# RISK MEASURES APPLICATIONS IN REGULATION AND INVESTMENT

**Georgios C. Zachos**

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# Contents

APPROVED BY SUPERVISING COMMITTEE:

Supervisors:

Professor Dimitrios G. Konstantinides,

Professor Angela Black,

and Professor Vakeroudis Stavros.

**To science...**

**and its magic!**

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# Publications

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# Abstract

Risk measures are inarguably the Economic Regulator Authorities' favorite tool for preventing crises and controlling the risk existing in Financial institutions and markets. Take for instance the Basel Committee of Banking Supervision's consultive documents (see for instance [**26**] and [**27**]) and it is clear that risk measures and their proper use are perceived as their most important instrument. Also academics are taking that into account and so they try to evolve their research on risk measures depending on the occasion and also they try to meet the ever changing needs. Last but no least we have the practitioners (for instance investors) who are always in pursuit for profit by utilizing the most advanced methods and tools and perhaps if we consider Adam Smith's invisible hand they indirectly contribute to the financial markets prosperity.

In addition we observe that we are leaving in a world of continuous economic crises, most of them interconnected in a global scale. The starting point of this era is the global crisis emerging from late 2007 as a subprime mortgage crisis in U.S.A.. Taking into account the most recent energy crisis and Food supply crisis, both because of the war in Ukraine, is evident that we are far from an efficient use of the tools that a regulator or a practitioner can utilize. One of the main reasons is the uncertainty and dependence between the markets that cause the so called Systemic Risk (from now abbreviated $SR$). We Recall that $SR$ refers to the instability of a financial system that can lead to its entire collapse [**75**]. Taking all the above into account, we give our effort at first on the robust theoretical basis for the utilization of risk measures. Second we consider how Risk Measures should be properly utilized in the presence of $SR$. Our effort is taking the form of four propositions of Risk Meaures' utilization for regulation and investment purposes.

The first utilization we propose is devoted to the investment strategies that combine asset pricing models and coherent risk measures. In particular, we utilize the theoretical framework of [**17**] , which suggests that simply by managing a portfolio of assets, an investor can achieve risk that converges to $-\infty$ and returns that converge to $+\infty$. We contribute on that framework by providing evidence that arise from the CAPM model, in regard to the efficient market hypothesis. In addition, our results suggest that an investor can exhibit returns that outperform the market index by managing a portfolio less volatile than the market.

The second utilization is devoted to the estimation of the insolvency probability. In addition we utilize dependence models that evaluate Systemic Risk (SR), as we contribute by proposing Euler contributions of risk in an environment that is regulated by a risk measure. Moreover the framework we are utilizing assumes that a component of the environment is in distress. Finally, we calculate the Insolvency Probability due to Systemic Risk and we suggest certain distribution classes under which our results are valid.

The third utilization is devoted to $SR$ and its potential depiction in the risk spectrum of Spectral Risk Measures. At first, we propose a fundamental way to quantify the existence of $SR$. Second we argue with common practice which suggests that risk spectrum should solely portray the utility function of the investor-regulator and thus be constructed only according to it. In addition we present and justify two conditions that the risk spectrum should satisfy in regard of the existence of $SR$ and to our knowledge such conditions have not been suggested by another academic or a practitioner. Moreover, we call the Spectral Risk Measures that have a risk spectrum that satisfy those conditions systemic risk aware and we discuss their practicality and applicability.

The fourth utilization is devoted to the Convergence of the so called Euler Risk Contributions when the underlying Risk Measures differ. To that end, our discussion is in regard of Euler contributions in a Risk Measure environment. In addition, we proceed by defining some conditions where the rate of convergence of Euler Risk Contributions in a Value at Risk regulation environment and Distortion Risk Measure regulation environment coincide. Finally, we generalize our findings in regard of the Expected Shortfall case.

**Keywords:** Market Efficiency; Predictive Ability; Coherent Risk Measures; Spectral Risk Measures; Risk Premium; Dependence Models; Systemic Risk; Euler Contributions; Distortion Risk Measures; Complete Measures; Adapted Measures; Rate of Convergence; Insolvency Probability; Ruin Probability;

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CHAPTER 1

# Introduction

Risk Measures are inarguably the Economic Regulator Authorities' favorite tool for preventing crises and controlling the risk existing in Financial institutions and markets. One can take as an example the Basel Committee of Banking Supervision's consultive documents (see for instance [**26**] and [**27**]) and it should be clear that risk measures and their proper use are perceived as their most important instrument. Their popularity relies heavily at some of the advantages they exhibit. For instance most of them are easy to compute, easy to interpret and they are intuitively friendly. For all that their multi-purposely use is apparent mostly for regulation and investment purposes. For instance risk measures can determine the cash or cash equivalent a Bank should hold in order to be safe from liquidity risk. For investing purposes a mutual fund manager can determine the levels of risk her/his clients are expose and determine their preferable portfolio according to their risk tolerance.

Also academics are taking into account the need of proper use of risk measures and so they try to evolve their research towards that direction and depending on the occasion. For instance they are interested on the properties that risk measures should preserve in order to be suitable (see for instance the seminal work of [**9**] on coherency and [**67**] on convexity). Recently the scholars focused their research interest also on the dependence existed among risks that are evaluated by risk measures (see [**58**] or [**14**]). Last but no least we have the practitioners (for instance investors) who are always in pursuit for profit by utilizing the most advanced methods and tools and perhaps if we consider Adam Smith's invisible hand they indirectly contribute to the financial markets prosperity.

In addition we observe that we are leaving in a world of continuous economic crises, most of them interconnected in a global scale. The starting point of this era is the global crisis emerging from late 2007 as a subprime mortgage crisis in U.S.A.. Taking into account the most recent energy crisis and Food supply crisis, both because of the war in Ukraine, is evident that we are far from an efficient use of the tools that a regulator or a practitioner can utilize. One of the main reasons is the uncertainty and dependence between the markets that cause the so called Systemic Risk (abbreviated $SR$). We Recall that $SR$ refers to the instability of a financial system that can lead to its entire collapse [**75**]. Taking all the above into account, we give our effort at first on the robust theoretical basis for the utilization of risk measures. Second we consider how Risk Measures should be properly utilized in the presence of $SR$. Our effort is taking the form of three propositions of Risk Meaures' utilization for regulation and investment purposes.

The first utilization we propose is devoted to the investment strategies that combine asset pricing models and coherent risk measures. In particular, we utilize the theoretical framework of [**17**] , which suggests that simply by managing a portfolio of assets, an investor can achieve risk that converges to $-\infty$ and returns that converge to $+\infty$. We contribute on that framework by providing evidence that arise from the CAPM model, in regard to the efficient market hypothesis. In addition, our results suggest that an investor can exhibit returns that outperform the market index by managing a portfolio less volatile than the market.

By this methodology we provide a way to examine the efficiency market hypothesis not solely with an empirical validation that depends on an equilibrium model, but we provide a mathematically solid framework. On top of that we find sufficient evidence of market inefficiency that is also apparent just by observing recent aforementioned crises.

In regard of the second utilization initially we consider that Systemic Risk (SR) is considered of high interest in academia. Recall that Systemic Risk refers to the instability of a financial system component that can lead to its entire collapse [**75**]. From the aforementioned statement one can intuitively understand that (SR) is closely related to the concept of Dependence. [**14**] state that the main purpose concerning systemic risk is to evaluate the financial distress of an economy as a consequence of the failure of one of its components. They also point out the importance of the Extreme Value Theory (EVT) in the analysis of systemic risk. similar to the aforementioned work we also mention [**10**], [**11**] and [**12**]. In addition [**32**] introduce SRISK to measure the capital shortfall of a firm conditional on a severe market decline.

Having the above in mind the second utilization is concerned of the estimation of the insolvency probability. In addition we utilize dependence models that evaluate Systemic Risk (SR), as we contribute by proposing Euler contributions of risk in an environment that is regulated by a risk measure. Moreover the framework we are utilizing assumes that a component of the environment is in distress. Finally, we calculate the Insolvency Probability due to Systemic Risk and we suggest certain distribution classes under which our results are valid.

The third utilization is devoted to $SR$ and its potential depiction in the risk spectrum of Spectral Risk Measures. We are mostly motivated by the fact that utility function of the investor, a concept related to the asset pricing theory (see [**40**], [**85**]) and portfolio diversification (see [**90**]) is the single factor that determines the risk spectrum of the measure. We argue with this practice and we propose that also systemic risk should be included in the depiction.

At first, we propose a fundamental way to quantify the existence of $SR$. Second we argue with common practice which suggests that risk spectrum should solely portray the utility function of the investor-regulator and thus be constructed only according to it. In addition we present and justify two conditions that the risk spectrum should satisfy in regard of the existence of $SR$ and to our knowledge such conditions have not been suggested by another academic or a practitioner. Moreover, we call the Spectral Risk Measures that have a risk spectrum that satisfy those conditions systemic risk aware and we discuss their practicality and applicability.

Finally, the fourth utilization is devoted to the Convergence of the so called Euler Risk Contributions when the underlying Risk Measures differ. With that in mind, our discussion is in regard of Euler contributions in a Risk Measure environment. Moreover, we proceed by defining some conditions where the rate of convergence of Euler Risk Contributions in a Value at Risk regulation environment and Distortion Risk Measure regulation environment coincide. Finally, we generalize our findings in regard of the Expected Shortfall case.

CHAPTER 2

# Basic Concepts

## 1. Risk Measures

Let consider $\Omega$ be a sample space. In addition, consider an investment over a single period of time, from 0 to $T$, where $X : \Omega \to \mathbb{R}$ is the money outcome of the investment. Moreover, consider a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and $\mathcal{G}$ is the set of real valued functions on $\Omega$.

Definition 1.1. *A risk measure is a function from the set $\mathcal{G}$ of risks $X$ to the real numbers $\rho : X \to \mathbb{R}$.*

Furthermore we take into account the future net worths that are accepted by an investor, namely the acceptance set $\mathcal{A}$. According to [**9**, Definition.2.3] an acceptance set $\mathcal{A}$ associated to a risk measure $\rho$ is denoted by $\mathcal{A}_\rho$ and is defined by

$$\mathcal{A}_\rho = \{X \in \mathcal{G} | \rho(X) \leq 0\}.$$

Moreover in [**9**, Definition 2.2] we find that given the total rate of return $r$ of a reference instrument (for instance we can consider $r$ as the total return of an investment), the risk measure that is associated to an acceptance set is the mapping from $\mathcal{G}$ to the real numbers, is denoted by $\rho_{\mathcal{A},r}$ and is defined by

$$\rho_{\mathcal{A},r}(X) = \inf\{m \in \mathbb{R} | m \cdot r + X \in \mathcal{A}\}.$$

For the past twenty years up to now risk measure concept (based on measure theory) is used widely for regulatory purposes (see for instance [**26**, p.20]). also, efforts like [**101**] depict Risk Measures analytically. Such a measure can determine the amount of currency (or other assets) a financial institution should keep in reserve depending on the financial risks it is exposed.

## 2. Properties of Risk Measures

Let introduce properties of risk measures that are associated with our work:

(1) Sub-additivity: $\rho(Z) + \rho(Y) \geq \rho(Z+Y)$ for any $Z, Y \in \mathcal{G}$.
(2) Homogeneity: $\rho(\lambda Y) = \lambda \rho(Y)$ for any $\lambda \geq 0$.
(3) Monotonicity: for any $Z, Y \in \mathcal{G}$ let consider $Z \geq Y$ at all scenarios, then $\rho(Z) \leq \rho(Y)$.
(4) Cash or translation invariance: for any $Z \in \mathcal{G}$ and let consider $M$, an investment with guaranteed risk free returns $m$, then $\rho(Z+M) = \rho(Z) - m$.
(5) Convexity: $\rho[\lambda Y + (1-\lambda)Z] \leq \lambda \rho(Y) + (1-\lambda)\rho(Z)$ for any$Y$, $Z \in \mathcal{G}$ and any$\lambda \in [0,1]$.
(6) Law Invariance: If $Z, Y$ have the same distribution under $\mathbb{P}$, then $\rho(Z) = \rho(Y)$.
(7) Comonotonic Additivity: If $Z, Y$ are comonotonic, then $\rho(Z) + \rho(Y) = \rho(Z+Y)$.

Analysis of properties (1)-(6) can be found in [**9**], [**68**], [**80**], [**109**] and [**69**], while analysis of (7) can be found in [**53**].

Definition 2.1. *A monetary risk measure is a risk measure satisfying monotonicity and translation invariance.*

Definition 2.2. *A coherent risk measure is a risk measure satisfying monotonicity, translation invariance, homogeneity and sub-additivity.*

The importance of coherency relies in the fact that when a risk measure exhibits them, then the acceptance set associated with this risk measure is closed and satisfies the 4 axioms suggested by [**9**, from Axiom 2.1 up to 2.4]. An in depth analysis concerning the relation between the axioms of Acceptance Sets and the properties of coherent risk measures can be found in [**9**, Proposition 2.1] [**9**, Proposition 2.2] [**9**, Proposition 2.3] and their proofs.

Recently, was suggested that there are certain risk scenarios where this set up need some relaxation. For instance [**52**] took into account property of sub-additivity and suggested that from the viewpoint of regulator a merger with adding the capitals should be preferred because the risk measure decreases. Furthermore, the concept, that leverage of a position rises the levels of the risk proportionally, cannot hold, since in reality the leverage can lead to disproportion risk changes. As a consequence, the weaker property of convexity, proposed by [**67**], come into consideration.

Definition 2.3. *A convex risk measure is a risk measure satisfying monotonicity, translation invariance and convexity.*

As [**67**] explains convexity suggest that diversification does not increase risk or rephrasing, a diversified position is less or equal to the weighted average of the individual risks. We observe that a convex risk measure that is homogenous and sub-additive is also coherent, as it already exhibits the properties of monotonicity and translation invariance (see above definition).

Also we define the expectation bounded risk measure(see [**97**]):

DEFINITION 2.4. *A risk measure that satisfies the inequality* $\rho(Y) \geq \mathbb{E}[-Y]$ *for any risk* $Y \in \mathcal{G}$ *is an expectation bounded risk measure.*

## 3. Important Risk Measures

Let us introduce the Value at Risk ($VaR_\alpha$).

DEFINITION 3.1. $VaR_\alpha(X) = \sup\{x \in \mathbb{R} \mid \mathbb{P}(X \geq x) > 1 - \alpha\}$.

We can describe the $VaR_\alpha$ as a threshold loss value of the investment, that has probability $1-\alpha$ to exceed this threshold in a given time horizon (assuming no additional trading). Important advantages of $VaR_\alpha$ is its straight computation while an important drawback is that is not coherent as its sub-additivity is questionable and depends on the suggested setting. Briefly we mention that sub-additivity holds when the investment returns are normally distributed so difficulties on that property may arise in cases of fat tails. Again, we consider [**47**, proposition 1], where we get that when two investments' returns are regularly decay (see Extreme Value Theory subsection for subtitles), then $VaR_\alpha$ preserves sub-additivity in the tail region. On the other hand sub-additivity is problematic in the case where returns exponentially decay (again on Extreme Value Theory subsection for details on exponentially decay).

DEFINITION 3.2. *Let introduce the expected shortfall as the risk measure given by* $ES_\alpha = \frac{1}{\alpha}\int_0^\alpha VaR_p(Y)\,dp$ *for any random variable* $Y$ *and any real number* $0 < \alpha < 1$.

For practical terms one can perceive it as the expected value of an investment, conditional that the value is less than the quantile $p$ of the worst outcomes. From [**81**] we have that $ES$ can be expressed in terms of the distribution of the investment's returns $F_X = \mathbb{P}(X \leq x)$:

$$ES_\alpha = -\frac{1}{\alpha} \int_0^\alpha F_X^{\leftarrow}(p)\, dp\,. \tag{3.1}$$

It is a coherent risk measure as it exhibits the four coherency properties (see [**3**]). Also, it is conservative in comparison to VaR, in the sense that $ES_\alpha(Y) \geq VaR_\alpha(Y)$ for any given risk $Y$. In addition a coherent risk is also bounded in the sense of $\rho(Y) \geq \mathbb{E}[-Y]$ for any risk $Y \in \mathcal{G}$ (see [**9**] p.219). In regard of Expected Shortfall we also mention the work of [**106**].

Let us introduce the class of distortion risk measures. If $\mathcal{H}$ is a set of increasing functions $h$ on $[0,1]$, where $h(0) = h(0^+) = 0$ and $h(1^-) = h(1) = 1$, then a distortion risk measure $\rho_h$ with distortion risk function $h \in \mathcal{H}$ is defined as

$$\rho_h(Z) = -\int_{\mathbb{R}} x dh\big(F_X(x)\big), \tag{3.2}$$

provided that the above integral exists for all $X \in \mathcal{G}$. Recall that $F_X$ is the distribution of the investment's returns. While we have followed the Distortion Risk Measure depiction of [**13**], nevertheless according to [**35**] when $h$ is continuous it can also be presented as

$$D(X) = \int_0^1 VaR_q(X)dh(q). \tag{3.3}$$

In addition a distortion risk measure is monotone, cash invariant, positively homogeneous, comonotonic additive and law-invariant [**35**]. Recall that [**106**] states that a rough interpretation of law invariance might be that estimations can occur out of statistical observations only (empirical data). In addition if $h$ is concave then $\rho_h$ is sub-additive and thus is coherent. As [**7**] state this is a whole class of risk measures, also known as the set of Concave Distortion Risk Measures. This subset of distortion risk measures exhibits resemblance with the Spectral Risk Measures (see below) as they have the same properties, yet with an importance difference: For quantifying risk, concave distortion risk measures modify the probability distribution (in accordance with the rest of the Distortion Risk Measures) while Spectral Risk Measures modify returns as in the expected utility framework. The scholar interest of Distortion Risk Measures can be found in many efforts like [**43**].

Let us introduce Spectral Risk Measures for which we rely on the theoretical framework of [**1**]. Consider that $\rho_i$ are coherent risk measures and $i = 1, 2, 3, ..., n$. If $\alpha_i \in \mathbb{R}^+$ and $\Sigma_i \alpha_i = 1$, then any convex combination $\rho = \Sigma_i \alpha_i \rho_i$ is also a coherent risk measure. Also, if we take that $\rho_\alpha$ represents an one-parameter family of coherent risk measures, then for any measure $dm(\alpha)$, where $\int dm(\alpha) = 1$, the statistic $\rho = \int \rho_\alpha dm(\alpha)$ is a coherent risk measure.

Taking (3.1) and suggesting a measure $dm(\alpha)$, where $\alpha \in [0, 1]$, then

$$M_m(X) = \int_0^1 \alpha ES_\alpha(X)dm(\alpha) = -\int_0^1 \int_0^\alpha F_X^{\leftarrow}(p)\ dp\ dm(\alpha), \tag{3.4}$$

is a coherent risk measure, provided $\int_0^1 \alpha dm(\alpha) = 1$ which stands as the normalization condition, holds. By interchanging the integrals (Fubini - Tonelli theorem) one gets that $M_m(X) \equiv M_{F_0}(X)$ ([**1**] equation 8 p.1508).

Also $dm(\alpha)$ can be expressed by a decreasing and positive $F_0(p)$, namely a risk spectrum, which equals $\int_p^1 dm(\alpha)$ and is normalized, thus $\int_0^1 F_0(p)dp = 1$ ([**1**, equation 9 p.1508]).

Definition 3.3. *Risk spectrum $F_0(p)$ is a function that equals $\int_p^1 dm(\alpha)$, or $F_0(p) = \int_p^1 dm(\alpha)$. In addition $\int_p^1 dm(\alpha)$ can be expressed by $F_0(p)$.*

Let us define the property of positivity for $F_0(p)$:

Definition 3.4. *$F_0(p)$ exhibits the property of positivity if for every $A \subseteq [0,1]$, $\int_A F_0(p)dp \geq 0$.*

Remark 3.5. *The property of positivity ensures that the risk spectrum will not give negative weights at some quantiles of the cumulative distribution function of the returns.*

Moreover, it is necessary to define the condition risk spectrum needs to be endowed with, in order to be decreasing:

Definition 3.6. *Condition for $F_0(p)$ to be decreasing is that for every $a \in (0,1)$ and for every $b > 0$, such that $[a-b, a+b] \subset [0,1]$, $\int_{a-b}^a F_0(p)dp \geq \int_a^{a+b} F_0(p)dp$.*

Remark 3.7. *Aforementioned definition ensures that the risk spectrum will give more weights as the losses in the cumulative distribution function are greater. In other words, a decreasing function depicts the risk aversion of the investor.*

REMARK 3.8. *A risk spectrum* $F_0(p) : [0,1] \to \mathbb{R}^+$ *that is decreasing and satisfy the properties of normalization and positivity gives a coherent risk measure that can be expressed by* (3.4)*, where* $dm(a) = -dF_0(a)$*. Such a risk spectrum is admissible (see* [**1**]*).*

PROPOSITION 3.9. *A Spectral Risk Measure is coherent if and only if it has an admissible risk spectrum.*

PROOF. See Appendix A □

Also, [**1**] states that the risk spectrum should be presented as an element $F_0$ of $L^1([0,1])$ normed space, where its element is depicted by a class of functions. Different depictions (functions) of the same element $F_0$ will define the same measure.

