(15,1/100,1/100,1/800,2)
my_fun6 <-F4(40,17)-F4(15,17)
```