In addition, we consider the real-worlds' risk management applications where ([**1**] equation 13 p.1509) is well defined and finite. There if $F_0(p)$ is positive, decreasing and normalized, then it is an admissible risk spectrum. Moreover a Spectral Risk Measure endowed with an admissible risk spectrum is a coherent risk measure and thus exhibits the properties of (1)-(4).

Some spectral risk measures developed in recent literature where their risk spectrum is admissible under certain conditions, are the following:

(1) Power: It came to prominence from [**56**], where they considered to utilize the properties existed in the power utility function. Its depiction is:

$$F_0(p,d) = dp^{d-1} \tag{3.5}$$

where $d \in (0,1]$.

(2) Exponential: Initially it was suggested by [**2**, p.178] and later was represented by [**44**] and [**55**] in the following form:

$$F_0(p) = \frac{Ae^{-Ap}}{1 - e^{-A}}$$

where $A \geq 0$ stands as the Arrow - Pratt coefficient of absolute risk aversion.

(3) Wang transform: It is suggested by [**111**] and despite the fact that Wang transform was developed before the concept of admissible risk spectrum, nevertheless it fulfills its requirements. Wang transform also fulfills the requirements of [**21**] in order to be a well defined risk measure.

## 4. Euler Allocation Principle

For this subsection we consider the framework of [**107**] and [**105**], where he discusses how Euler risk contributions can be estimated for risk measures and in order to decompose portfolio-wide capital into a sum of risk contributions by sub-portfolios of solitary exposures (for a thorough read on Euler allocation principle see [**91**, Section 6.3]).

Let consider an economic entity (for instance a portfolio) with $n \in \mathbb{N}$ assets. In addition those assets' profits/losses are the real r.v. $X_1, X_2, ..., X_n$ and $X$ is the economic entity's profit/loss, where

$$X = \sum_{i=1}^{n} X_i. \tag{4.1}$$

In addition the capital that is required by this economic entity is determined with a risk measure $\rho(X)$ (see relevant subsection).

By introducing variables $u = (u_1, u_2, ..., u_n)$ it occurs a useful representation of (4.1):

$$X(u) = X(u_1, u_2, ..., u_n) = \sum_{i=1}^{n} u_i X_i. \tag{4.2}$$

Clearly $u_i$ stands for the amount of capital that is invested in the asset which has $X_i$ profit/loss. Also in [**107**] are considered some variations of the $u$ and therefore they introduce the following function

$$f_{\rho,X}(u) = \rho\big(X(u)\big). \tag{4.3}$$

By dropping $X$, then the left side of (4.3) can equivalently be written $f_\rho(u)$.

Before proceeding with the properties of risk contributions we have to present two fundamental definitions, the homogeneous of degree $t$ risk measure and the total portfolio Return on Risk Adjusted Capital (RORAC)

Definition 4.1. *A risk measure is homogeneous of degree $t$ if for any $h > 0$ the following equation obtains:*

$$\rho(hX) = h^t \rho(X). \tag{4.4}$$

In addition we set that $\mu_i = \mathbb{E}[X_i]$ and we give the following definition:

Definition 4.2. (1) *The total portfolio (RORAC) is defined by the following:*

$$RORAC(X) = \frac{\mathbb{E}[X]}{\rho(X)} = \frac{\sum_{i=1}^{n} \mu_i}{\rho(X)}.$$

(2) *The $i^{th}$ asset's portfolio based (RORAC) is defined by the following:*

$$RORAC(X_i|X) = \frac{\mathbb{E}[X_i]}{\rho(X_i|X)} = \frac{\mu_i}{\rho(X_i|X)}.$$

Furthermore we consider the economic entity's profit/loss as depicted in (4.1) and we state the following definition:

Definition 4.3. (1) *If*

$$\sum_{i=1}^{n} \rho(X_i|X) = \rho(x),$$

*then risk contributions $\rho(X_1|X), ..., \rho(X_n|X)$ to portofolio wide risk $\rho(X)$ satisfy the full allocation property.*

(2) *If there are some $e > 0$, such that*

$$RORAC(X_i|X) > RORAC(X) \Rightarrow RORAC(X + hX_i) > RORAC(X)$$

*for all $0 < h_i < e_i$, then $\rho(X_i|X)$ are RORAC compatible.*

Moreover in [**107**, Proposition 2.1] we find that if $f_\rho$ is continuously differentiable and there are risk contributions $\rho(X_1|X), ..., \rho(X_n|X)$ that are RORAC compatible according to above definition for arbitrary expected values $\mu_1, ..., \mu_n$ of $X_1, ..., X_n$, then $\rho(X_i|X)$ is uniquely determined according to the following:

$$\rho_{Euler}(X_i|X) = \frac{d\rho}{dh}(X + hX_i)|_{h=0} = \frac{\partial f_\rho}{\partial u_i}(1, ..., 1). \tag{4.5}$$

Furthermore we consider [**107**, Remark 2.1] where is stated that for homogeneous $\rho$ the risk contributions, as defined in (4.5) are Euler contributions, they satisfy full allocation property and they satisfy RORAC compatibility property. It should be clear that from an economic perspective the properties of full allocation and RORAC compatibility are desirable. Finally assigning capital to sub-portfolios by calculating Euler contributions is named Euler allocation.

Before concluding with this subsection it worth mentioning that in [**107**] is highlighted the relation between Euler allocation principle and sub-additivity property of a risk measure (see relevant subsection), where we have that the latter property can be written in terms of Euler contribution, or

$$\rho_{Euler}(X_i|X) \leq \rho(X_i).$$

## 5. Extreme Value Theory

For the best possible understanding of the contribution section we need to present the concept of Extreme Value Theory (EVT). For doing that we consider the work of [**41**] where it is considered that there is a sequence of $X_1, ..., X_n$ independent r.v. with common distribution function V. The interest is focused on $M_n = max\{X_1, ..., X_n\}$ where $n \to \infty$. Also a linear normalization of $M_n$ is needed and so $M_n^* = \frac{M_n - b_n}{\alpha_n}$ for sequences of constants $\{\alpha_n > 0\}$ and $\{b_n \in (-\infty, \infty)\}$. With this approach we are interested in the correct selections of $\{\alpha_n\}$ and $\{b_n\}$ rather than $M_n^*$, that will allow us to seek limit distributions for $M_n^*$.

Under that concept we consider the [**41**, theoem 3.1] where it presents the well documented Fisher-Tippett theorem (see [**66**]) and suggests that for $\{\alpha_n > 0\}$ and $\{b_n \in (-\infty, \infty)\}$ sequences where

$$\lim_{n\to\infty} \mathbb{P}\left[\frac{M_n - b_n}{\alpha_n} \leq y\right] \to G(y), \tag{5.1}$$

and $G$ in (5.1) is a non-degenerate distribution function, then $G$ belongs either the Gumbel family, the Fréchet family or the Weibull family (their presentations will follow).

Also these three distribution families can be grouped into the Generalized Extreme Value (GEV) distribution. A presentation of (GEV) is achievable if we consider that it has three parameters $\mu, \sigma, \xi$. Then we have that $x = \frac{y-\mu}{\sigma}$, we set that $\xi$ is the shape parameter and we get that

$$G(x) = \exp\big(\exp[-x]\big) \tag{5.2}$$

is the Gumbel family where we have for (5.2) that $\xi = 0$. Moreover we get that

$$G(x) = \exp\big(-1[1+\frac{x}{\alpha}]^{-\alpha}\big) \tag{5.3}$$

is the Fréchet family where we have for (5.3) that $\xi = \frac{1}{\alpha} > 0$. Finally we get that

$$G(x) = \exp\big(-1[1-\frac{x}{\alpha}]^{\alpha}\big) \tag{5.4}$$

is the Weibull family where we have for (5.4) that $\xi = -\frac{1}{\alpha} < 0$.

In addition $V$ is in the domain of attraction of Gumbell if and only if

$$\lim_{x\to\infty} \frac{1 - V\big(x + tb(x)\big)}{1 - V(x)} = e^{-t}, \tag{5.5}$$

for all $t > 0$ in (5.5) and exponentially decay in the tail of $V$ which is symbolized as $V \in \mathcal{R}_{-\infty}$ in [**14**]. Moreover $V$ is in the domain of attraction of Fréchet if and only if

$$\lim_{x\to\infty} \frac{1 - V(tx)}{1 - V(x)} = t^{-\frac{1}{\xi}}, \tag{5.6}$$

for all $t > 0$ in (5.6) and regularly decay in the tail of $V$ which is symbolized as $V \in \mathcal{R}_{-t}$ in [**14**]. It is beyond the scope of this work to present when $V$ is in the domain of attraction of Weibull and consequently we omit it. In regard of the importance of Extreme Value theory we mention the report of [**31**], where they discussed securitized insurance product like insurance linked securities(for instance Catastrophe Bonds and Industry Loss Warranties) or even [**77**]. To that end they proposed - utilized Extreme Value Theory to characterize the catastrophe risks. We refer the reader in regard of this theory to [**102**].

## 6. Copula Concept-Diversification Index

One of the most common tools in probability theory to describe the dependence among r.v. are the copulas. Scholar interest can be found in many works like [**24**]. As a preliminary description, copula is a multivariate probability distribution for which the marginal probability distribution of each variable is uniform. For introducing a solid mathematical frame, we rely initially on [**94**, Definition 2.2.1] of a 2-dimensional subcopula, who suggested that a 2-dimension subcopula is a function $C$ that exhibits the following properties:

(1) Domain of $C$ is the $S_1 \times S_2$, where $S_1$ and $S_2$ are subsets of $[0, 1]$.
(2) $C$ is grounded and 2-increasing (see [**94**] for an in-depth analysis of those properties).
(3) for every $u \in S_1$ and every $v \in S_2$, we have that $C(u, 1) = u$ and $C(1, v) = v$.

Furthermore, [**94**, Definition 2.2.2] denotes that in order to get 2-dimensional copula we need to get from the first aforementioned property that $C \in [0, 1] \times [0, 1]$.

We also need to present the so-called Sklar's theorem as depicted in [**94**, Definition 2.3.3]. There is stated that if we have a joint distribution function $H$ with $F$ and $G$ margins, then exists a copula such that

$$H(x, y) = C\big(F(x), G(y)\big), \tag{6.1}$$

for all $x, y \in \mathbb{R}$. Furthermore if $F$ and $G$ are continuous then $C$ of (6.1) is unique.

Also for every $C$ we have that

$$W(u,v) \leq C(u,v) \leq M(u,v), \tag{6.2}$$

where in (6.2) we have that $M(u,v) = min(u,v)$ and $W(u,v) = max(u+v-1,0)$. $M$ is called the Fréchet-Hoeffding upper bound and $W$ is named the Fréchet-Hoeffding lower bound while (6.2) is the Copula setting of Fréchet-Hoeffding bounds inequality. Since Copulas are considered mostly as a tool for measuring dependence of r.v. it is important to discuss the notion of Concordance-Discordance. As a rule of thumb one can keep in mind that two r.v. are concordant if big values of one variable are associated with big values of the other and also the same thing occurs with small values. Fréchet-Hoeffding bounds can prove very useful in situations where we need to have concordance ordering as the Fréchet-Hoeffding lower bound copula $W$ is smaller than every copula, and the Fréchet-Hoeffding upper bound copula $M$ is larger than every copula.

The classical theory of Copulas also suggets Kendall's tau and Spearman's rho as two main tools for evaluating the Concordance-Discordance between r.v.. While the reader can find thorough analysis in [**94**] yet it is beyond the scope of this chapter to further elaborate on the topic. On the contrary we are dealing with a similar measurement of dependence which is the diversification index, suggested by [**107**]. There is suggested that for (4.1) if $\rho$ is a risk measure such that $\rho(X_1), ..., \rho(X_n)$ are defined, then

$$DI_\rho(X) = \frac{\rho(X)}{\sum_{i-1}^{n} \rho(X_i)} \tag{6.3}$$

is the Diversification Index (DI) of $X$ with respect to $\rho$. Also

$$DI_\rho(X_i|X) = \frac{\rho_{Euler}(X_i|X)}{\rho(X_i)}. \tag{6.4}$$

stands as the marginal diversification index of $X_i$ with respect to $\rho$.

## 7. Jensen's Alpha

Recall the classical Capital Asset Prising model (Abbreviated CAPM) formula, or

$$r = r_f + \beta(r_m - r_f) + \alpha, \tag{7.1}$$

where $r_f$ is the risk free rate, $r$ is an one-period investment returns and $r_m$ is the market portfolio returns. According to [**73**], market portfolio consists of an investment in each asset in the market in proportion to its fraction of the total value of all assets in the market. Next, $(r - r_f)$ is the risk premium or the excess return of the investment and $(r_m - r_f)$ is the market risk premium. The beta ($\beta$) coefficient given by the formula $\beta = Covariance(r, r_m)/Variance(r_m)$ represents the sensitivity of the asset returns compared to market returns. Moreover, the $\alpha$ stands as the intercept of security characteristic line which is extracted from the CAPM theory (see for instance [**90**]).

CAPM was initially developed by Bill Sharpe and Jack Treynor (1963). They developed their models each one independently and for different reasons, but both were trying to find solution on how to quantify risk in an investment and afterwards investigate relationship between risk and return in marketplace, Treynor from real economy and Sharpe from capital markets perspective. The crucial determinant in both cases concerning asset valuation was covariance with common factor. Moreover the two models have similarities in terms of assumptions in order to function in equilibrium. These assumptions are included in theory of CAPM model and concern investors and markets. Particularly investors aim to maximize economic utility. Furthermore they are rational and risk averse. Moreover they cannot influence stock prices and manipulate markets. In addition investors can borrow unlimited and under the risk free rate. Besides the above, markets don't have transaction costs, the securities in markets are divisible and finally all information is available at the same time. Model's purpose is to locate the rate of return of a single asset that is added in an already diversified portfolio, if asset's systematic (non-diversifiable risk) is given.

For explaining the importance of $\beta$, which is actually the slope in the regression, we recall the two components of risk according to CAPM's theoretical background, thus systematic and unsystematic (idiosyncratic) risk (see [**100**]). Also, $\beta$ denotes how volatile and thus risky is an investment compare to the market risk. Moreover, $\beta$ demonstrates to which extend an investment is exposed to systematic risk. An investment with $\beta = 1$ suggests same risk with market portfolio, $\beta < 1$ denotes an investment less volatile and thus risky compare to market portfolio and finally a $\beta > 1$, suggest a more volatile and thus risky investment compare to the market. In addition, an investment with $\beta < 1$ is the choice of a loss-risk averse investor while an investment with $\beta > 1$ is more suitable for someone more tolerate to risk. We also mention that $\beta$'s price is not fluctuating considerably. Actually is not taking prices less than zero (consider that a negative price is not having a adequate meaning financially) and exceeds a price of 2 quite seldom. To give a fundamental example, a beta of 0.5 suggests a very risk-less investment while a price of 1.5 suggests that the investment is very volatile and thus risky.

Also, in the papers [**62**] and [**99**] we find usage of the $\alpha$ factor, where the abnormal returns are evinced when $\alpha > 0$. Further, it is used as a measure of fund performance, which gives evidence whether a market is efficient of any form, saying weak - semi or strong form efficient. We recall that those forms where proposed by [**64**] according to sets of available information.

Also in [**73**] we find that:

(1) if $\alpha$ is positive and statistically significant then the investment performs better relative to benchmark index and worth selling, as it is overpriced.
(2) If $\alpha$ is positive and statistically insignificant then any good performance occurs due to mere random chance.
(3) Negative $\alpha$ means that the investment performs worst relative to market, is undervalued and worth buying.
(4) If $\alpha$ equals 0 the investment is correctly priced and no portfolio adjustment is necessary.

## 8. Sharpe Ratio

Let us introduce the classical Sharpe ratio, suggested by [**98**],

$$SharpeRatio = \frac{r - r_f}{\sigma}. \tag{8.1}$$

As we observe the numerator is the risk premium or the investments excess return over the return of the risk free rate, and the denominator is the standard deviation of the excess return. We observe that Sharpe ratio can be utilized mostly in a comparative manner as the higher the Sharpe ratio, so better for the investor. Rephrasing, if an investor has to select among two investments with same $\sigma$, she/he will end up with the one that has higher risk premium in order to get a higher Sharpe ratio. In addition, between two investments with the same risk premium, an investor should choose the one with the lower $\sigma$ in order to get the best possible Sharpe ratio.

A good alternative of Sharpe ratio is the Treynor ratio. It also evaluates fund performance and compare to Sharpe ratio, as denominator it uses the beta of the fund instead of its standard deviation. Taking as an advantage the simplicity those ratios provide, on the other hand they have caveats of their own. For instance Sharpe ratio takes into account the whole risk and does not separates it into components the portfolio theory suggests, thus systematic and non systematic.

## 9. Efficient Market Hypothesis

Inarguably one of the most fundamental concepts in modern portfolio theory is the Efficient Market Hypothesis. Suggesting that efficient is a market where a security price is an unbiased estimate of its true value, it is one of the most prolific and valid issues academic society scrutinizes. When EMH holds then prices reflect all available information and any deviations from the true values of investments are random leaving little abnormal returns for speculators. Moreover [**64**] proposed three forms of market efficiency according to sets of available information:

(1) The weak -form where investment prices reflect all available information included in the history of security prices.
(2) The semi-strong where current investment prices instantly reflect all publicly available information about securities.
(3) The strong-form where investment prices reflect all publicly and privately available information concerning securities.

As [**85**] explains researchers are putting to the test the financial markets for about half a century and found evidence that in a plethora of occasions weak-form tends to hold while there are very few occasions where semi-strong holds and finally strong-form efficiency does not tend to hold.

CHAPTER 3

# Exhibiting Abnormal Returns Under a Risk Averse Strategy

## 1. Preliminaries

In this chapter we are concerned on portfolio optimization by examining the existence of investment strategies that appear to have returns that converge to $+\infty$ and risk that converges to $-\infty$. Such concept suggests that there is an unbounded risk premium that challenges the existence of market efficiency.

Portfolio optimization with risk control is an important topic for academia. At first we consider [**96**] who suggest that Expected Shortfall can be combined with analytical and scenario based methods in order to optimize portfolios. Also in [**8**] the economic implications of mean - VaR framework are examined. They point out that in two mean - variance efficient portfolios, the one with higher variance might get a less VaR and as consequence, a global portfolio that minimizes VaR might be the empty set. They also denote that a mean-VaR efficient set is a subset of mean variance efficient set, as mean-VaR resulted in a smaller efficient frontier compare to mean variance efficient frontier. Also mean-VaR efficient set can also be empty. The paper [**25**] demonstrated that the mean-expected shortfall optimization problem, can be solved as a convex optimization problem. In addition, the sample mean-shortfall portfolio optimization problem can be solved as a linear optimization problem. Moreover [**23**] approach to the topic is to minimize the risk of the fund issuer under constraint that is dictated by a buyer who is willing to enter the transaction only if the risk level remains below a given threshold. Their optimization problem concerns on the investor's attempt to maximize the expected utility of her/his global terminal wealth. In addition,[**70**] suggest a method of calculating a portfolio which gives the smallest Value at Risk (VaR) among those portfolios that yield at least some desired expected return. This approach, enables them to calculate the complete mean - VaR efficient frontier.

Also, one of the most fundamental concepts in modern portfolio theory is the Efficient Market Hypothesis (EMH) which suggests that efficient is a market where a security price is an unbiased estimate of its true value. When EMH holds, then prices reflect all available information and any deviations from the true values of securities are random, leaving little space for abnormal returns. Concerning market efficiency there is a vast number of papers that examine the topic, mostly by evaluating the performance of fund managers. For instance [**45**], or [**76**] suggest that fund managers are able to beat markets indexes, but there is also equally large number, if not more, of papers who deny that conclusion. For example in [**62**] and [**46**] was suggested that there are very few funds with sufficient returns to cover their costs. All the aforementioned works are based mostly on empirical validation that might exhibit biases like incubation bias suggested by [**61**], or the survivor bias (see for instance [**57**]).

In doing so, our findings are in agreement with those of [**76**] and [**45**] but instead of empirical validation, we rely on the theoretical framework of [**17**] which suggest that sequences of investment strategies that are characterized as good deals, according to asset pricing theory, occur. In addition, we focus on implementation potential that requires a strategy that follows both an asset pricing model that is unbounded and a bounded, coherent risk measure.

Also, we consider the [**100**] and [**86**] CAPM model and we contribute to the findings, by suggesting that there is evidence of abnormal profits when an investor follows a risk averse strategy. We are also motivated by the fact that CAPM is still a very popular model despite its longevity and its evolution through the years, see for instance [**63**] three factor model, or the [**37**] four factor model.

If we need an intuitive interpretation of the findings, we can assume that it is possible to follow a risk averse strategy and at the same time achieve abnormal returns. More interestingly, most of the studies that examine the performance of investments, like [**62**], tend to focus on the performance of active management, where $\beta \geq 1$ quite often, and they assume that passive investments will have a zero $\alpha$. Our findings, on the contrary, denote that a less risky investment is a prerequisite for abnormal returns. Moreover, our result is in full compliance with the notion that an investor can achieve strategies whose risk turns to $-\infty$ and returns converges to $+\infty$ (more on the theoretical framework).

We suggest that an investor is able to manage a portfolio of investment that is less risky than the market and at the same time achieve abnormal returns, which is by all means an odd conclusion according to portfolio theory who requires extra risk for extra returns. Such interpretation gives adequate theoretical framework that a market may not be efficient, not even weak form.

The structure of the chapter is the following: next section includes is devoted to the theoretical framework suggested by [**17**]. Section 3 includes our contribution to the theoretical framework and in section 4 are the concluding remarks.

## 2. Theoretical Framework

Let consider the probability space $(\Omega, \mathcal{F}, \mathbb{P})$, where $\Omega$ is the set of the states of the world, $\mathcal{F}$ is the $\sigma$-algebra of events and $\mathbb{P}$ is the probability measure. We also consider a time interval $[0, T]$ and we are concerned on every payoff $y \in L^2$ Hilbert space (the set of square integrable $L^2$ functions on space $(\Omega, \mathcal{F}, \mathbb{P})$) at time $T$. Also we consider that every payoff $y$ can occur via portfolios that are self financed.

In addition [**17**] denotes the existence of a pricing rule $P : L^2 \to \mathbb{R}$ which gives the price of every payoff $y$ at time 0. Broadly speaking, a pricing rule can perform pricing adjustments to an investment. In addition those prices are applicable if certain conditions are satisfied. Also, the equality

$$P(k) = ke^{-r_f T} \tag{2.1}$$

holds for $k \in \mathbb{R}$ and $r_f$ denotes the risk free rate. Taking into account the Riesz representation theorem, for every $y$ there is a unique $z_\pi \in L^2$. In addition for the unique $z_\pi$ we have that

$$P(y) = e^{-r_f T}\mathbb{E}[yz_\pi]\,, \tag{2.2}$$

where $z_\pi$ can be perceived as the Stochastic Discount Factor (SDF) and $\mathbb{E}$ stands for the expected value. Also, no arbitrage opportunities are assumed and thus $z_\pi > 0$ holds almost surely. Moreover, (2.2) and (2.1) draw the conclusion that $\mathbb{E}[z_\pi] = 1$, as they imply that $ke^{-r_f T} = e^{-r_f T}k\mathbb{E}[z_\pi]$ , which can only occur if $\mathbb{E}[z_\pi] = 1$.

For controlling the levels of risk, let us consider that the investor uses coherent and expectation bounded risk measure $\rho : L^2 \to \mathbb{R}$, of the form

$$\rho(y) = \max\{-\mathbb{E}(yz)\ ;\ z \in \Delta_\rho\}, \tag{2.3}$$

and we have that for $\Delta_\rho$ of (2.3) that

$$\Delta_\rho \subset \{z \in L^2\ ;\ \mathbb{P}(z \geq 0) = 1,\ \mathbb{E}(z) = 1\}. \tag{2.4}$$

Moreover, $\Delta_\rho \subset L^2$ is the subgradient of $\rho$ (see [**95**, page 272] for a definition of subgradient).

Also, we consider that $C$ represents the investment that have to be made in order to control the risk $\rho(y_f)$, which is the risk of the investor's final wealth ($y_f$ is investor's final wealth). In addition we take as given that $C > 0$. Furthermore we are interested in the present value $C$ and thus for the $Ce^{-r_f T} > 0$. When this investment is risk free we can take into account the property of coherent risk measures (translation invariance) and thus the risk would be $\rho(y_f + C) = \rho(y_f) - C$. Now let us assume that the best possible reduction of risk may not be necessarily achieved by a risk free investment. Then the optimization problem is

$$\min\{\rho(y + y_f - \mathbb{E}(y\, z_\pi))\ ;\ \mathbb{E}(y\, z_\pi) \leq C,\ y \geq 0\}. \tag{2.5}$$

(2.5) is a universal risk level that an investor has to consider in order to include value $\mathbb{E}(yz_\pi)$ of the new investment in the portfolio. Rephrasing, the investor's wealth reduction is the value $\mathbb{E}(yz_\pi)$, for achieving a risk level of $\rho(y + y_f - \mathbb{E}(y\, z_\pi)$, or the investor binds the least possible capital $\mathbb{E}(yz_\pi)$ in order to accomplish $\rho(y + y_f - \mathbb{E}(y\, z_\pi)$ level of risk.

[**19**, Lemma 4] and [**19**, Theorem 5]) can be employed and lead to the dual optimization problem, a methodology described in [**88**], that gives

$$\max\{-C\lambda - \mathbb{E}(y_f z)\ ;\ z \leq (1+\lambda)z_\pi,\ \lambda \in \mathbb{R},\ \lambda \geq 0,\ z \in \Delta_\rho\} \tag{2.6}$$

where $\lambda$ and $z$ are the decision variables.

We also consider that the investor can choose negative payoffs. If this is the case, the optimization problem would be the following:

$$\min\{\rho(y + y_f - \mathbb{E}(y\, z_\pi))\ ;\ \mathbb{E}(y\, z_\pi) \leq C\} \tag{2.7}$$

and its dual (depicted in (2.6)) would be

$$\max\{-\mathbb{E}(y_f z)\ ;\ z = z_\pi,\ z \in \Delta_\rho\}. \tag{2.8}$$

By Theorem 2.1 in [**17**] there is no duality gap between the primal optimization problems and their dual ones. This occurs due to the fact that the Slater's sufficient condition for strong duality holds (see [**19**, Proof of Lemma 3]). Recall that duality gap is the difference between the primal and dual solutions. Consequently when there is no duality gap then this difference equals 0 and strong duality holds. Additionally, a solution requires that (2.5), or in case of negative payoffs (2.7), are bounded.

Let consider that there exist a $y^* \in L^2$ and $(\lambda^*, z^*) \in (\mathbb{R} \times L^2)$ that solves the aforementioned optimization problem. According to Theorem 2.1 in [**17**], this could only occur if the following Karush - Kuhn - Tucker conditions are satisfied

$$
\begin{cases}
\lambda^*[C - \mathbb{E}(y^* z_\pi)] = 0 \\
C - \mathbb{E}(y^* z_\pi) \geq 0 \\
\mathbb{E}[(y^* + y_f)z] \geq \mathbb{E}[(y^* + y_f)z^*], \text{ for all } z \in \Delta_\rho \\
\mathbb{E}[((1+\lambda^*)z_\pi - z^*)y^*] = 0 \\
(1+\lambda^*)z_\pi - z^* \geq 0 \\
y^* \in L^p, \; y^* \geq 0, \; \lambda \in \mathbb{R}, \; \lambda \geq 0, \; z^* \in \Delta_\rho
\end{cases} \tag{2.9}
$$

Under conditions of (2.9) and suppose that $\lambda^*$ and $z^*$ solve the optimization problem (2.6), then if $\lambda^* = 0$ we have that $z^* = z_\pi$. Also if $y^*$ solves (2.5) and $\mathbb{P}(y^* \geq 0) = 1$, then $\lambda^* = 0$ and $z^* = z_\pi$ (see [**17**, Corollary 2.2]).

Now suppose that for risk control the investor is utilizing a risk measure where all the elements $z \in \Delta_\rho$ are bounded. As an example of such measure, we give the Expected Shortfall: if $\rho = ES_\alpha$ for $\alpha \in (0,1)$ then

$$
\Delta_{ES_\alpha} = \left\{ z \in L^2 \; ; \; \mathbb{E}(z) = 1, 0 \leq z \leq \frac{1}{1-\alpha} \right\}.
$$

With that hypothesis in mind, [**17**, Remark 2.3] denotes that solution $y^*$ for (2.5) is often a risky asset, because a risk free asset would lead to $z_\pi \in \Delta_\rho$. On the other hand, there is no restriction that an investor should choose a pricing model that requires a bounded $z_\pi$. For instance, as [**17**, p.33] mentions, she/he could choose a stochastic volatility model where $z_\pi$ might exhibit a heavy tailed distribution.

The above frame suggests that $z_\pi$ does not necessarily belong to $\Delta_\rho$. Consequently if $z_\pi \notin \Delta_\rho$, (2.8) is not feasible. In addition (2.7) is unbounded, suggesting that an investor has the opportunity to construct sequences of payoffs $(y_n)_{n=1}^{\infty}$, $n \in \mathbb{N}$, where

$$\begin{cases} \rho(y_n + y_f - \mathbb{E}(y_n z_\pi)) \mapsto -\infty \ , \\ \mathbb{E}(y_n z_\pi) \leq C. \end{cases} \tag{2.10}$$

Lemma 2.1. $\mathbb{E}(y_n + y_f - \mathbb{E}(y_n z_\pi)) \mapsto \infty$ *(extends Remark 2.4 of* [**17**] *).*

Proof. Recall that $\rho$ is expectation bounded, hence $\rho(Y) \geq -\mathbb{E}[Y]$ according to Definition 3.4, that means $\rho(y_n + y_f - \mathbb{E}(y_n z_\pi)) \geq -\mathbb{E}(y_n + y_f - \mathbb{E}(y_n z_\pi))$ and this is equivalent to $\mathbb{E}(y_n + y_f - \mathbb{E}(y_n z_\pi)) \geq -\rho(y_n + y_f - \mathbb{E}(y_n z_\pi))$ which finally gives the result. □

Taking into consideration the previous statements, an investor is able to construct sequences of payoffs $(y_n)_{n=1}^{\infty}$, $n \in \mathbb{N}$ where

$$\begin{cases} \mathbb{E}(y_n + y_f - \mathbb{E}(y_n z_\pi)) \mapsto \infty \\ \rho(y_n + y_f - \mathbb{E}(y_n z_\pi)) \mapsto -\infty \\ \mathbb{E}(y_n z_\pi) \leq C \text{ for } n \in \mathbb{N} \end{cases}$$

hold, as long as she/he prefers a pricing model with unbounded SDF, and a coherent and expectation bounded risk measure. The term for those sequences will be good deals [**40**]. In terms of $\rho$ in (2.10), an investor can choose not only the Expected Shortfall, but also a Spectral Risk Measure which is coherent.

Also, by [**17**, Lemma 2.8] we obtain that if $y^* \in L^2$ and $z^* \in \Delta_{ES_\alpha}$, then they satisfy the third Karush - Kuhn - Tucker condition of (2.9), if there exists $\gamma \in \mathbb{R}, \gamma_1, \gamma_2 \in L^2$

and a measurable partition $\Omega = \Omega_0 \cup \Omega_1 \cup \Omega_2$ such that the following conditions hold

$$(2.11) \qquad \begin{cases} y^* + y_f = \gamma - \gamma_1 + \gamma_2 \\ \gamma_i \geq 0, i = 1, 2 \\ \gamma_1 = \gamma_2 = 0, \text{ on } \Omega_0 \\ z^* = \dfrac{1}{1-\alpha}, \gamma_2 = 0 \text{ on } \Omega_1 \\ z^* = 0, \gamma_1 = 0 \text{ on } \Omega_2. \end{cases}$$

Clearly, this Lemma has a profound practical application as it suggests that the third condition in (2.9) can be replaced in order to solve (2.6) and (2.5).

## 3. Main Contribution

The theoretical framework proves that there is a solution $y^*$, outperforming risk free investments, and when it does so there are sequences of payoffs $(y_n)_{n=1}^{\infty}, n \in \mathbb{N}$ satisfying (2.10), that is known as "good deals", according to [**40**] terminology. In practical terms, for calculating the good deals, we need to extract $y^*$, and we also need to use a risk measure with bounded $z$. To that end we pick $\rho = ES$. Then according to [**17**, Theorem 2.9], there exist $\gamma$ and $\delta \in \mathbb{R}$ , such that

$$(3.1) \qquad y^* = \begin{cases} \gamma - y_f & \text{if } \delta < y_f \leq \gamma \\ 0 & \text{if } y_f \leq \delta \text{ or } y_f > \gamma. \end{cases}$$

As investors final wealth is a random variable one can take that

$$y_f(\omega) = F^{\leftarrow}(\omega)$$

where $\omega$ is an element of set $\Omega$, that follows the uniform distribution taking values on $(0,1)$, and by $F^{\leftarrow}$ we denote the inverse cumulative distribution of the random variable $y_f$ and $y_f$ is continuous and strictly increasing. Also $z_\pi$ can be perceived as a function with domain $(0,1)$, where $\omega \overset{\prime}{\longrightarrow} z_\pi(\omega) \in (0,+\infty)$. Moreover, we assume that $z_\pi$ is not the risk free asset. In addition [**17**] denote that if

$$\operatorname{ess\,sup}(z_\pi) > \frac{1}{1-\alpha}$$

holds, then if either $z_\pi$ is continuous and strictly decreasing, or $y_f$ maximizes the Sharpe Ratio, then there is $\gamma, \delta$ as proposed in (3.1). In addition $\delta \leq \gamma$ and (3.1) holds.

Having in mind aforementioned framework we contribute with the Theorem 3.1 stated below, in a twofold manner:

(1) We find that an investor can achieve good deals by utilizing both Sharpe ratio and $ES_a$ for her/his investing strategy.
(2) We examine market efficiency. For that we utilize the CAPM model to find that once achieving those good deals, then the portfolio should be less risky compare to the market risk, or for achieving a Jensen's $\alpha > 0$ one should have that $\beta \leq 1$.

The second part of the contribution questions the efficiency of the market where the investment is taking place, which oppose to certain empirical modern portfolio analysis that tends to consider that markets are efficient (see [**62**]). The remark below describes our motivation for this chapter.

REMARK 3.1. *There are many studies that question the existence of abnormal returns (that means $\alpha > 0$), see for instance* [**62**]*, and conclude that markets are efficient. On the contrary, we observe that by utilizing the framework of* [**17**] *(see theorem below and its proof), abnormal returns may occur under a less risky strategy, compare to the market. Such conclusion questions the aforementioned market efficiency.*

THEOREM 3.2. *Let consider that $\rho = ES_a$, there is $y^*$ that solves* (2.5) *and there is $z_\pi$ that satisfies*

$$\operatorname{ess\,sup}(z_\pi) > \frac{1}{1-\alpha}$$

*Assume that $y_f$ maximizes Sharpe Ratio. Then:*

(1) *if an investment strategy is less risky compare to the market, then it performs better relative to the market and the market is not efficient.*
(2) *if an investment strategy performs better relative to benchmark index, then it is less risky compare to the market and the market is not efficient,*

*and exist $\gamma, \delta \in \mathbb{R}$ for which $\gamma > \delta$ and* (3.1) *holds.*

PROOF. Let us prove the existence of $\gamma$ and $\delta$ and that (3.1) holds. For that, we initially prove that if $y_f$ maximizes Sharpe Ratio, then $z_\pi$ is continuous and strictly decreasing. Following the proof of theorem 6 of [**17**] in appendix I section, we consider the $l\{1, z_\pi\} \subset L^2$, which is the linear subspace composed of the risk free asset and the $z_\pi$. The orthogonal projection of $y_f$ in $l\{1, z_\pi\}$, which is $q_a + q_b z_\pi$ satisfies

$$y_f = q_a + q_b z_\pi + \varepsilon. \tag{3.2}$$

In addition, $\mathbb{E}(\varepsilon) = 0$ and $\mathbb{E}(z_\pi \varepsilon) = 0$, implying that mathematical expectation and price of $q_a + q_b z_\pi$ equals the mathematical expectation and price of $y_f$. In addition, $Variance_{y_f}$ is not higher than $Variance_{q_a+q_b z_\pi}$ , (3.2) implies that $\varepsilon = 0$ and thus,

$$y_f = q_a + q_b z_\pi. \tag{3.3}$$

(3.3) suggests that in order to prove that $z_\pi$ is continuous and strictly decreasing, we only have to prove $q_b < 0$, since $y_f$ is continuous and strictly increasing. We proceed by proving that the assumption $q_b \geq 0$ does not hold:

$q_b = 0$ does not hold as $y_f$ is not a risk free investment

$q_b > 0$, (2.2) and $\mathbb{E}[z_\pi] = 1$ suggest that $P(y_f) = e^{-r_f T}[(q_a + q_b \mathbb{E}(z_\pi^2)]$, which coincides with risk free investment whose price is $q_a + q_b \mathbb{E}(z_\pi^2)$. In addition, the expected payoff of the risk free asset is higher because $\mathbb{E}[z_\pi] = 1$, which draws the following conclusions: first, $\mathbb{E}(q_a + q_b z_\pi) = q_a + q_b$ and second $\mathbb{E}(z_\pi^2) = Variance(z_\pi) + 1 > 1$. A contradiction in the assumption is apparent, since $Variance$ of the risk free asset is lower than $Variance_{q_a+q_b z_\pi}$.

Let us prove the existence of $\gamma$ and $\delta$. Again, we rely on [**17**] and the proof demonstrated in the Appendix I. We consider the dual solution $(\lambda^*, z^*) \in (\mathbb{R} \times L^2)$. In addition, $\Delta_{ES}$ is bounded, while $z_\pi$ is not. As stated (2.8) is possible only if $\lambda = 0$ so since it is not, then $\lambda^* > 0$. Since $z_\pi$ is continuous and strictly decreasing, also $(1+\lambda^*)z_\pi$ is continuous and strictly decreasing and as a conclusion, there is a $\gamma_a$ such that

$$\begin{cases} (1+\lambda^*)z_\pi(\gamma_a) = \dfrac{1}{1-\alpha} \\ (1+\lambda^*)z_\pi(\omega) > \dfrac{1}{1-\alpha} \text{ for } \omega \in (0,\gamma_a) \\ (1+\lambda^*)z_\pi(\omega) < \dfrac{1}{1-\alpha} \text{ for } \omega \in (\gamma_a, 1) \end{cases}$$

hold. Consider the fourth and fifth condition of (2.9). Since $(1+\lambda^*)z_\pi(\omega) > z^*(\omega)$ in $(0,\gamma_a)$, this implies that $y^*(\omega) = 0$ in $(0,\gamma_a)$. Consider that $y_f$ is continuous and strictly increasing and take $\delta = y_f(\gamma_a)$. Moreover, $\delta \geq y_f$ if and only if $(0,\gamma_a] \ni \omega$. The existence of $\delta$ is proved.

Let us prove that there exist $\tilde{\gamma_a} \leq \gamma_a$ such that $\Omega_1 = (0,\gamma_a]$. Recall the partition in (2.11), which is $\Omega = \Omega_0 \cup \Omega_1 \cup \Omega_2$. Also consider

$$z^* = \frac{1}{1-\alpha},$$

$\gamma_2 = 0$ on $\Omega_1$ in (2.11), and $(1+\lambda^*)z_\pi - z^* \geq 0$ in (2.9). Those two equations lead to $\Omega_1 \subset (0,\gamma_a]$. Consider $y^* + y_f = \gamma - \gamma_1 + \gamma_2$, which is the first equation in (2.11). Notice that $y_f = \gamma - \gamma_1$ in $\Omega_1$, since $\gamma_2 = 0$ in $\Omega_1$ and also $y^*$ vanishes in $(0,\gamma_a]$. Also, $y_f = \gamma + \gamma_2$ in $(0,\gamma_a]\backslash\Omega_1$, since $\gamma_1$ vanishes outside $\Omega_1$. We take into account that $\gamma_i \geq 0, i = 1,2$ from second equation in (2.11), and that $y_f$ is continuous and strictly increasing, so we conclude that $y_f$ increases from $\Omega_1$ to $(0,\gamma_a]\backslash\Omega_1$ and that exist $\tilde{\gamma_a} \leq \gamma_a$ such that $\Omega_1 = (0,\gamma_a]$.

We also prove that $(\tilde{\gamma_a},\gamma_a] \subset \Omega_2$. Consider a non-null subset of $(\tilde{\gamma_a},\gamma_a]$ and that $\gamma_2$ vanishes outside $\Omega_2$ and we get that $y_f = \gamma + \gamma_2 \Rightarrow y_f = \gamma$ which does not hold as $y_f$ is continuous and strictly increasing. Finally $(\tilde{\gamma_a},\gamma_a] \subset \Omega_2$ hold.

Moreover, we prove that $\Omega_0 \neq \emptyset$. Consider for a moment that $\Omega_0 = \emptyset$, and bear in mind that $\Omega_2 = (\tilde{\gamma_a}, 1)$ and $z^* = 0$ in $(\tilde{\gamma_a}, 1)$, according to fifth condition in (2.11). In addition, since $(1+\lambda^*)z_\pi > 0$, then condition $\mathbb{E}[((1+\lambda^*)z_\pi - z^*)y^*] = 0$ of (2.9) implies that $y^* = 0$ in $(0, 1)$. Since $C > 0$ and $\lambda^* > 0$ then $\lambda^*[C - \mathbb{E}(y^* z_\pi)] = 0$ in (2.9) does not hold and consequently $\Omega_0 \neq \emptyset$.

Now, let us prove that $\tilde{\gamma_a} = \gamma_a$. We are aware that $\Omega_0 \subset (\gamma_a, 1)$. We fix $\lambda^*$ and according to (2.8) and $(1 + \lambda^*)z_\pi - z^* \geq 0$, which is the fifth constraint in (2.9), then $z^*$ should solve

$$\min\{\mathbb{E}(y_f z); z \leq (1 + \lambda^*)z_\pi, z \in \Delta_\rho\} \tag{3.4}$$

If $\tilde{\gamma_a} < \gamma_a$ then

$$\tilde{z} = \begin{cases} z^*, \omega \in \Omega_1 = (0, \tilde{\gamma_a}] \\ z^*(\omega + \inf(\Omega_0) - \tilde{\gamma_a}), \tilde{\gamma_a} < \omega < \tilde{\gamma_a} + \sup(\Omega_0) - \inf(\Omega_0) \\ 0, \text{otherwise} \end{cases}$$

$\tilde{z}$ trivially satisfies constraints of (3.4), $z^*$ vanishes on $\Omega_2$ and $z_\pi$ is strictly decreasing. On the other hand, $\mathbb{E}(y_f \tilde{z}) < \mathbb{E}(y_f z^*)$ trivially holds as $y_f$ is continuous and strictly increasing, $z^*$ is cannot solve (3.4), $\tilde{\gamma_a} < \gamma_a$ does not hold and thus $\tilde{\gamma_a} = \gamma_a$.

Similarly, we can prove the existence of $\gamma_b \geq \gamma_a$ where $\Omega_0 = (\gamma_a + \gamma_b)$. Additionally, $y^* = \gamma - y_f$ in $(\gamma_a, \gamma_b)$, which implies that $y_f(\omega) \leq \gamma$, when $\omega \in (\gamma_a, \gamma_b)$, as $y^* \geq 0$. Recall that $y_f$ is continuous and strictly increasing so $0 < \delta = y_f(\gamma_a) < y_f(\gamma_b) \leq \gamma \Rightarrow 0 < \delta < \gamma$.

Moreover, when $y_f > \gamma$ then $\omega > \gamma_b$, $\omega \in \Omega_2$ and according to fifth constraint in 2.11, $z^* = 0$. Also $(1+\lambda^*)z_\pi - z^*) > 0$, which is the fifth condition of (2.9), yet with strict inequality. Also the fourth constraint in (2.9), which is $\mathbb{E}[((1+\lambda^*)z_\pi - z^*)y^*] = 0$ suggests that $y^*$ vanishes.

The existence of $\gamma, \delta \in \mathbb{R}$ for which:

(1) $\gamma > \delta$
(2) (3.1) holds,

is proved.

In addition, we contribute by proving that a positive Jensen's $\alpha$ denotes a CAPM coefficient $\beta \leq 1$ and vice versa. Recall the formula of the intercept of the security characteristic line (as extracted from (6.4)), where

$$\alpha = r - r_f - \beta(r_m - r_f).$$

On the theorem we assume that $y_f$ maximizes the Sharpe Ratio (see (8.1)),which is

$$SharpeRatio = \frac{r - r_f}{\sigma}.$$

Next, $\sigma > 0$, as it denotes as the standard deviation of the investment $r$. So, $SharpeRatio$ can take either positive value, negative value or even zero in case $r = r_f$. In fact we have $SharpeRatio \in \mathbb{R}$. As $SharpeRatio$ is maximized, we observe that $r - r_f > 0 \Rightarrow r > r_f$ .

Also, $y_f$ represents the investors final wealth and its existence suggest that the returns of investment beat those of the market or a benchmark which leads to $r > r_m$. Subtracting $r_f$ in both sides we get $r - r_f > r_m - r_f$.

Let assume that $\alpha > 0$. Then the assumption implies that $r - r_f - \beta(r_m - r_f) > 0 \Rightarrow r - r_f > \beta(r_m - r_f) \Rightarrow (r - r_f)/\beta > r_m - r_f$. The above inequality holds if: $1/\beta \geq 1 \Rightarrow \beta \leq 1$.

Additionally, let consider that $\beta \leq 1$

Then $r_m - r_f \geq \beta(r_m - r_f)$, but $r - r_f > r_m - r_f$ which leads to $r - r_f > \beta(r_m - r_f) \Rightarrow r - r_f - \beta(r_m - r_f) > 0 \Rightarrow \alpha > 0$

The assumption that $y_f$ maximizes Sharpe Ratio denotes a positive $\alpha$ provided there is a CAPM $\beta \leq 1$ and $\beta \leq 1$, suggests an investment strategy less risky compare to the risk incorporated in the market portfolio as explained in basic concept section.

Finally, the maximization of $y_f$ denotes that CAPM's $\beta \leq 1$, provided $\alpha > 0$. Recall that an investment has $\alpha > 0$ when it performs better relative to benchmark index and worth selling, as it is overpriced.

□

Bear in mind that while we are using a different approach compare to [**62**], as they examine if the aggregate of mutual funds' performance in a market achieve abnormal returns, yet our work has the same interest for examining the market efficiency. In addition, we contribute by observing that those abnormal returns are achievable if an investor chooses the right tools for his strategy. Consequently such conclusion is evidence of market inefficiency.

REMARK 3.3. *If $y_f$ maximizes Sharpe Ratio, then according to proof of Theorem 3.1 $z_\pi$ is continuous and strictly decreasing, hence the first condition in 2.15 theorem of* [**17**], *is sufficient but not necessary.*

**3.1. Practical Implementation of Theoretical Findings.** There is a significant amount of papers that examine the portfolio performance and they verify their findings empirically with real data. While for this chapter we are interested in the theoretical justification of our finding, yet we consider that a work with empirical data in the near future is also required to investigate if the initial finding is true and to which extent. For that, we will have to utilize the Theorem 3.1 in practice. Apart from its somewhat confusing presentation, in fact a practical validation can be implemented in a two stage procedure. At first an investor can attempt to get an optimal good deal by picking a portfolio of options from an option market according to a combination of two parameters: First parameter is the maximization of classical Sharpe ratio:

$$SharpeRatio = \frac{r - r_f}{\sigma}.$$

In order to do that the Sharpe Ratio should get a value of at least equal or better, compare to a Sharpe ratio value of a market portfolio index the investor attempts to beat. In portfolio analysis beating an index implies better performance (returns) compare to that index. As index, one can select future indexes like the Standard & Poor's 500 in case she/he invests in U.S.'s international index or the DAX 30 index in case she/he is interested in Frankfurt's stock exchange.

In [**18**] and [**20**] is demonstrated that this maximization can be achieved if one combines a stochastic volatility model with a coherent and expectation bounded risk measure. In terms of stochastic volatility model we could utilize a Generalized Autoregressive Conditional heteroscedasticity (GARCH) model with CAPM, for instance the CAPM-EGARCH (1.1). It is beyond the scope of this chapter to analyze the model, yet we mention that it is a conditional volatility model.

Second parameter is the risk measure which would be $ES_a$ as proposed by the Theorem 3.1

$$\Delta_{ES_\alpha} = \{z \in L^2 \; ; \; \mathbb{E}(z) = 1, 0 \leq z \leq \frac{1}{1-\alpha}\} \, .$$

More details on the exact methodology of first stage can also be found in [**17**] p.43. The difference compare to our methodology is that we will utilize a Generalized Autoregressive Conditional heteroscedasticity (GARCH) model with CAPM instead of the Black-Scholes model.

In the second stage we can derive the portfolio's beta and alpha out of the CAPM-EGARCH (1.1) model. In case our theoretical findings are proven in practical terms, we expect to get a $\beta \leq 1$ when $\alpha > 0$. in addition we expect to get an $\alpha > 0$ when $\beta \leq 1$. Finally, in order to evaluate our findings the best possible way, we believe that we should select a case with data from a mature market with solid indexes and less fluctuations (e.g. Frankfurt's DAX 30) and also one with data from an emerging one (e.g. Vienna's ATX index).

# 4. Concluding Remarks

This chapter is concerned on the unbounded risk premium that might occur if there is combination of asset pricing model which has an unbounded stochastic discount factor, and a coherent and bounded risk measure. Suggesting an intuitive explanation of the findings, one can assume that is possible to follow a risk averse strategy, as $\beta \leq 1$ and at the same time achieve abnormal returns, as $\alpha > 0$. Rephrasing, it appears to be that this framework is suitable for an investor who is loss averse. On top of that, it seems that a loss averse investor is the ideal candidate to earn a risk premium compare to someone who is more tolerate to risk. Summing up, our findings are not favoring the opinion that market efficiency tends to hold.

CHAPTER 4

# Estimation of Insolvency Probability Under Systemic Risk

## 1. Preliminaries

Systemic Risk (SR) is considered of high interest in academia. Recall that Systemic Risk refers to the instability of a financial system component that can lead to its entire collapse [**75**]. From the aforementioned statement one can intuitively understand that $SR$ is closely related to the concept of Dependence. Most importantly there is not yet a consensus among academics on a common way to evaluate the $SR$ and so there is a prolific scholar discussion towards that direction. [**54**] proposed some sufficient conditions for two random vectors to be ordered by the so-called Conditional distortion risk measures and demonstrated how these risk measures are quantifying $SR$. [**28**] specifies a framework for $SR$ measures via multidimensional acceptance sets and aggregation functions. In addition, while usually $SR$ measures are mostly interpreted as the minimal amount of cash needed before aggregating individual risks, their approach suggests that $SR$ measures are minimal amount of cash needed before aggregating individual risks. While our effort exhibits similarities with all the aforementioned yet there are some fundamental differences. We consider that under the assumption that one component (entity) of an economic environment is in distress it is evaluated the Expected Capital Shortfall of this financial market (more on the relevant subsection).

[**14**] states that the main purpose concerning systemic risk is to evaluate the financial distress of an economy as a consequence of the failure of one of its components. They

also point out the importance of Extreme Value Theory (EVT) in the analysis of systemic risk, which we also will make extensive use. In addition [**32**] introduces SRISK to measure the capital shortfall of a firm conditional on a severe market decline. This approach is quite similar with the one we also utilize. Also, in regard of Insolvency probability, which may also be addressed as Ruin Probability under the Insurance theory perspective, we are primarily motivated and we follow methodologically up to an extent the paper of [**50**]. Similar to the aforementioned effort we also mention [**82**]. . In addition, similar to our research interest, we also mention the work of [**16**], [**34**] and [**74**].

Having all the above in mind we contribute by calculating the Insolvency Probability due to Systemic Risk, when variation variable is less than one. The structure of chapter is the following: next section is devoted to the theoretical framework that concern the $SR$ and Risk Contributions. The next one includes our contribution to the theoretical framework and finally section 4 concludes.

## 2. Theoretical Framework

**2.1. Systemic Risk.** For our analysis we choose the $SR$ definition, as proposed by [**5**] and can also be found in [**32**]. There, under the assumption that one component (entity) of an economic environment is in distress it is evaluated the Expected Capital Shortfall of this financial market. For instance, the term component (or the term entity) may refer to a company of a particular business sector, or a bank that is regulated by a regulatory organization. It can even be an investment that is consisted in a portfolio. Consequently an economic environment can be perceived as a business sector or a financial market where its banking system is regulated by a regulatory organization. It can as well be a portfolio that its risk is controlled by a portfolio manager who is utilizing a risk measure. Similar to this effort is the one proposed by [**6**]. There is presented an economic model of systemic risk where the whole business sector is under-capitalized and consequently it harms the real economy. In terms of similarity we also mention [**33**]. In this paper is developed a framework for measuring, allocating and managing systemic risk.

For the best possible understanding of the distress of a component one can consider a situation where a risk measure is utilized for controlling risk of that component. Then distress in the component occurs when its price, dictated by the risk measure is exceeded. In other words, the risk measure gives the level of loss that should not exceed and if this happens, then financial distress emerges. In a formal mathematical framework we consider the probability space $(\Omega, \mathcal{F}, \mathbb{P})$, and the set of positive r.v. with infinite upper points $L_+(\mathbb{P})$. In addition let $X, Z \in L_+(\mathbb{P})$ and the equation [**14**, equation 1.1], or $\rho_{X,Z}(q) := \mathbb{E}\Big[\Big(X - t_1(q)\Big)_+ |Z > t_2(q)\Big]$, where $t_1(q)$ and $t_2(q)$ are two positive functions with $q \in (0, 1)$ and $\lim_{q \to 1} t_2(q) = \infty$. Clearly, $t_1(q)$ and $t_2(q)$ can be perceived as risk measures that a regulator (or an investor) is utilizing for controlling the levels of risk. The economical interpretation of the components of this conditional probability is quite straightforward as $\big(X - t_1(q)\big)_+$ represents the total liabilities of the financial market (or investment or bank sector) minus total capital allocated to the market and can be determined by risk measure. Also $Z > t_2(q)$ represents the fact that there is crisis in a component of the market as it should be clear that $t_2(q)$ represents the capital, via a risk measure that is allocated to the entity that bears the liability $Z$. In regard how we utilize all the above, we refer the reader to Euler contribution for distortion risk measures subsection.

## 3. Contribution

**3.1. Euler Contributions for Risk Measures.** Initially we establish that Euler Allocation is applicable in an economic environment that is regulated by a Distortion Risk Measure. Moreover we consider that $SR$ has a significant impact on that economic environment. Let consider that there are $k$ economic entities with $X_1, X_2, ...X_k$ random variables, $F_1, F_2, ...F_k$ marginal distribution functions and $k \in \mathbb{N}$. In addition consider that the regulator allocates for every $X_i$ entity $C_i$ capital. Recall that in Euler Allocation Principle subsection is stated that $C_i$ is the capital that is required by the economic entity $X_i$ and is determined with a risk measure (see relevant sub-section). Moreover the total capital that is allocated to all the economic entities would be $\sum_{i=1}^{k} C_i$ and we also set that $S_k := \sum_{i=1}^{k} X_i$ with $I$ distribution function. By assuming that the first economic entity exhibits financial distress, then without loss of generality $SR$ is defined to have the following value

$$SR := \mathbb{E}\big[\big(S_k - \sum_{i=1}^{k} C_i\big)_+ | X_1 > C_1\big] \tag{3.1}$$

in terms of aggregate risk and is depicted in [**14**] and [**5**]. Moreover $SR$ contribution to the $n^{th}$ economic entity is defined to have the following value $SR_n := \mathbb{E}\big[\big(X_n - C_n\big)_+ | X_1 > C_1\big]$, where $n \in \{1, 2, ...., k\}$ and is also depicted in [**14**].

**3.2. A Calculation of Systemic Risk's Insolvency Probability, when Variation Variable is less then 1.** Another important topic that should be addressed is insolvency due to Systemic Risk. By recalling the definition of $SR$ as depicted in (3.1), it is only natural to consider that the probability of insolvency or $\Psi$ can be defined as the following:

$$\Psi_{(k,n)} := \mathbb{P}\Big[\bigvee_{i=1}^{n} S_k > \sum_{i=1}^{k} C_i\Big] \text{ as } \sum_{i=1}^{k} C_i \to \infty, \tag{3.2}$$

where in accordance with [**104**, eq.1.2] and our Euler Allocation Principle subsection, we have that $S_k$ is the randomly weighted sum that can be decomposed into primary n real-valued independent random variables $X_1, ..., X_k$ and $u_1, u_2, ..., u_k$ nonnegative random variables, independent of the primary. By setting $\sum_{i=1}^{k} C_i = x$, then (3.2) is depicted in the following form:

$$\Psi_{(k,n)} := \mathbb{P}\left[\bigvee_{i=1}^{n} S_k > x\right] \text{ as } x \to \infty. \tag{3.3}$$

Regardless of the indicators that may appear due the mathematical structure we consider that $\Psi$ stands in this paper for the Insolvency Probability. In addition, by introducing positive real numbers $u = (u_1, u_2, ..., u_k)$ as presented in (4.2), and by considering [**104**, eq.1.1 and eq.1.2] it occurs a useful representation of $X$ and specifically:

$$\sum_{i=1}^{k} u_i X_i = S_k(u).$$

In accordance with [**104**], $u_1, u_2, ..., u_k$ can be perceived as positive real numbers that capture dependence. Although such interpretation is not per se useful in the sense that dependence is not our scope for this subsection, nevertheless $u_1, u_2, ..., u_k$ will prove very useful for our mathematical framework. Having the above in mind, we consider [**104**, eq.1.3] and [**103**] where we get that if $X_1, ..., X_k$ iid by a Sub-exponential distribution (see [**49**, Definition 2] for the definition of Sub-exponential distribution, denoted as $\mathcal{S}$) and $u_1, u_2, ..., u_k$ are in $(0, b]$ for some constant $b$ and $0 < b \leq \infty$, then

$$\mathbb{P}\left[\bigvee_{i=1}^{n} S_k(u) > x\right] \sim \mathbb{P}\big[S_k(u) > x\big] \sim \mathbb{P}\left[\bigvee_{i=1}^{n} u_i X_i > x\right] \sim \sum_{i=1}^{k} \mathbb{P}\big[u_i X_i > x\big]. \tag{3.4}$$

Remark 3.1. (3.4) *suggests that the heavy tails of the* $X_1, ..., X_k$ *random variables vanishes the dependence presented from* $u_1, u_2, ..., u_k$. *Such notion is in tandem with the principle of a single big jump in the presence of random weights (one may consult* [**49**] *for subtleties).*

In tandem with the rest of our contribution subsection, we have that there are $k$ economic entities with $X_1, X_2, ...X_k$ continuous non negative random variables, $F_1, F_2, ...F_k$ marginal distribution functions that vary regularly unless we state otherwise and $k \in \mathbb{N}$.

Also, in accordance with [**60**] we define the positive truncated mean function, or

$$m_+(x) := \int_0^x \big[1 - F(y)\big] dy.$$

In addition, we define the integrand $J_-$, or

$$J_- := J_-(X) = \int_{-\infty}^{0^-} \frac{\mid x \mid}{m_+(\mid x \mid)} dF(x).$$

A key assumption is that $\bigvee_{i=1}^n S_k(u)$ is finite almost surely, which occurs iff $S_k(u) \to \infty$ as k $\to \infty$ with probability 1 [**65**, Chapter XII, Section 2, Theorem 1]. Also, we have from [**60**, Corollary 1 (a)], [**60**, Theorem 2 (c)] and the remark that follows in the same context that:

$$\text{if } \mathbb{E}|X_i| = \infty, \text{ then } S_k(u) \to \infty \text{ a.s. as } k \to \infty \text{ iff } J_- < \infty, \tag{3.5}$$

which is the [**60**, Corollary 1 (a)], as it will be utilized in this paper.

Remark 3.2. *Initially* (3.5) *assures that* $\bigvee_{i=1}^n S_k$ *is finite and thus a proper r.v..Furthermore, by attempting to give an intuitive explanation of the* (3.5) *condition, it suggests that the right tail of* $F_i$ *is heavier than the left one.*

Definition 3.3. *A distribution function* $F$ *on* $\mathbb{R}$ *is dominatedly-varying tailed, or* $F \in \mathcal{D}$ *when its right tail satisfies* $\overline{F}(xz) = O\big(\overline{F}(x)\big)$ *for all* $0 < z < 1$.

In addition, we consider [**49**, Definition 1] for the definition of Long-tailed distribution, denoted as $\mathcal{L}$. Also,we state from [**22**] the following:

Definition 3.4. *A distribution function $F$ on $\mathbb{R}$ has an extended rapidly varying tail or belongs to class $\mathcal{E}_{\mathcal{R}}$, when its right tail satisfies $\limsup_{x\to\infty} \frac{\overline{F}(xz)}{\overline{F}(x)} < 1$ for some $z > 1$.*

$\mathcal{E}_{\mathcal{R}}$ is also met in other works like [**78**, eq.4], where is addressed under the notation $\mathcal{PD}$ (positively decreasing-tailed). Also, we set that $F_*(z) = \liminf \overline{F}(xz)/\overline{F}(x)$, $F^*(z) = \limsup \overline{F}(xz)/\overline{F}(x)$, and for a distribution function $F$ with an ultimate right tail we have that the upper Matuszewska index, or $\alpha_F$ is defined as

$$\alpha_F = \inf\{-(\log F_*(z)/\log z) : z > 1\} \in [0, \infty],$$

and the lower Matuszewska index, or $\beta_F$ is defined as

$$\beta_F = \sup\{-(\log F^*(z)/\log z) : z > 1\} \in [0, \infty].$$

For the remaining of the paper we assume that $z$ is positive. Now, let us proceed with the following Lemma, which is based on [**50**, Corollary 1]:

Lemma 3.5. *Let consider that $\mathbb{E}|X_i| = \infty$,* [**60**, Corollary 1 (a)] *holds, $\overline{F}(z)$ is regularly varying with varying variable $\beta$ and $m(z)$ is regularly varying with varying variable $1-\alpha$, where $0 < \alpha \leq 1$. If $\alpha < \beta$ and $\beta \in \mathbb{R}_+$, then*

$$\Psi(x) \sim \frac{\Gamma(\beta-\alpha)}{\Gamma(\beta)\Gamma(2-\alpha)} \frac{x\overline{F(x)}}{m(x)}, \text{ as } x \to \infty$$

*where $\Gamma$ is the Gamma function.*

Remark 3.6. *Since* [**60**, Corollary 1 (a)] *holds, we have that due to the fact that $J_- < \infty$ that $\Psi(x) \sim \frac{\Gamma(\beta-\alpha)}{\Gamma(\beta)\Gamma(2-\alpha)} \frac{x\overline{F(x)}}{m(x)}$, as $x \to \infty$ is not meaningless.*

Proof. Initially we consider [**50**, Theorem 2]. There is addressed the case where $m(z)$ is regularly varying and $\overline{F}$ is long tailed and index $1-\alpha \in [0, 1]$. Specifically in [**50**, eq.6] we have that

$$\Psi(x) \sim \frac{\overline{G_a}(x)}{\Gamma(1+\alpha)\Gamma(2-\alpha)}, \tag{3.6}$$

where $\alpha$ is the varying variable of $m$. Moreover, we consider [**50**, (7)] which is a special case of $\overline{G_a}(x)$ once we have that $0 < \alpha \leq 1$ and is

$$\overline{G_a}(x) = \min\left(1, \int_1^\infty \frac{\overline{F}(x+z)}{m(z)} dz\right). \tag{3.7}$$

By considering (3.6) and (3.7), we get that

$$\Psi(x) \sim \frac{\overline{G}(x)}{\Gamma(\alpha)\Gamma(2-\alpha)}, \tag{3.8}$$

By taking a closer look at (3.7) and (3.8) we realize that we primarily need to calculate $\int_1^\infty \frac{\overline{F}(x+z)}{m(z)} dz$. For doing so, we initially consider the case where both $m(x)$ and $\overline{F}$ are varying regularly. Now, let us fix $e > 0$, $E > 0$ where $E > e$. Moreover, we consider the partition of $[1, \infty]$ into $[1, e]$,$[e, E]$ and $[E, \infty]$. Accordingly we get from [**50**, eq.21] that

$$\int_1^{ex} \frac{\overline{F}(x+z)}{m(z)} dz \leq \overline{F}(x) \int_1^{ex} \frac{1}{m(z)} dz \sim \frac{\overline{F}(x)}{\alpha} \frac{ex}{m(ex)}, \text{ as } x \to \infty, \tag{3.9}$$

while (3.9) stems from the fact that

$$\frac{d}{dx} \frac{z}{m(z)} = \frac{\alpha + o(1)}{m(z)} \text{ as } z \to \infty,$$

(see [**50**, p.24]), and thus

$$d\left(\frac{z}{m(z)}\right) \sim \frac{\alpha + o(1)}{m(z)} dz,$$

$$\int_1^{ex} d\left(\frac{z}{m(z)}\right) \sim \int_1^{ex} \frac{\alpha + o(1)}{m(z)} dz,$$

$$\frac{z}{m(z)}\bigg|_1^{ex} \sim \int_1^{ex} \frac{\alpha + o(1)}{m(z)} dz,$$

$$\frac{ex}{m(ex)}\frac{1}{\alpha} \sim \int_1^{ex} \frac{1}{m(z)}dz,$$

and

$$\frac{ex}{m(ex)}\frac{\overline{F}(x)}{\alpha} \sim \overline{F}(x)\int_1^{ex} \frac{1}{m(z)}dz.$$

By considering [**60**, Corollary 1 (a)] we get that $\int_1^{ex} \frac{\overline{F}(x+z)}{m(z)}dz$ is bounded by a finite quantity.

Remark 3.7. *While* (3.9) *is initially met in* [**50**, eq.21], *yet we further elaborate on the proof of the statement this equation stands for.*

We also have that

$$\int_{Ex}^{\infty} \frac{\overline{F}(x+z)}{m(z)}dz \leq \int_{Ex}^{\infty} \frac{\overline{F}(z)}{m(z)}dz \sim \frac{1}{\beta-\alpha}\frac{Ex\overline{F}(Ex)}{m(Ex)}, \text{ as } x \to \infty, \tag{3.10}$$

where (3.10) stems from the fact that we know that asymptotically $\overline{F}(z) = z^{-\beta}$ and $m(z) = z^{1-\alpha}$. Moreover:

$$\int_{Ex}^{\infty} \frac{\overline{F}(z)}{m(z)}dz$$

$$\sim \int_{Ex}^{\infty} \frac{z^{-\beta}}{z^{1-\alpha}}dz,$$

$$\sim \int_{Ex}^{\infty} t^{-\beta+\alpha-1}dz,$$

$$\sim \left.\frac{z^{-\beta+\alpha}}{-\beta+\alpha}\right|_{Ex}^{\infty},$$

and consequently

$$\int_{Ex}^{\infty} \frac{\overline{F}(z)}{m(z)}dz \sim \frac{1}{Ex^{\beta-\alpha}}\frac{1}{\beta-\alpha}. \tag{3.11}$$

Again, we consider (3.10) and we have the following:

$$\frac{1}{\beta-\alpha}\frac{Ex\overline{F}(Ex)}{m(Ex)}$$

$$\sim\frac{1}{\beta-\alpha}\frac{Ex(Ex)^{-\beta}}{Ex^{1-\alpha}},$$

$$\sim\frac{1}{\beta-\alpha}\frac{(Ex)^{-\beta+1}}{(Ex)^{1-\alpha}},$$

$$\sim\frac{1}{\beta-\alpha}\frac{1}{(Ex)^{1-\alpha}(Ex)^{-1+\beta}},$$

and consequently

$$\frac{1}{\beta-\alpha}\frac{Ex\overline{F}(Ex)}{m(Ex)}\sim\frac{1}{\beta-\alpha}\frac{1}{(Ex)^{\beta-\alpha}}. \tag{3.12}$$

It is clear that by considering (3.11) and (3.12) we get (3.10) and in tandem with Condition [**60**, Corollary 1 (a)] we get that $\int_{Ex}^{\infty}\frac{\overline{F}(x+z)}{m(z)}dz$ is bounded by a finite quantity.

REMARK 3.8. (3.10) *is again stated in* [**50**, eq.22], *and we illustrate on the necessary proof that contributes to its validity.*

Now, in regard of $[e,E]$ recall that we initially examine the case of Regular Variation. Moreover, we set that there is an interval $n\in[e,E]$ and we have that $\int_{ex}^{Ex}\frac{\overline{F}(x+z)}{m(z)}dz=\frac{\overline{F}(x)}{m(x)}\int_{ex}^{Ex}\frac{\overline{F}(x+z)}{\overline{F}(x)}\frac{m(x)}{m(z)}dz$, $\int_{ex}^{Ex}\frac{\overline{F}(x+z)}{m(z)}dz=\frac{x\overline{F}(x)}{m(x)}\int_{e}^{E}\frac{\overline{F}(x(1+n))}{\overline{F}(x)}\frac{m(x)}{m(xn)}dn$,

and, by considering Uniform Convergence Theorem for r.v. functions (see [**30**, Theorem 1.5.2]) we have that

$$\frac{\overline{F}(x(1+n))}{\overline{F}(x)}\frac{m(x)}{m(xn)}\to\frac{(1+n)^{-\beta}}{n^{1-\alpha}}, \tag{3.13}$$

as $x \to \infty$ uniformly in $n \in [e, E]$ and finally

$$\int_{ex}^{Ex} \frac{\overline{F}(x+z)}{m(z)} dz \sim \frac{x\overline{F}(x)}{m(x)} \int_{e}^{E} \frac{(1+n)^{-\beta}}{n^{1-\alpha}} dn \text{ as } x \to \infty. \tag{3.14}$$

once we let $[1, e]$ become negligible in the sense that $e \to 0$, let $[E, \infty]$ also become negligible in the sense that $E \to \infty$, then from (3.8),(3.9) and (3.14) we get that $\Psi(x) \sim \frac{\overline{G}(x)}{\Gamma(\alpha)\Gamma(2-\alpha)}$, $\frac{\overline{G}(x)}{\Gamma(\alpha)\Gamma(2-\alpha)} = \frac{1}{\Gamma(\alpha)\Gamma(2-\alpha)} \times \frac{x\overline{F}(x)}{m(x)} \int_0^\infty \frac{(1+n)^{-\beta}}{n^{1-\alpha}} dn$ and finally

$$\Psi(x) \sim \frac{1}{\Gamma(\alpha)\Gamma(2-\alpha)} \times \frac{x\overline{F}(x)}{m(x)} \int_0^\infty (1+n)^{-\beta} n^{\alpha-1} dn. \tag{3.15}$$

In addition, we consider that $\int_0^\infty (1+n)^{-\beta} n^{\alpha-1} dn$ is the representation of the Beta function $B(\alpha, \beta-\alpha) = \int_0^1 t^{\alpha-1}(1-t)^{\beta-\alpha-1} dt$, once we put $t = \frac{n}{1+n}$, and so

$$\Psi(x) \sim \frac{B(\alpha, \beta-\alpha)}{\Gamma(\alpha)\Gamma(2-\alpha)} \times \frac{x\overline{F}(x)}{m(x)}. \tag{3.16}$$

Now we consider the relation of $B$ function and $\Gamma$ function where

$$B(\alpha, \beta-\alpha) = \frac{\Gamma(\beta-\alpha)\Gamma(\alpha)}{\Gamma(\beta)}. \tag{3.17}$$

Finally, we plug (3.17) into (3.16) and so,

$$\Psi(x) \sim \frac{\Gamma(\beta-\alpha)\Gamma(\alpha)}{\Gamma(\alpha)\Gamma(\beta)\Gamma(2-\alpha)} \times \frac{x\overline{F}(x)}{m(x)}$$

and thus,

$$\Psi(x) \sim \frac{\Gamma(\beta-\alpha)}{\Gamma(\beta)\Gamma(2-\alpha)} \times \frac{x\overline{F}(x)}{m(x)} \text{ as } x \to \infty,$$

and proof is complete. □

Now, let us expand our work when $\overline{F}(z) \in \mathcal{D} \cap \mathcal{E}_{\mathcal{R}}$ and $m(z) \in \mathcal{D} \cap \mathcal{E}_{\mathcal{R}}$, and proceed with the following Theorem

THEOREM 3.9. *Let consider that* $\mathbb{E}|X_i| = \infty$, [**60**, Corollary 1 (a)] *holds,* $\overline{F}(z) \in \mathcal{D} \cap \mathcal{E}_{\mathcal{R}}$ *and* $m(z) \in \mathcal{D} \cap \mathcal{E}_{\mathcal{R}}$*, where* $0 < \alpha_l \leq 1$ *and* $0 < \alpha_u \leq 1$*. If* $\alpha_u < \beta_u$ *and* $\alpha_l < \beta_l$*, then there exist constants* $C$ *and* $D$ *such that*

$$\frac{\Gamma(\beta_l - \alpha_l)}{\Gamma(\beta_l)\Gamma(2-\alpha_l)} \frac{Dx\overline{F}(x)}{m(x)} \lesssim \Psi(x) \lesssim \frac{\Gamma(\beta_u - \alpha_u)}{\Gamma(\beta_u)\Gamma(2-\alpha_u)} \frac{Cx\overline{F}(x)}{m(x)}$$

*as* $x \to \infty$*, where* $\Gamma$ *is the Gamma function.*

PROOF. Initially, we consider for $\mathcal{D}$ [**36**, eq.3.2] and [**48**, Theorem 3, A, a1 and a2 of unpublished Appendix titled Positive increase and bounded increase for non-decreasing functions]. It is not difficult to realize that in regard of $\mathcal{D}$ we get an upper bound for $\frac{\overline{F}(xz)}{\overline{F}(x)}$ which is depicted in (3.18) and also an upper bound for $\frac{m(xz)}{m(x)}$, depicted in(3.19).

Also for $\mathcal{E}_{\mathcal{R}}$ we consult [**48**, Theorem 3, B, b1 and b2 of unpublished Appendix titled Positive increase and bounded increase for non-decreasing functions]. The aforementioned we can get a lower bound for $\frac{\overline{F}(xz)}{\overline{F}(x)}$ which is depicted in (3.18) and also a lower bound for $\frac{m(xz)}{m(x)}$, depicted in(3.19). In addition, by considering [**22**, p.124] we have for $\overline{F}(z)$ that there exist a $C_F$ for each $Z > 1$ and for some $D_F > 0$ and all $Z > 1$ as so that:

$$D_F z^{-\beta_l} \lesssim \frac{\overline{F}(xz)}{\overline{F}(x)} \lesssim C_F z^{-\beta_u} \text{ uniformly in } z \in [1, Z], \tag{3.18}$$

as $x \to \infty$. Similarly, for $m$ we have that:

$$C_m z^{1-\alpha_l} \lesssim \frac{m(xz)}{m(x)} \lesssim C_m z^{1-\alpha_u} \text{ uniformly in } z \in [1, Z], \tag{3.19}$$

as $x \to \infty$. By considering (3.13), we get that the upper bound is

$$\frac{\overline{F}(x(1+n))}{\overline{F}(x)} \times \frac{m(x)}{m(xn)} \lesssim \frac{(1+n)^{-\beta_u} C_F}{n^{1-\alpha_u} C_m}, \tag{3.20}$$

as $x \to \infty$. Notice that $m$'s asymptotic behavior is depicted in the denominator of (3.13). Furthermore, we set that $C_F/C_m = C$, and by plugging (3.20) into (3.15) we get

$$\Psi(x) \lesssim \frac{1}{\Gamma(\alpha_m)\Gamma(2-\alpha_m)} \times \frac{Cx\overline{F}(x)}{m(x)} \int_0^\infty (1+n)^{-\beta_u} n^{\alpha_u - 1} dn,$$

as $x \to \infty$. Working as in the previous Lemma, we get that $\int_0^\infty (1+n)^{-\beta_u} n^{\alpha_u-1} dn$, is the representation of the Beta function $B(\alpha_u, \beta_u - \alpha_u)$, once we put $t = \frac{n}{1+n}$, and in tandem with (3.16) we get

$$\Psi(x) \lesssim \frac{B(\alpha_u, \beta_u - \alpha_u)}{\Gamma(\alpha_u)\Gamma(2-\alpha_u)} \times \frac{Cx\overline{F}(x)}{m(x)}, \tag{3.21}$$

as $x \to \infty$. Now we consider the relation of $B$ function and $\Gamma$ function, similarly with that depicted in (3.17), and by plugging it to (3.21) we get that

$$\Psi(x) \lesssim \frac{\Gamma(\beta_u - \alpha_u)}{\Gamma(\beta_u)\Gamma(2-\alpha_u)} \times \frac{Cx\overline{F}(x)}{m(x)}, \tag{3.22}$$

as $x \to \infty$, which is the upper bound of $\Psi(x)$.

Similarly we have to work for the lower bound. Specifically, we consider (3.18), (3.19) and (3.13) and we have that

$$\frac{(1+n)^{-\beta_l} D_F}{n^{1-\alpha_l} D_m} \lesssim \frac{\overline{F}(x(1+n))}{\overline{F}(x)} \times \frac{m(x)}{m(xn)}, \tag{3.23}$$

as $x \to \infty$. Notice once again that $m$'s asymptotic behavior is depicted in the denominator. Furthermore, we set that $D_F/D_m = D$, and by plugging (3.23) into (3.15) we get

$$\frac{1}{\Gamma(\alpha_l)\Gamma(2-\alpha_l)} \times \frac{Dx\overline{F}(x)}{m(x)} \int_0^\infty (1+n)^{-\beta_l} n^{\alpha_l - 1} dn \lesssim \Psi(x),$$

as $x \to \infty$ which is the lower bound of $\Psi(x)$. Working as in the previous Lemma, we get that $\int_0^\infty (1+n)^{-\beta_l} n^{\alpha_l-1} dn$, is the representation of the Beta function $B(\alpha_l, \beta_l - \alpha_l)$, once we put $t = \frac{n}{1+n}$, and in tandem with (3.16) we get

$$\frac{B(\alpha_l, \beta_l - \alpha_l)}{\Gamma(\alpha_l)\Gamma(2-\alpha_l)} \times \frac{Dx\overline{F}(x)}{m(x)} \lesssim \Psi(x) \tag{3.24}$$

as $x \to \infty$. Now we consider the relation of $B$ function and $\Gamma$ function, similarly with that depicted in (3.17), and by plugging it to (3.24) we get that

$$\frac{\Gamma(\beta_l - \alpha_l)}{\Gamma(\beta_l)\Gamma(2-\alpha_l)} \times \frac{Dx\overline{F}(x)}{m(x)} \lesssim \Psi(x), \tag{3.25}$$

as $x \to \infty$, which is the lower bound of $\Psi(x)$. By combining (3.22) and (3.25) we get that

$$\frac{\Gamma(\beta_l - \alpha_l)}{\Gamma(\beta_l)\Gamma(2-\alpha_l)} \frac{Dx\overline{F}(x)}{m(x)} \lesssim \Psi(x) \lesssim \frac{\Gamma(\beta_u - \alpha_u)}{\Gamma(\beta_u)\Gamma(2-\alpha_u)} \frac{Cx\overline{F}(x)}{m(x)},$$

as $x \to \infty$, and the proof is complete. □

Let us state also the following Lemma, which is heavily influence by [**104**, Theorem 3.1].

Lemma 3.10. *If $u_1, u_2, ..., u_k$ are bounded from above, $i = 1, 2, ..., k$ and $F \in \mathcal{L} \cap \mathcal{D}$, then*

$$\mathbb{P}\Big[\bigvee_{i=1}^{n} S_k(u) > x\Big] \sim \mathbb{P}\big[S_k(u) > x\big] \sim \mathbb{P}\Big[\bigvee_{i=1}^{n} u_i X_i > x\Big] \sim \sum_{i=1}^{k} \mathbb{P}\big[u_i X_i > x\big], \text{ as } x \to \infty.$$

Proof. we already know from (3.4) that $u_1, u_2, ..., u_k$ are in $(0, b]$ for some constant $b$ and $0 < b \leq \infty$. By having that $u_1, u_2, ..., u_k$ are bounded from above, we consider that there are [**103**, Type II bound of r.v.]. Now, one can consult [**103**, Corollary 3.1] and it is easy to understand that relation of [**103**, Theorem 3.1] also holds for the setting, suggested in the Lemma in the sense that in both case $F \in \mathcal{L} \cap \mathcal{D}$. For proving the above we initially observe that we have the following basic ordering:

$$\mathbb{P}\Big[\bigvee_{i=1}^{k} S_k(u) > x\Big] \leq \mathbb{P}\big[S_k(u) > x\big],$$

$$\mathbb{P}\big[S_k(u) > x\big] \leq \mathbb{P}\Bigg[\bigvee_{i=1}^{k} u_i X_i > x\Bigg],$$

and

$$\mathbb{P}\Bigg[\bigvee_{i=1}^{k} u_i X_i > x\Bigg] \leq \sum_{i=1}^{k} \mathbb{P}\big[u_i X_i > x\big]$$

is also true due to Bonferroni's inequality where $\mathbb{P}\Bigg[\bigcup_{i=1}^{k} u_i X_i > x\Bigg] \leq \sum_{i=1}^{k} \mathbb{P}\big[u_i X_i > x\big]$ and also because $\mathbb{P}\Bigg[\bigvee_{i=1}^{k} u_i X_i > x\Bigg] \leq \mathbb{P}\Bigg[\bigcup_{i=1}^{k} u_i X_i > x\Bigg]$. Now for achieving that (3.4) is true as $x \to \infty$ we have to prove that

$$\mathbb{P}\big[S_k > x\big] \geq \sum_{i=1}^{k} \mathbb{P}\big[u_i X_i > x\big], \tag{3.26}$$

and

$$\mathbb{P}\Bigg[\sum_{i=1}^{k} u_i X_i > x\Bigg] \leq \sum_{i=1}^{k} \mathbb{P}\big[u_i X_i > x\big], \tag{3.27}$$

as $x \to \infty$ are true. We already have that $u_1, u_2, ..., u_n$ are bounded from above so without loss of generality we assume that they are bounded from above by 1. Under that setting we can verify that

$$\sum_{1 \leq j \neq k \leq m} \mathbb{P}(u_j X_j > x, u_m X_m > x) = o(1) \sum_{i=1}^{k} \mathbb{P}(u_i X_i > x). \tag{3.28}$$

Also, (3.28) stems from the fact that from the left side we have independence suggests that we are dealing with a product of r.v., that are bounded. To that end the left side becomes negligible compare to the quantity on the right hand side, as $x \to \infty$.

For proving (3.26) we recall initially that $X_i \in L^1_+$. Now, we only have to consider that $S_k(u) \geq \bigvee_{i=1}^{k} u_i X_i$, the fact that (3.28) holds and thus we get that also (3.26) holds.

Now, let us prove that (3.27) holds. For that, we first consider the case where $u_1, u_2, ..., u_n$ are positive. Initially we consider an arbitrary subset $I \subset \{1, ..., k\}$, we have

$I^c = \{1, ..., k\} \setminus I$ and also $\Omega_I^\epsilon(u) = \{\omega : u_i > \epsilon$ for $i \in I$ and $u_j \le \epsilon$ for $j \in I^c\}$ for $0 < \epsilon < 1$.

To that end we consider that $\sum_{i=1}^k u_i X_i = S_k(u)$ and obtain the following inequality

$$\mathbb{P}\bigg(\sum_{i=1}^k u_i X_i > x\bigg) \le \sum_{I \subset \{1,...,k\}} \mathbb{P}\bigg(\sum_{i \in I} u_i X_i + \sum_{j \in I^c} \epsilon X_j > x, \Omega_I^\epsilon(u)\bigg). \tag{3.29}$$

By considering [**104**, Lemma 5.1] we have that the right side of (3.29) equals

$$\sum_{i \in I} \mathbb{P}(u_i X_i > x, \Omega_I^\epsilon(u)) + \sum_{j \in I^c} \mathbb{P}(\epsilon X_j > x) \mathbb{P}(\Omega_I^\epsilon(u))$$

$$= \sum_{i \in I} \mathbb{P}(u_i X_i > x, \Omega_I^\epsilon(u)) + \sum_{j \in I^c} \mathbb{P}(\epsilon X_j > x, u_j > \epsilon) \frac{\mathbb{P}(\Omega_I^\epsilon(u))}{\mathbb{P}(u_j > \epsilon)}.$$

By plugging the above to(3.29) and interchanging summation order we get

$$\mathbb{P}\bigg(\sum_{i=1}^k u_i X_i > x\bigg)$$

$$\le \sum_{i=1}^k \sum_{I : i \in I \subset \{1,...,k\}} \mathbb{P}(u_i X_i > x, \Omega_I^\epsilon(u)) + \sum_{j=1}^k \sum_{I : j \notin I \subset \{1,...,k\}} \mathbb{P}(u_j X_j > x) \frac{\mathbb{P}(\Omega_I^\epsilon(u))}{\mathbb{P}(u_j > \epsilon)},$$

$$= \sum_{i=1}^k \mathbb{P}(u_i X_i > x, u_i > \epsilon) + \sum_{j=1}^k \mathbb{P}(u_j X_j > x) \frac{\mathbb{P}(u_j \le \epsilon)}{\mathbb{P}(u_j > \epsilon)},$$

$$\le \bigg(1 + \max_{1 \le j \le k} \frac{\mathbb{P}(u_j \le \epsilon)}{\mathbb{P}(u_j > \epsilon)}\bigg) + \sum_{j=1}^k \mathbb{P}(u_j X_j > x).$$

Recall that we have that $u_j$ is positive, and as $\epsilon \to 0$ we obtain (3.27). Now when $u_1, ..., u_k$ may take value 0 with positive probability, we recall $I$,$I^c$ and also $\Omega_I^0(u) = \{\omega : u_i > 0$ for $i \in I$ and $u_j = 0$ for $j \in I^c\}$, and consequently we obtain

$$\mathbb{P}\bigg(\sum_{i=1}^k u_i X_i > x\bigg) = \sum_{\phi \ne I \subset \{1,...,k\}} \mathbb{P}(\sum_{i \in I} u_i X_i > x, \Omega_I^0(u))$$

$$\mathbb{P}\bigg(\sum_{i=1}^k u_i X_i > x\bigg) \le \sum_{\phi \ne I \subset \{1,...,k\}} \sum_{i \in I} \mathbb{P}(u_i X_i > x, \Omega_I^0(u))$$

$$\mathbb{P}\bigg(\sum_{i=1}^{k} u_i X_i > x\bigg) \le \sum_{i\in I}^{k} \mathbb{P}(u_i X_i > x),$$

(3.27) is true and the proof is complete. □

We also need to state the following Definition

Definition 3.11. *A distribution function $F$ on $\mathbb{R}$ belongs to class $\mathcal{A}$ when it belongs to $\mathcal{S}$ and its right tail satisfies* $\limsup\limits_{x\to\infty} \frac{\overline{F}(xz)}{\overline{F}(x)} < 1$ *for some* $z > 1$.

Now, we can state the following Theorem.

Theorem 3.12. *Let consider that* $\mathbb{E}|X_i| = \infty$, $u_1, u_2, ..., u_k$ *are bounded from above,* $i = 1, 2, ..., k$, [**60**, Corollary 1 (a)] *holds,* $\overline{F}(z) \in \mathcal{D} \cap \mathcal{A}$ *and* $m(z) \in \mathcal{D} \cap \mathcal{A}$, *where* $0 < \alpha_F \le 1$ *and* $0 < \alpha_m \le 1$. *If* $\alpha_F < \alpha_m$ *and* $\beta_F < \beta_m$, *then there exist constants $C$ and $D$ such that*

$$\Psi(x) \sim \mathbb{P}\bigg[\bigvee_{i=1}^{n} S_k(u) > x\bigg] \sim \mathbb{P}\big[S_k(u) > x\big] \sim \mathbb{P}\bigg[\bigvee_{i=1}^{n} u_i X_i > x\bigg], \ \textit{as } x \to \infty,$$

*where $\Psi(x)$ is bounded by the quantities determined in Theorem 3.9.*

Proof. Initially, we consider that $\mathcal{S} \subset \mathcal{L}$. With that in mind, straightforwardly from Lemma 3.10 we have that If $u_1, u_2, ..., u_k$ are bounded from above, $i = 1, 2, ..., k$ and $F \in \mathcal{S} \cap \mathcal{D}$, then

$$\mathbb{P}\bigg[\bigvee_{i=1}^{n} S_k(u) > x\bigg] \sim \mathbb{P}\big[S_k(u) > x\big] \sim \mathbb{P}\bigg[\bigvee_{i=1}^{n} u_i X_i > x\bigg] \sim \sum_{i=1}^{k} \mathbb{P}\big[u_i X_i > x\big], \ \text{as } x \to \infty.$$

We also have from Definition 3.3 that $\mathcal{A}$ belongs to $\mathcal{S}$, from which we get that $\mathcal{A} \subset \mathcal{S} \subset \mathcal{L}$ which implies that If $u_1, u_2, ..., u_k$ are bounded from above, $i = 1, 2, ..., k$ and $F \in \mathcal{A} \cap \mathcal{D}$, then

$$\mathbb{P}\bigg[\bigvee_{i=1}^{n} S_k(u) > x\bigg] \sim \mathbb{P}\big[S_k(u) > x\big] \sim \mathbb{P}\bigg[\bigvee_{i=1}^{n} u_i X_i > x\bigg] \sim \sum_{i=1}^{k} \mathbb{P}\big[u_i X_i > x\big], \ \text{as } x \to \infty.$$

Finally we have from Definition 3.3 and Definition 3.4 that $\mathcal{A} \subset \mathcal{E}$, which allows us to utilize Theorem 3.9 in order to conclude that

$$\frac{\Gamma(\beta_l - \alpha_l)}{\Gamma(\beta_l)\Gamma(2-\alpha_l)} \times \frac{Dx\overline{F}(x)}{m(x)} \lesssim \Psi(x) \sim \mathbb{P}\Big[\bigvee_{i=1}^{k} S_k(u) > x\Big] \sim \mathbb{P}\big[S_k(u) > x\big] \sim \mathbb{P}\Big[\bigvee_{i=1}^{k} u_i X_i > x\Big] \lesssim \frac{\Gamma(\beta_u - \alpha_u)}{\Gamma(\beta_u)\Gamma(2-\alpha_u)} \times \frac{Cx\overline{F}(x)}{m(x)}$$

holds as $x \to \infty$ and the proof is complete. □

Remark 3.13. *Aforementioned Theorem has a profound importance since we are interested in calculating the Insolvency probability as $x \to \infty$.*

Remark 3.14. *In the model we suggest Insolvency Probability can be calculated without any restriction in terms of the Risk Measure that is responsible for regulating/mitigating risk.*

Remark 3.15. *Our research could be utilized in many cases, like for instance when we deal with Pareto type distributions and the mean is undefined.*

Remark 3.16. *Also, findings of Theorem 3.9 complies with the principle of a single big jump as presented in* [**49**].

## 4. Conclusion

The main interest of this chapter is the estimation of Insolvency Probability, under the fact that Systemic Risk is present. To that end, we proceed be setting risk contributions in an economic environment that is regulated by a Distortion Risk Measure. By considering that many important classes of Risk Measures, like the class of Spectral Risk Measures under some conditions, can be expressed in terms of Distortion Risk Measures we can also conclude that some important generalizations can be extracted out of our initial findings. Moreover, we mostly contribute be concluding the so called Insolvency Probability (Ruin Probability in Insurance Theory) due to the fact that systemic Risk is present and the variation variable is less than one.

Apparently, it is of great convenience the fact that in the model we suggest Insolvency Probability can be calculated without any restriction in terms of the Risk Measure that is responsible for regulating/mitigating risk. Finally we point out that our findings, which are consistent with the principle of a single big jump could be utilized in many cases, like for instance when we deal with Pareto type distributions and the mean is undefined.

CHAPTER 5

# A note on Spectral Risk Measures when Systemic Risk is present

## 1. Preliminaries

Spectral risk measures are considered of high interest in academia. It is widely accepted that the so called risk spectrum can depict the level of risk aversion of the investor and notable efforts are taken place on the portrayal of that factor. For instance [**56**] and [**81**] mention some very important classes of risk measures, like the exponential or the power spectral risk measures. There is also the work of [**110**] and [**111**] which suggest the so called Wang transform and although it emerges from insurance theory, yet it falls in the category of spectral risk measures. Apart form the academic interest there is also the practical aspect which is as important. As [**26**] suggests Spectral risk measures are a hopeful generalization of Expected shortfall, implying that it may be utilized in the near future.

In addition, there is criticism on the theoretical ground of Spectral Risk Measures, see for instance [**21**] work on the properties of completeness (necessary condition), exhaustion and finally, adaptability (necessary and sufficient condition). Apparently, exponential and power spectral risk measure does not exhibit all of those properties. Also [**58**] denoted the robust backtesting problems of Spectral Risk Measures as they are not elicitable. On the other hand, those classes are very easy to be constructed and the theoretical background leaves room for limitless options, as long as the risk preferences of an investor can be exposed somehow.

We on the other hand are motivated by the fact that utility function of the investor, a concept related to the asset pricing theory (see [**40**], [**85**]) and portfolio diversification (see [**90**]) is the single factor that determines the risk spectrum of the measure. We argue with this practice and we propose that also systemic risk should be included in the depiction. Recall that systemic risk refers to the instability of a financial system that can lead to its entire collapse [**75**]. In addition, there is academic interest for the implications of systematic risk. For instance [**14**] state that the main purpose concerning systemic risk is to evaluate the financial distress of an economy as a consequence of the failure of one of its components. They also point out the importance of the Extreme Value Theory (EVT) in the analysis of systemic risk. Moreover [**87**] discuss that the Financial institutions can generate systemic risk. Also [**4**] addressed the same topic. In addition [**32**] introduce SRISK to measure the capital shortfall of a firm conditional on a severe market decline. An interesting comparison would be between our approach and the one of [**39**]. He utilizes the income distribution in order to capture how the stochastic process of the income share is taking place among certain groups. We on the other hand use the returns' distribution of a certain investment in order to highlight how cautious an investor needs to be due to the risk of the economic environment.

Having the practitioners in mind, we attempt to give a different perspective on how risk spectrum should be comprehended. Rephrasing, instead of attempting to picture an investor or a managers' aversion profile, which by definition is rather subjective, we suggest the degree of aversion she/he should have depending on the markets' underlying risk. Moreover, we attempt to suggest, not only a theoretically robust, but also an applicable method.

Furthermore, we are motivated by the fact that situations like the global crisis emerging from late 2007 as a subprime mortgage crisis in U.S. and LIBOR scandal, triggered the need to reassess the tools a professional or a regulator should employ (see for instance [**83**] and [**84**] work on liquidity risk, [**89**] work on worst case scenario and [**14**] work on systemic risk). This reassessment is taking place up to now and a milestone example is the decision of [**27**], as it suggests the Expected Shortfall instead of the Value at Risk as a standard risk measure.

Unfortunately, there is criticism on the regulatory logic of using a unique risk assessment tool in order to have a homogeneous benchmark, see for instance [**58**], as it lucks the notion that every market exhibits certain characteristics that influence its underlying investment risk that cannot be diversified away. As [**1**] suggests different portfolios' risk can be better detected by different risk measures and also, a unique risk measure can be a risk generator itself. A notable attempt to personalize the risk measure is the specific risk, a component added to Value at Risk, due to issuer's specific price movement [**91**] . Our point of view is in compliance with this criticism. Moreover, by this chapter we denote a way to categorize each market according to the systemic risk it bears. Also, we propose a way to quantify systemic risk via macroeconomic factors as indicators. Consequently, each market will utilize a risk measure that corresponds to its systemic risk.

We mention that this work is in tandem with our previous work in risk (see for instance [**79**]). The structure of chapter is the following: next section includes an example that gives further insight to the motivation. The next section is devoted to our contribution, the next section provides an application that can be empirically validated and final section provides concluding remarks.

## 2. Motivation

We give a theoretical example in order to polish our motivation the best possible way. Consider two markets $M_1, M_2$, and three elements that are responsible for the systemic risk $\{r_1, r_2, r_3\}$. Also, consider that those elements are associated with fundamental macroeconomic factors, for instance $r_1$ with the Real Gross Domestic Product YoY change in %, $r_2$ with the unemployment rate and $r_3$ with the inflation rate (according to [**92**], GDP measures everyone's income in an economy, inflation rate is associated with the fraction of labour capacity that is out of work and inflation measures how fast prices are rising).

We consider that there is a threshold corresponding to each element that the economies acknowledge. Exceeding a threshold for an economy signifies an increase in systemic risk. For the shake of argument we set that the thresholds are $r_1 = -10\%$, $r_2 = 20\%$ and $r_3 = 5\%$. Moreover, the macroeconomic factors for $M_1, M_2$ and for a three year period are the following:

| | | $M_1$ | |
|---|---|---|---|
| Year | $r_1$ | $r_2$ | $r_3$ |
| 2015 | $+5\%$ | $1\%$ | $3\%$ |
| 2016 | $+1\%$ | $5\%$ | $6\%$ |
| 2017 | $-3\%$ | $4\%$ | $4\%$. |

$M_2$

| Year | $r_1$ | $r_2$ | $r_3$ |
|---|---|---|---|
| 2015 | $+1\%$ | $15\%$ | $6\%$ |
| 2016 | $-11\%$ | $22\%$ | $4\%$ |
| 2017 | $-12\%$ | $21\%$ | $3\%$. |

Bear in mind that the elements we have chosen are not at all restrictive. On the contrary one can chose of any element that is associated with the robustness of a market. For instance the income (see [**38**, p.54]) for details) which is also of academic interest (see [**39**]) can prove quite useful. We follow the analytical approach of [**112**] where we can present for each year the three elements as an ordered triplet of real numbers, in a three dimensional cartesian coordinate system. So, $[r_1, r_2, r_3]$ is a vector for each year and for each market with $r_1, r_2, r_3$ as the components of the vector. Under that framework we can represent the macroeconomic factors for each market $M_1$ and $M_2$ for years 2015 up to 2017 in $3X3$ matrices. Specifically:

$$M_1 = \begin{bmatrix} 5 & 1 & 3 \\ 1 & 5 & 6 \\ -3 & 4 & 4 \end{bmatrix}$$

$$M_2 = \begin{bmatrix} 1 & 15 & 6 \\ -11 & 22 & 4 \\ -12 & 21 & 3 \end{bmatrix}$$

where the rows correspond to each year and the columns correspond to each macroeconomic factor. Obviously there are indications that $M_2$ bears more systemic risk for all three years compare to $M_1$ ($M_2$ has one element that exceeds the threshold in 2015, two in 2016 and two in 2017 while $M_1$ has one element that exceeds a benchmark only in 2016). A mathematical approach would be to calculate the magnitude of each years vector (see [**112**, Definition 2 p.5]), then we get:

Vectors' Magnitudes

| Year | $\mid M_1 \mid$ | $\mid M_2 \mid$ |
|---|---|---|
| 2015 | 5.92 | 29.61 |
| 2016 | 7.87 | 24.92 |
| 2017 | 6.40 | 24.37 |

The magnitudes of the vectors representing $M_2$ are in all occasions much higher than $M_1$, which denotes a basic ordering in terms of systemic risk (rephrasing $M_2$ is more exposed to systemic risk). Since we are having in our example square matrices we may also utilize the determinants, where $\det M_1 = 15$ and $\det M_2 = 135$. Consider a geometrical approach for the vectors $r_1, r_2, r_3$, assume that they are linearly independent and consider the (see [**93**, theorem p.216]). Then the absolute value of determinants are the volume of the parallelepiped determined by vectors $r_1, r_2, r_3$. Clearly the value of $M_2$ parallelepiped is much greater than the value of $M_1$, which also denotes in terms of a basic ordering that $M_2$ is exposed to more systemic risk.

Now consider a risk averse investor who is investing to both $M_1$ and $M_2$. Moreover, this investor utilizes a spectral risk measure to dictate his investing attitude. Consequently the risk spectrum of that measure portrays his aversion towards risk. Under that framework, we propose that apart from the risk aversion of the investor the risk spectrum should also depict the systemic risk that the market is exposed. It is obvious that the investor's attitude towards risk needs to be more conservative in case he chooses to invest in $M_2$ (compare to $M_1$). For instance if the investor is planning to invest in 2018 then he should take into account the macroeconomic factors of 2017, which suggest that $M_2$ bears more systemic risk compare to $M_1$ (two macroeconomic factors exceed the benchmark in $M_2$ and none in $M_1$).

With the use of this setting, we observe that the depiction of systemic risk is of major importance. In addition, this depiction makes the risk spectrum aware of the systemic risk underlined in a market. Equally important is the fact that an investor or a regulator is able to quantify the systemic risk, as for each macroeconomic factor that exceeds a threshold, the more risk averse the investor will be (more on contribution section).With this simple example we provide an insight of our proposition but in order to mathematically formalize our work we need to demonstrate the theoretical foundation we utilize.

## 3. Contribution

Considering the example of motivation section, it is apparent that whether a macroeconomic factor exceeds the threshold that signifies more systemic risk or not can be perceived as random experiment. We take that $N_1, N_2, ..., N_k$ are $k$ economic factors that are responsible for depicting the robustness of an economy (how much systemic risk it bears). In addition $k \in \mathbb{N}$ and $k$ is of course deterministic. Moreover we consider that each trial corresponds to each year's figures of those economic factors. Also $a_1, a_2, ..., a_k$ are $k$ thresholds as described in motivation section, where each $a_i$ for every $i \in \{1, 2, 3, ...., k\}$ corresponds to the $N_i$ economic factor. For regulation purposes every $a_i$ could be a threshold that concern macroeconomic indexes while for investing purposes it could be a limit of the capitalization of an index.

Under that perspective we set that for every $i$ random experiment there is an $X_i$ random variable that follow the discrete binomial distribution, or

$$X_i \sim B(n_i, p_i).$$

Each experiment for every year is a Bernoulli trial where $n_i$ is the number of trials and $p_i$ is the probability for success (the probability that the $a_i$ threshold will be exceeded). Also, the probability for failure $q_i = 1 - p_i$ is the complement of the probability for success. Also, its experiment for every year is statistically independent. A similar assumption is met in [**39**], where the model's assumption suggest that the distribution will tend towards a unique equilibrium distribution dependent upon the stochastic matrix but not on the initial distribution. Recall that a stochastic matrix used to describe the transitions of a Markov chain which describes a sequence of events according only to their current state (the events are memoryless).

The distribution function of each random variable $X_i$ is:

$$P_{X_i}(x_i) = \binom{n_i}{x_i} p_i^{x_i}(1-q_i)^{n_i - x_i},$$

where $x_i$ are the number of successes for $n_i$ trials ($n$ years). The sample space $\Omega = \{(N_i > a_i | \prod_{i=1}^{k} N_i \in \mathbb{R}\}$, and also $\mathbf{P}(\Omega) = 1$, since each factor can either exceed or fail to exceed the threshold that corresponds to it. In addition $X_i$'s range is $\{0, 1, 2, ..., n_i\}$. Following the above setting of our experiment we also set its probability space $(\Omega, \mathcal{F}, \mathbb{P})$ where $\mathcal{F}$ is the set of events which includes the subset $\mathcal{X}$ of all $x_i$ succeeded experiments. Also, the indicator function of a subset $\mathcal{X}$ of a set $\mathcal{F}$ is a function $1_{\mathcal{X}} : \mathcal{F} \to \mathbb{R}_+$ defined as

$$1_{\mathcal{X}(x_i)} =: \begin{cases} > 0 \text{ if } x_i \in \mathcal{X} \\ \\ = 0 \text{ if } x_i \notin \mathcal{X}. \end{cases}$$

Consider the following random variable:

$$Y = 1 + \sum_{i=1}^{k} 1_{\mathcal{X}(x_i)}, \tag{3.1}$$

where $\sum_{i=1}^{k} 1_{\mathcal{X}(x_i)}$ is the sum of aforementioned indicator functions and it is trivial to see that (3.1) is non-negative.

Clearly in each experiment ideally we would get that $N_i \leq a_i$ or, that each $N_i$ should have an upper bound $a_i$, in order to be free of systemic risk. Attempting a further discussion on the thresholds, it is obvious that there are many ways to be determined, yet they have to be acknowledged by all the markets they wish to utilize them. For regulation purposes the limits (macroeconomic factors) that European economies need to reach in order to be inducted in European monetary union can be serve as thresholds. For investing purposes a numeraire can prove quite useful. In any case, this thresholds will present the ideal and free of systemic risk economy. In order to further illustrate this matter we suggest an application at the relevant section. Moreover (3.1) is not determining if all components are equally weighted. While we do that for simplicity, this may not be always the case and it is a topic that can be further addressed.

Consider the Power Spectral risk measure as depicted in (3.5). Now assume that for the r.v. of $X$ in (3.1) it holds that $\mathbb{E}[X^+] < \infty$, $\mathbb{E}[X^-] < \infty$, $\sup_{p[0,1]} F_0(p) < \infty$ and the observations of the returns of the investment that is represented in (3.1) $\to \infty$. then according to [1, Theorem 5.4] a Spectral Risk Spectrum with (3.5) converges to (3.4) with probability 1.

In addition when the $F_0(p)$ is a known function, then according to [56] the equivalent properties for a risk spectrum to be admissible would be:

(1) $F_0(p) \geq 0$,
(2) $F_0^{'}(p) \leq 0$,
(3) $\int_0^1 F_0(p)dp = 1$.

To be precise, [56] states the increasing property but this occurs when the distributions in which loss outcomes are given positive values. On the contrary [1] is implying negative values. Now consider the following function:

$$\psi(Y,p,d) = Y\left[dp^{d-1}\right] - (Y-1), \text{ where } d \in \left[\frac{Y-1}{Y}, 1\right] \text{ and } p \in (0,1]. \tag{3.2}$$

PROPOSITION 3.1. (3.2) *is an admissible risk spectrum.*

PROOF. We consider that for $\psi(Y,p,d) \geq 0$ we should get that $Y\left[dp^{d-1}\right] - (Y-1) \geq 0$ which leads to $dp^{d-1} \geq \frac{Y-1}{Y}$. Moreover we consider that

$$\begin{cases} 0 < p \leq 1 \\ d-1 \leq 0 \end{cases}$$

which leads to $p^{d-1} \geq 1 \Rightarrow dp^{d-1} \geq d, \forall p \in (0,1]$. From the aforementioned we get that $dp^{d-1} \geq d \geq \frac{Y-1}{Y}$ or, if we need $\psi(Y,p,d)$ to be positive then we should have that $d \in \left[\frac{Y-1}{Y}, 1\right]$.

Now let us prove that $\psi^{'}(Y,p,d)<0$ (differentiating with respect to $p$). For that we get $\Big[Ydp^{d-1}-(Y-1)\Big]^{'}=Yd(d-1)p^{d-2}$ which is negative because $Y$,$d$ and $p^{d-2}$ are positive while $d-1$ is negative.

Finally for normalization we should get that $\int_0^1\psi(Y,p,d)dp=1$ which is true because $\int_0^1 Ydp^{d-1}-(Y-1)dp=\int_0^1 Ydp^{d-1}dp-\int_0^1 Y-1dp=Y\Big[p^d\Big]_0^1-(Y-1)\Big[p\Big]_0^1=Y-Y+1=1$. □

We refer the reader to the Appendix, where we present a specific example and its proof of the risk spectrum we propose.

Now we assume that there are always two markets $M_a$ and $M_b$ where market $M_a$ includes more systemic risk, in terms that it has more $x_i$ number of successes. Let consider that $Y_a$ and $Y_b$ are the (3.1) functions of $M_a$ and $M_b$ respectively.

Proposition 3.2. *For every two markets $M_a$ and $M_b$ where market $M_a$ includes more systemic risk compare to $M_b$, $Y_a>Y_b$ is true.*

Proof. We have to prove that $Y_a>Y_b\Rightarrow\Big[1+\sum_{i=1}^k 1_{\mathcal{X}(x_i)}\Big]_a>\Big[1+\sum_{i=1}^k 1_{\mathcal{X}(x_i)}\Big]_b\Rightarrow\Big[\sum_{i=1}^k 1_{\mathcal{X}(x_i)}\Big]_a>\Big[\sum_{i=1}^k 1_{\mathcal{X}(x_i)}\Big]_b$. We have from the setting that $(x_i)_a>(x_i)_b$ which leads to $\Big[\sum_{i=1}^k 1_{\mathcal{X}(x_i)}\Big]_a>\Big[\sum_{i=1}^k 1_{\mathcal{X}(x_i)}\Big]_b$ and thus $Y_a>Y_b$ is true. □

Also let consider that $\psi_a(Y_a,p,d)$ and $\psi_b(Y_b,p,d)$ are the (3.2) functions of $M_a$ and $M_b$ respectively. In addition we set that $\psi(Y,p,d)-\phi(p,d)=\epsilon$. Moreover one can consider the $p$ as an increasing function where $p_{(x)}=x$ and $x\in[0,1]$. Let us consider the real numbers $\gamma,\delta\in[0,1]$ such that $\gamma>\delta$, then $p_\gamma>p_\delta$. Now we are ready to give the following definition:

DEFINITION 3.3. *A Spectral Risk Measure is Systemic risk aware when it satisfies the following conditions:*

(1) *For every* $Y_a > Y_b \Rightarrow \psi_a(p,d) > \psi_b(p,d)$.
(2) *For* $Y > 1$, $\psi(p_\gamma, d) - F_0(p_\gamma, d) = \epsilon_\gamma$ *and* $\psi(p_\delta, d) - F_0(p_\delta, d) = \epsilon_\delta$, $\epsilon_\gamma < \epsilon_\delta$ *is true.*

REMARK 3.4. *The first conditions assures that between two markets that both utilize the same Risk Measure and both exhibit SR, the market that has more SR (according to the quantification we propose) will have a greater risk spectrum and will lead the utilizer to be more conservative.*

*The second condition assures that the magnitude of the difference between a risk spectrum that is not endowed with r.v.* $Y$ *and the one that is endowed with it, is greater when we are considering a worst outcome in the returns' distribution function.*

We present a numerical example in the Appendix for the possible understanding of the above Remark.

THEOREM 3.5. *For* (3.2) *the conditions of Definition 3.3 hold, and* (3.2) *is a systemic risk aware and admissible risk spectrum.*

PROOF. We have from proof of Proposition 3.1 that $\psi(p)$ is an admissible risk spectrum. Consequently we take as proven that both $\psi(Y,p,d)$ and $\phi(p,d)$ are positive, convex and decreasing functions.

For first condition we should get that $\psi_a(Y,p,d) > \psi_b(Y,p,d) \Rightarrow Y_a\phi(p,d) - Y_a + 1 > Y_b\phi(p,d) - Y_b + 1 \Rightarrow Y_a\phi(p,d) - Y_a > Y_b\phi(p,d) - Y_b \Rightarrow Y_a\big(\phi(p,d) - 1\big) > Y_b\big(\phi(p,d) - 1\big) \Rightarrow Y_a > Y_b$ which is true and thus second condition is met for (3.2).

For second condition, one should get that $\epsilon_\gamma < \epsilon_\delta$. So $\epsilon_\gamma < \epsilon_\delta \Rightarrow \psi(Y, p_\gamma, d) - \phi(p_\gamma, d) < \psi(Y, p_\delta, d) - \phi(p_\delta, d) \Rightarrow \psi(Y, p_\gamma, d) + \phi(p_\delta, d) < \psi(Y, p_\delta, d) + \phi(p_\gamma, d) \Rightarrow Y\phi(p_\gamma, d) - Y + 1 + \phi(p_\delta, d) < Y\phi(p_\delta, d) - Y + 1 + \phi(p_\gamma, d) \Rightarrow Y\phi(p_\gamma, d) + \phi(p_\delta, d) < Y\phi(p_\delta, d) + \phi(p_\gamma, d) \Rightarrow Y\phi(p_\gamma, d) - \phi(p_\gamma, d) < Y\phi(p_\delta), d - \phi(p_\delta, d) \Rightarrow \phi(p_\gamma, d)\big(Y - 1\big) < \phi(p_\delta, d)\big(Y - 1\big) \Rightarrow \phi(p_\gamma, d) < \phi(p_\delta, d)$ which is true as $\phi(p)$ is decreasing and thus third condition is met. □

COROLLARY 3.6. *Any admissible spectral risk spectrum that also exhibits the conditions of Definition 3.3 is an admissible and systemic risk aware Spectral Risk Spectrum.*

PROOF. Assume that there is an admissible risk spectrum other than (3.2) and a variable other than (3.1). In addition assume that this variable preserves for the risk spectrum the properties of Definition 3.3. With those assumptions and setting we observe that the admissible risk spectrum is systematic risk aware. □

So far we have proposed three desirable conditions that the risk spectrum needs to satisfy once there is Systemic Risk. We also have found a way to quantify the systemic risk and embody it to the risk spectrum with the utilization of a variable. In tandem we preserved the properties the risk spectrum needed in order to be admissible. In terms of practicality preliminaries, we use basics of set theory (see for instance [**71**]) to ensure that the set of coherent and systemic risk aware measures is not the null set. Consider that $\mathcal{C}$ stands as the set of all coherent risk measures, then:

LEMMA 3.7. *A spectral risk measure $M_{F_0}(X)$ that has an admissible risk spectrum $F_0$, is a subset of coherent risk measures, or $M_{F_0}(X) \subset \mathcal{C}$.*

PROOF. Consider an $M_{F_0}(X)$ that is a spectral risk measure and has an admissible risk spectrum $F_0(p)$. We know that $M_{F_0}(X) \neq \emptyset$, as we already mentioned Power Spectral Risk

Measures, Exponential Spectral Risk Measures and Wang transform which are spectral risk measures that have an admissible risk spectrum. Consequently, since $M_{F_0}(X)$ is a coherent risk measure, $M_{F_0}(X) \subseteq \mathcal{C}$.

In addition, we are also aware of $ES_a$, which is a risk measure that is coherent, yet its risk spectrum is not admissible. This suggests that $M_{F_0}(X) \neq \mathcal{C}$ and so $M_{F_0}(X) \subset \mathcal{C}$. □

Lemma 3.8. *A spectral risk measure $M_{F_0}(X)$ that has an admissible risk spectrum $F_0(p)$ that satisfies condition of Definition 3.3, is a coherent risk measure and so $M_{F_0}(X) \subset \mathcal{C}$. In addition $M_{F_0}(X) \neq \emptyset$.*

Proof. Lemma 3.7 suggests that a $M_{F_0}(X)$ is coherent and so $M_{F_0}(X) \subseteq \mathcal{C}$. Also, $ES_a \subset \mathcal{C}$ and does not have an admissible risk spectrum. Consequently $M_{F_0}(X) \neq \mathcal{C}$ and thus $M_{F_0}(X) \subset \mathcal{C}$.

For proving $M_{F_0}(X) \neq \emptyset$ we need to suggest at least one coherent spectral risk measure that has a risk spectrum that satisfies conditions of Definition 3.3. For that we consider (3.2), where the property of admissibility hold, the risk measure is coherent and $M_{F_0}(X) \neq \emptyset$. □

## 4. A Suggested Application

As aforementioned, the limits (macroeconomic factors) that European economies need to reach in order to be inducted in European monetary union can serve as thresholds. To that end if a regulator needs to quantify the systemic risk that exists in a country, she/he can utilize the five convergence criteria ([**42**], [**72**]) that needs to be met in order to join the eurozone:

(1) First criterion is interested in the stability of an economy. An economy meets that criterion if its inflation is not more than 1.5 percentage points above the rate of the three best performing member states.

(2) Second critrion examines the robustness of the public finance of the economy. The criterion is met if the government deficit is not more than 3% of the GDP.

(3) Third criterion is about the sustainability of public finances. If an economy needs to meet the criteria, the debt of the government should not exceed the 60% of the GDP.

(4) Fourth criterion is interested in the durability of convergence. To that end the criterion is that the long term interest rate is not more than 2% above the rate of the three best performing Member States.

(5) Fifth and final criterion is the stability of the exchange rate. The criterion for that is that the inductee economy is participating in ERM II for at least 2 years without severe tensions.

Of course those criteria can serve directly as $N$ factors for an investor who wishes to invest in economies either in the eurozone or planning to adopt the euro. Its elementary to see that $k = 5$ and there are five $a$ correspond to each criterion. Now one can directly proceed by utilizing (3.1) of the economy she/he is interested to invest. Afterwards she/he can calculate the systemic risk aware risk spectrum that is proposed from (3.2) and utilize it accordingly for her/his investing strategy. The apparent benefit is a Spectral Risk Measures that is dictated not only by the risk aversion of the investor but also by the quantified systemic risk underlined in its economy. Moreover if someone chooses to invest in a country outside the European union then she/he simply has to exclude the fifth criterion and set $k = 4$.

## 5. Conclusion

By this chapter we tackle the issue that risk spectrum of a Spectral risk measure is portraying solely the preferences of an investor/regulator on her/his risk tolerance. We argue by suggesting that also the systemic risk of market should be embodied in the risk spectrum. In line with the above suggestion, we attempt to solve that issue by suggesting two conditions that a risk spectrum should satisfy. We gave an example where we demonstrated how systemic risk can be quantified from one market to another. In the same example fundamental macroeconomic indexes prove quite useful as they can expose in a sense "how much" systemic risk a market bears. In addition, we proposed a risk spectrum class that satisfy those conditions. Summing up, we mention that there are limitations concerning the Spectral Risk Measures we propose. For instance we need to reconsider if those factors should be equally weighted. We need to rethink that depending on the extend of the poor performance of a factor, maybe it should be weighted more than the others. Also we consider that one could use the determinants that we mentioned in the motivation section for calculating the r.v. $Y$ of each market. That way the random experiment could be avoided but some other conditions should be utilized. Although it is beyond the scope of this work to address such matters nevertheless we believe that there should be a study with respect to those topics in the near future. We also believe that an empirical validation of the proposed application should take place with investments within economies that are either members of the eurozone or scheduling to join.

CHAPTER 6

# A Note on the Convergence of Euler Contributions

## 1. Preliminaries

This chapter is devoted to the Convergence of Euler contributions that use different Risk Measures. Initially, our discussion is in regard of Euler contributions in a Risk Measure environment. The utilization of Euler contributions for our case, primarily relies to the fact that they decompose portfolio-wide capital into a sum of risk contributions (sub-portfolios of of solitary exposures). Our work is in tandem with that of [**15**]. Also, we contribute in the aforementioned effort by generalizing their findings in regard of the Expected Shortfall. Having all the above in mind we contribute by defining some conditions where the rates of convergence of Value at Risk and Distortion Risk Measure coincide.

The structure of chapter is the following: Section 2 is devoted to the theoretical framework that concerns the Rate of Convergence. Section 3 includes our contribution to the theoretical framework and finally section 4 concludes.

## 2. Theoretical Framework

**2.1. Rate of Convergence.** A practitioner is usually interested in allocating capital non-parametrically. In other words $\rho_{Euler}(X_i|S_k)$ is determined via non-parametric estimators (see for instance [**15**], or [**108**]). In such cases we have that depending on the selected risk measure the rate of convergence of $\rho_{Euler}(X_i|S_k)$ may differ. Nevertheless, under certain condition those rates may converge. Since the same phenomenon may occur also when Distortion Risk Measures are utilized, we are interested in the conditions where we could get that the rate of convergence is the same for $VaR_q$, and Distortion Risk Measures. Rephrasing, we are interested to get when

$$\frac{VaR_q(X_i|S_k)}{D_{q,h}(X_i|S_k)} \to 1, \tag{2.1}$$

as $q \to 1$ and $X_i$ and $S_k$ are unbounded. We omit the $h$ distortion function for simplicity of the notation and thus

$$D_q(X_i|S_k)$$

stands for the asymptotic approximation of $\rho_{Euler}(X_i|S_k)$ when Distortion Risk Measure is used. Also by considering (3.3) where we get the Distortion Risk Measure, the definition of $VaR_q$ and the unique determination of $\rho_{Euler}(X_i|S_k)$ we have the following for $D_q(X_i|S_k)$ (one can also consult [**15**] and [**91**] for subtleties):

$$D_q(X_i|S_k) := \mathbb{E}\big[X_i|S_k \geq \sup\{x \in \mathbb{R} \mid \mathbb{P}(S_k \geq x) > 1-q\}\big]. \tag{2.2}$$

Bear in mind that (2.2) refers to the class of Distortion Risk Measures. Moreover,

$$VaR_q(X_i|S_k)$$

is $\rho_{Euler}(X_i|S_k)$ for the asymptotic approximation when $VaR_q$ is utilized. In tandem with (2.2), we can get the following

$$VaR_q(X_i|S_k) := \mathbb{E}\big[X_i|S_k = \sup\{x \in \mathbb{R} \mid \mathbb{P}(S_k \geq x) > 1-q\}\big]. \tag{2.3}$$

## 3. Contribution

**3.1. Rates of Convergence for Different Risk Measures.** Throughout the whole contribution section we will dealing with non-negative risks. Also, in this subsection we are interested in some conditions where we could get that the rate of convergence of the $X_i$ profit/loss is the same for $VaR_q$ and Distortion Risk Measures environment. Naturally, we need to focus on the worst case measurement where we have that $q \to 1$, for instance where we are measuring risk in an insurance company that is wealthy in the sense that initial capital $\to \infty$. Moreover $X_i$ and $S_k \in L^1_+$ are bounded from below and unbounded from above or $VaR_q(X_i) \to \infty$, $VaR_q(S_k) \to \infty$ as $q \to 1$.

For our analysis we consider that both $VaR_q$ and Distortion Risk Measure (because of their homogeneity) are satisfying the full allocation property. Consequently

$$\sum_{i=1}^{k} VaR_q(X_i|S_k) = VaR_q(S_k)$$

and

$$\sum_{i=1}^{k} D_q(X_i|S_k) = D_q(S_k). \tag{3.1}$$

Also, let us define $q^*$ as the following

$$q^* := inf\{t \in [0,1] : D_t(S_k) \geq VaR_q(S_k)\},$$

$S_k \in L^1$, both $q$ and $q^* \in (0,1)$ and $q^*$ is dependent on both $S_k$ and $q$, while (3.1) holds for all $q$. Moreover, we define $x$ as the following $x := VaR_{q^*}(S_k)$. Under that framework, the following theorem demonstrates some assumptions where (2.1) holds.

Theorem 3.1. *Let $X_i, S_k \in L^1_+$, be bounded from below, unbounded from above, integrable on probability space $(\Omega, \mathcal{F}, \mathbb{P})$, $S_k$ is continuously distributed, absolutely continuous* *with respect to* $\mathbb{P}$ *and assume that* $\mathbb{E}(S_k) \leq VaR_q(S_k)$. *If* $\lim\limits_{q\to 1} \frac{VaR_q(X_i|S_k)}{x} > 0$, *exists and is finite, then*

$$\frac{VaR_q(X_i|S_k)}{D_{q^*}(X_i|S_k)} \to 1 \text{ as } q \to 1, \tag{3.2}$$

*holds.*

REMARK 3.2. *Since the* [**15**, Theorem 5.1 (i)] *is interested in the assumptions under which* $ES_{q^*}(X_i|S_k)$ *and* $VaR_q(X_i|S_k)$ *converge for a larger* $q$, *we generalize those findings (with the use of* [**15**, Proposition 2.1]*) and suggest that with the same assumptions also* $D_{q^*}(X_i|S_k)$ *and* $VaR_q(X_i|S_k)$ *converge for a larger* $q$.

PROOF. Let us define $y(x) := \mathbb{E}\big[S_k|S_k > x\big]$. Now, for the denominator of (3.2) we get from (2.2) that $D_{q^*}(X_i|S_k) = \mathbb{E}\big[X_i|S_k > \sup\{x \in \mathbb{R} \mid \mathbb{P}(S_k \geq x) > 1 - q^*\}\big]$, or $D_{q^*}(X_i|S_k) = \mathbb{E}\big[X_i|S_k > VaR_{q^*}(S_k)\big]$, and finally

$$D_{q^*}(X_i|S_k) = \mathbb{E}\big[X_i|S_k > x\big]. \tag{3.3}$$

For the nominator of (3.2) we get from (2.3) that $VaR_q(X_i|S_k) = \mathbb{E}\big[X_i|S_k = \sup\{x \in \mathbb{R} \mid \mathbb{P}(S_k \geq x) > 1 - q\}\big]$ or $VaR_q(X_i|S_k) = \mathbb{E}\big[X_i|S_k = VaR_q(S_k)\big]$. By considering [**15**, proposition 2.1] and the setting of the theorem it should be clear that $y(x) = D_{q^*}(S_k)$ holds, $y(x) = VaR_q(S_k)$ holds and thus,

$$VaR_q(X_i|S_k) = \mathbb{E}\big[X_i|S_k = y(x)\big] \tag{3.4}$$

is true. Having in mind (3.3) and (3.4), it should be clear that (3.2) is equivalent to the following:

$$\frac{\mathbb{E}\big[X_i|S_k = y(x)\big]}{\mathbb{E}\big[X_i|S_k > x\big]} \to 1 \text{ as } x \to \infty. \tag{3.5}$$

Also, by assumption $\lim\limits_{q\to 1} \frac{VaR_q(X_i|S_k)}{x} > 0$, exists and it is finite and can also be rephrased as $\lim\limits_{x\to\infty} \frac{VaR_q(X_i|S_k)}{x} > 0$. By considering (3.4) the aforementioned assumption implies that $\frac{\mathbb{E}\big[X_i|S_k=y(x)\big]}{x} > 0$. There are certain occasions where this condition is satisfied, like when asymptotic dependence occurs with similar individual tail risks. By recalling [**15**, proposition 2.1] we have that $y(x) = VaR_q(S_k)$. We also recall that $x := VaR_{q^*}(S_k)$ which draw the conclusion that as $x \to \infty$, then $y(x) \to \infty$ and so $\frac{\mathbb{E}\big[X_i|S_k=x\big]}{x} > 0$ exists and it is finite, as $x \to \infty$. Moreover, without loss of generality we further assume that $\frac{\mathbb{E}(X_i|S_k=x)}{x} \to m$ as $x \to \infty$, where $m \in (0, \infty)$. Consequently $\frac{\mathbb{E}\big[X_i|S_k=x\big]}{xm} \to 1$, as $x \to \infty$. Moreover, it is not difficult to verify from the assumptions of the Theorem that $\mathbb{E}\big[X_i|S_k = x\big] > 0$ and $xm > 0$. Now for any $\delta > 0$ exists a sufficiently large $x_0(\delta)$, such that $(1-\delta)m < \frac{\mathbb{E}\big[X_i|S_k=x\big]}{x} < (1+\delta)m$, for all $x > x_0(\delta)$.

Also recall that $X$ is integrable on probability space $(\Omega, \mathcal{F}, \mathbb{P})$, is continuously distributed and absolutely continuous with respect to $\mathbb{P}$. Moreover, we have that $\{S_k > x\} \in \mathcal{F}$. In addition with $x_1(\delta) > x_0(\delta)$, by integrating with respect to $S_k$ on the interval $(x_1(\delta), \infty)$ we have that

$$\frac{\mathbb{E}\big[X_i|S_k > x_1(\delta)\big]}{\mathbb{E}\big[S_k|S_k > x_1(\delta)\big]} = \frac{\int_{x_1(\delta)}^{\infty} \mathbb{E}[X_i|S_k = x]\mathbb{P}(S_k \in dx)}{\int_{x_1(\delta)}^{\infty} x\mathbb{P}(S_k \in dx)} \in \big((1-\delta)m, (1+\delta)m\big). \tag{3.6}$$

Observing the denominators of (3.6), it should be clear that they stem from the fact that by integrating $\mathbb{E}\big[S_k|S_k > x_1(\delta)\big]$ with respect to $X$ on the interval $(x_1(\delta), \infty)$ we have that $\int_{x_1(\delta)}^{\infty} \mathbb{E}\big[S_k|S_k = x\big]\mathbb{P}(S_k \in dx) = \int_{x_1(\delta)}^{\infty} x\mathbb{P}(S_k \in dx)$.

Also, one can consult [**29**, property ii, p.445] and its relevant Definition in order to assure the existence of the integrals depicted in (3.6).

Since we have that $\delta$ is arbitrary, we can get that when $\delta \to 0$ then $x_1(\delta) \to \infty$ and $\frac{\mathbb{E}\left[S_k|S_k>x_1\right]}{\mathbb{E}\left[X_i|S_k>x_1\right]} \to m^{-1}$ as $x \to \infty$. Now we can get from (3.5) that $\frac{VaR_q(X_i|S_k)}{D_{q^*}(X_i|S_k)} \to 1$ as $q \to 1$, since

$$\frac{VaR_q(X_i)|S_k)}{D_{q^*}(X_i|S_k)} \to 1 \text{ as } q \to 1 \text{ is equal to } \frac{\mathbb{E}\left[X_i|X = y(x)\right]}{\mathbb{E}\left[X_i|S_k > x\right]} \to 1 \text{ as } x \to \infty.$$

Moreover, by multiplying in both nominator and denominator $y(x) := \mathbb{E}\left[S_k|S_k > x\right]$ we get

$$\frac{\mathbb{E}\left[X_i|S_k = y(x)\right]}{y(x)} \times \frac{\mathbb{E}\left[S_k|S_k > x\right]}{\mathbb{E}\left[X_i|S_k > x\right]}. \tag{3.7}$$

Since we already concluded that as $x \to \infty$, then $y(x) \to \infty$ we get that $\frac{\mathbb{E}\left[X_i|S_k=y(x)\right]}{y(x)} \to m$ as $y(x) \to \infty$. Finally we have for (3.7) that

$$\frac{\mathbb{E}\left[X_i|S_k = y(x)\right]}{y(x)} \times \frac{\mathbb{E}\left[S_k|S_k > x\right]}{\mathbb{E}\left[X_i|S_k > x\right]} =$$

$$m \times m^{-1} = 1 \text{ as } x \to \infty.$$

The proof is now complete. □

Corollary 3.3. *Let $X_i, S_k \in L^1_+$, be bounded from below, unbounded from above, integrable on probability space $(\Omega, \mathcal{F}, \mathbb{P})$, $S_k$ is continuously distributed, absolutely continuous with respect to $\mathbb{P}$ and assume that $\mathbb{E}(S_k) \leq VaR_q(S_k)$. If $\lim\limits_{x\to\infty} \frac{VaR_q(X_i|S_k)}{x} > 0$, exists and is finite, then*

$$\frac{ES_{q^*}(X_i|S_k)}{D_{q^*}(X_i|S_k)} \to 1 \text{ as } q \to 1, \tag{3.8}$$

*holds.*

PROOF. From the above theorem and with the same setting we proved that

$$\frac{VaR_q(X_i|S_k)}{D_{q^*}(X_i|S_k)} \to 1 \text{ as } q \to 1 \tag{3.9}$$

holds. Also, we have from [**15**, Theorem 5.1 (i)] that with the same setting

$$\frac{VaR_q(X_i|S_k)}{ES_{q^*}(X_i|S_k)} \to 1 \text{ as } q \to 1 \tag{3.10}$$

holds. It should be clear that by combining (3.9) and (3.10), one concludes that (3.8) holds. □

## 4. Conclusion

This chapter is devoted in defining some conditions where the rates of convergence of Euler contributions of a Value at Risk environment and Distortion Risk Measure environment coincide. At first, our discussion is in regard of Euler contributions in a Risk Measure environment. Moreover, our work is in tandem with that of [**15**]. Also, we contribute in the aforementioned effort by generalizing our findings in regard of the Expected Shortfall.

Finally, we consider that there is a heated discussion in terms of practice, for the most suitable risk measure for mitigating risk (see for instance [**26**]). Apparently, our findings suggest that once we employ Euler allocation principle the differences between risk measures appear to be less important under certain conditions.

CHAPTER 7

# Conclusion

This thesis' primarily is concerned on the Risk Measures and how those can be utilized properly from both regulation authorities and investors. In addition we are giving emphasis on the $SR$ that is present in the economic environment and is responsible for crises and their interrelation. Also, our research is particularly interested on the Spectral Risk Measures, an important class of Distortion Risk Measures, due to the desired properties they exhibit. To that end our effort is in the form of four propositions of Risk Meaures' utilization for regulation and investment purposes:

The first proposition is concerned on the unbounded risk premium that might occur if there is combination of asset pricing model which has an unbounded stochastic discount factor, and a coherent and bounded risk measure. An intuitive explanation of the findings, suggests that is possible to follow a risk averse strategy, as $\beta \leq 1$ and at the same time achieve abnormal returns, as $\alpha > 0$. In other words, it appears to be that this framework is suitable for an investor who is loss averse. On top of that, it seems that a loss averse investor is the ideal candidate to earn a risk premium compare to someone who is more tolerate to risk. Summing up, our findings are not favoring the opinion that market efficiency tends to hold.

Next proposition is concerned of the estimation of the insolvency probability. In addition we utilize dependence models that evaluate Systemic Risk (SR), as we contribute by proposing Euler contributions of risk in an environment that is regulated by a risk measure.

Moreover the framework we are utilizing assumes that a component of the environment is in distress. Finally, we calculate the Insolvency Probability due to Systemic Risk and we suggest certain distribution classes under which our results are valid.

In the third proposition we examine the issue that risk spectrum of a Spectral risk measure is portraying only the preferences of an investor/regulator on her/his risk tolerance. We contribute by suggesting that also the systemic risk of market should be embodied into the risk spectrum. In line with the above suggestion, we attempt to solve that issue by suggesting two desired conditions. We gave an example where we demonstrated a mathematical way to quantify $SR$. In addition, we proposed a risk spectrum class that satisfy those conditions.

The final proposition is devoted to the Convergence of the so called Euler Risk Contributions when the underlying Risk Measures differ. With that in mind, our discussion is in regard of Euler contributions in a Risk Measure environment. Moreover, we proceed by defining some conditions where the rate of convergence of Euler Risk Contributions in a Value at Risk regulation environment and Distortion Risk Measure regulation environment coincide. Finally, we generalize our findings in regard of the Expected Shortfall case.

## 1. Future research

In regard of the first proposition, we propose the following for future research:

(1) Practical Implementation of Theoretical Findings. For the best possible results we need to evaluate our theoretical findings with data from a mature capital market and an emerging one.

(2) We should also consider that our theoretical setting implies market completeness. To that end it would be very interesting to examine what the results would be in case of incomplete Markets.

Second proposition is in regard of the Estimation of Insolvency Probability Under Systemic Risk. To that end the interest should be focused in regard of the how this estimation deviates if we alter the conditions we propose. Also it would be interested to examine how the estimation behaves in case of different distribution classes.

For the third proposition we propose the following future research:

(1) We observe that there are limitations concerning the Spectral Risk Measures we propose. For instance we need to reconsider if those factors should be equally weighted. We need to rethink that depending on the extend of the poor performance of a factor, maybe it should be weighted more than the other. Rephrasing, depending on the extend of the poor performance of a factor, maybe it should be weighted more than the others.
(2) Also we consider the utilization of the determinants that we mentioned in the motivation section for quantifying how much systemic risk a market bears.

Finally, fourth proposition deals with the Convergence of Euler Contributions, Depending on the Underlying Risk Measure. Clearly the future research should be in regard of different underlying Risk Measures, and to which extend Convergence is still applicable.

# APPENDIX A

Proof of Proposition 3.9.

Proof. Let consider that $F_0(p)$ is not positive and according to Definition 3.4, $\int_A F_0(p)dp < 0$. Also, consider a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, where the events on $\Omega$ would be $\omega_1, \omega_2, \omega_3$, two random variables on the probability space $Y$, $E$ and $Y > E$. The probability $\mathbb{P}$ is:

| $\omega$ | $\mathbb{P}(\omega)$ | $E(\omega)$ | $Y(\omega)$ |
|---|---|---|---|
| $\omega_1$ | $a_1$ | $E_1$ | $Y_1 = E_1$ |
| $\omega_2$ | $a_2 - a_1$ | $E_2$ | $Y_2 = E_2 - b$ |
| $\omega_3$ | $1 - a_2$ | $E_3$ | $Y_3 = E_3$. |

We set that $b > 0$, $Y_1 < Y_2 < Y_3$ and $E_1 < E_2 < E_3$. Then:

| $p$ | $F_E^{\leftarrow}(p)$ | $F_Y^{\leftarrow}(p)$ |
|---|---|---|
| $p \in (0, a_1]$ | $E_1$ | $Y_1$ |
| $p \in (a_1, a_2]$ | $E_2$ | $Y_2$ |
| $p \in (a_2, 1]$ | $E_3$ | $Y_3$. |

Calculating $M_{F_0}(Y) - M_{F_0}(E)$ we get:

$$M_{F_0}(Y) - M_{F_0}(E) = -b\int_0^1 F_0(p)\big(F_E^{\leftarrow}(p) - F_E^{\leftarrow}(p)\big)dp = -b\int_0^1 F_0(p)dp$$

As $b > 0$ and $\int_A F_0(p)dp < 0$ we get that $M_{F_0}(Y) - M_{F_0}(E) > 0 \Rightarrow M_{F_0}(Y) > M_{F_0}(E)$. This violates the property of monotonicity (see relevant subsection), which proves that $\int_A F_0(p)dp > 0$ and thus the necessity for positivity is proven.

For the necessity of monotonicity we recall from Definition 3.6 that for $F_0(p) \in [c,d]$, $F_0(p)$ is decreasing, if for every $a \in [c,d]$ and for every $b \geq 0$, such that $[a-b, a+b] \subset [c,d]$, $\int_{a-b}^{a} F_0(p)dp \geq \int_a^{a+b} F_0(p)dp$. Let consider that $\int_{a-b}^{a} F_0(p)dp < \int_a^{a+b} F_0(p)dp$, a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ where the events on $\Omega$ would be $\omega_1, \omega_2, \omega_3, \omega_4$ and three random variable on the probability space where $Y + E = Z$ . The probability $\mathbb{P}$ would be:

| $\omega$ | $\mathbb{P}(\omega)$ | $Y(\omega)$ | $E(\omega)$ | $Z(\omega)$ |
|---|---|---|---|---|
| $\omega_1$ | $a-b$ | $Y_1$ | $E_1$ | $Z_1 = Y_1 + E_1$ |
| $\omega_2$ | $b$ | $Y_2$ | $E_3$ | $Z_2 = Y_2 + E_3$ |
| $\omega_3$ | $b$ | $Y_3$ | $E_2$ | $Z_3 = Y_3 + E_2$ |
| $\omega_4$ | $1-a-b$ | $Y_4$ | $E_4$ | $Z_4 = Y_4 + E_4.$ |

subscripts denote the ordering of every outcome, for instance $Z_3 > Z_2$. in terms of distribution functions, there is the following tabular:

| $p$ | $F_Y^{\leftarrow}(p)$ | $F_E^{\leftarrow}(p)$ | $F_Z^{\leftarrow}(p)$ |
|---|---|---|---|
| $p \in (0, a-b] \equiv A_1$ | $Y_1$ | $E_1$ | $Z_1$ |
| $p \in (a-b, a] \equiv A_2$ | $Y_2$ | $E_2$ | $Z_2$ |
| $p \in (a, a+b] \equiv A_3$ | $Y_3$ | $E_3$ | $Z_3$ |
| $p \in (a+b, 1] \equiv A_4$ | $Y_4$ | $E_4$ | $Z_4$. |

By assuming coherency, the property of sub-additivity holds and thus , $Z = Y + E \Rightarrow M_{F_0}(Z) \leq M_{F_0}(Y) + M_{F_0}(E) \Rightarrow M_{F_0}(Z) - M_{F_0}(Y) - M_{F_0}(E) \leq 0$. Moreover, the setting suggests that:

$$M_{F_0}(Z) - M_{F_0}(Y) - M_{F_0}(E) =$$

$$-\int_0^1 \left(F_Z^{\leftarrow}(p) - F_Y^{\leftarrow}(p) - F_E^{\leftarrow}(p)\right) F_0(p)dp \ =$$

$$-\sum_{i=1}^{4} \int_{A_i} (Z_i - Y_i - E_i) F_0(p)dp.$$

then, for $i = 1$

$$\int_{A_1} (Z_1 - Y_1 - E_1) F_0(p)dp \ = \int_{A_1} (Y_1 + E_1 - Y_1 - E_1) F_0(p)dp \ = 0.$$

for $i = 4$

$$\int_{A_4} (Z_4 - Y_4 - E_4) F_0(p)dp \ = \int_{A_4} (Y_4 + E_4 - Y_4 - E_4) F_0(p)dp \ = 0.$$

for $i = 2$

$$\int_{A_2} (Z_2 - Y_2 - E_2) F_0(p) dp \ = \int_{A_2} (Y_2 + E_3 - Y_2 - E_2) F_0(p) dp = \int_{A_2} (E_3 - E_2) F_0(p) dp.$$

for $i = 3$

$$\int_{A_3} (Z_3 - Y_3 - E_3) F_0(p) dp \ = \int_{A_3} (Y_3 + E_2 - Y_3 - E_3) F_0(p) dp = -\int_{A_3} (E_3 - E_2) F_0(p) dp.$$

So:

$$M_{F_0}(Z) - M_{F_0}(Y) - M_{F_0}(E) =$$

eguals:

$$-\big( \int_{A_2} (E_3 - E_2) F_0(p) dp - \int_{A_3} (E_3 - E_2) F_0(p) dp \big) =$$

$$-\big( \int_{A_2} (E_3 - E_2) F_0(p) dp - \int_{A_3} (E_3 - E_2) F_0(p) dp \big) =$$

$$-(E_3 - E_2)\big( \int_{A_2} F_0(p) dp - \int_{A_3} F_0(p) dp \big)$$

This equation suggests that

$$M_{F_0}(Z) - M_{F_0}(Y) - M_{F_0}(E) > 0.$$

There is a contradiction and thus, necessity of monotonicity (decreasing) is proven.

Finally, for the necessity of normalization, one can consider the property of translation invariance, as suggested in relevant subsection: if $M$ is an investment with guaranteed risk free returns $m$, then for any $Z \in X$, $\rho(Z+M) = \rho(Z) - m$. Also In terms of distribution function we note that $F^{\leftarrow}_{Z+M}(p) = F^{\leftarrow}_{Z} + m$.

$$M_{F_0}(Z+M) = -\int_0^1 F_0(p) F^{\leftarrow}_{Z+M}(p)dp = M_{F_0}(Z) - m\int_0^1 F_0(p)dp$$

.

Finally, the risk measure can be coherent if and only if $\int_0^1 F_0(p)dp = 1$ and thus the necessity of final property is proven.

$\square$

# APPENDIX B

Specific example and its proof of the risk spectrum we propose.

EXAMPLE 0.1. *Let us take* (3.2) *and set that* $d = 0.9$ *and* $k \leq 89$ *or* $\psi(Y,p) = Y\left[0.9p^{-0.1}\right] - (Y-1)$. *Now we can prove that it is admissible:*

PROOF.

$$\psi(Y,p) = (\big(dY\big)p^{d-1} - (Y-1)) \Rightarrow (\big(0.9Y\big)p^{-0.1} - (Y-1)) \Rightarrow \frac{9Y}{10\sqrt[10]{p}} - Y + 1$$

Calculating for first derivative with respect to $p$ we get:

$$\psi(Y,p)^{'} = \frac{d}{dp}\left[\frac{9Y}{10\sqrt[10]{p}} - Y + 1\right] = \frac{9Y}{10}\frac{d}{dp}\left[\frac{1}{\sqrt[10]{p}}\right] + \frac{d}{dp}[-Y] + \frac{d}{dp}[1] =$$

$$\frac{9\big(-\frac{1}{10}\big)p^{-\frac{1}{10}-1}f(p_i)}{10} = -\frac{9Y}{100p^{\frac{11}{10}}},$$

where $100p^{\frac{11}{10}} > 0$ as $p \in (0,1]$ and $9Y > 0$ as (3.1) is non-negative. Also, $k \leq 89 \Rightarrow 1 \leq Y \leq 9.9$. Recall that from the setting we observe that $Y$ has range $[1, 9.9]$ and so $d \in \left[\frac{Y-1}{Y}, 1\right]$ is satisfied. Summing up, $\psi(Y,p)^{'} < 0 \Rightarrow \psi(Y,p)$ is decreasing in $(0,1]$ and second condition is met for $\psi(Y,p)$ to be admissible. Calculating for second derivative we get:

$$\psi(p)^{''} = \frac{d}{dp}\left[-\frac{9Y)}{100p^{\frac{11}{10}}}\right] = -\frac{9Y}{100}\frac{d}{dp}\left[\frac{1}{p^{\frac{11}{10}}}\right] = -\frac{9\big(-\frac{11}{10}\big)p^{-\frac{11}{10}-1}Y}{100} = \frac{99Y}{1000p^{\frac{21}{10}}},$$

where $1000p^{\frac{21}{10}} > 0$ as $p \in (0,1]$, $99Y > 0$ as $\psi(Y,p)$ is non negative. Concluding $\psi(Y,p)^{''} > 0$ and thus $\psi(p)$ is convex. Moreover, for proving the positivity, we consider the decreasing property and we take $\psi(0^+)$:

$$\psi(0^+) = \big(Y0.9\big)0^{+(-0.1)} - Y + 1,$$

and we consider that $\psi(0^+) > 0 \Rightarrow \big(Y0.9\big)0^{+(-0.1)} - Y + 1 > 0 \Rightarrow \big(Y0.9\big)0^{+(-0.1)} > Y - 1$ which is true since $\big(Y0.9\big)0^{+(-0.1)} = \infty$ and $Y \leq 9.9$. For the latter recall that we set $k < 89$ or, there is a limit of macroeconomic factors that are responsible for generating systemic risk. In case that all experiments are succeed then $Y = 9.9$. Moreover, $\psi(1) = \big(Y0.9\big)1^{-0.1} - Y + 1 \Rightarrow 0.9Y - Y + 1 > 0 \Rightarrow 0.9Y - Y > -1 \Rightarrow -0.1Y > -1 \Rightarrow 0.1Y < 1 \Rightarrow \frac{Y}{10} < 1 \Rightarrow Y < 10$, which is true. Since $\psi(0^+) > 0$ and $\psi(1) > 0$, the positivity condition is proven. Concerning Normalization condition one should have that $\int_0^1 \psi(Y,p)dp = 1$ and so:

$$\int_0^1 \psi(Y,p)dp = \int_0^1 \left[\frac{9Y}{10\sqrt[10]{p}} - Y + 1\right]dp = \frac{9Y}{10}\int_0^1 \frac{1}{\sqrt[10]{p}} + 1 - Y dp =$$

$$\left[\big(1 - Y\big)p + Yp^{\frac{9}{10}} + C\right]_0^1 = \left[p - Yp + Yp^{\frac{9}{10}} + C\right]_0^1 =$$

$$\left[1 - Y + Y\right] - \left[0 + 0 + 0\right] = 1$$

normalization condition holds and thus is an admissible risk spectrum. $\square$

# APPENDIX C

Numerical example for the understanding of the Remark 3.4

EXAMPLE 0.1. *Consider the equation in the above example. If we set* $p = 0.001$, *where the greater losses are then* $\phi_{0.001} = 2.01640$ *while* $\psi_{0.001} = 2.21968$. *On the other side of the distribution where* $p = 0.999$ *we get that* $\phi_{0.999} = 1.01069$ *while* $\psi_{0.999} = 1.01282$. *We take into account the properties of* $\phi(p)$ *and* $\psi(p)$ *and we observe that when systemic risk emerges then* $\psi(p) > \phi(p)$ *for almost the whole* $p$ *interval and first condition is met. Also the second condition is met as* $2.21968 - 2.01640 > 1.01282 - 1.01069$ *is true.*

*Moreover, compare to* $\phi(p)$ *the* $\psi(p)$ *guides a practitioner to be more conservative in the presence of systemic risk as losses become greater. On the contrary when losses diminish* $\psi(p)$ *grants similar weight like* $\phi(p)$. *Trivially we observe that* $Y$ *forces* $\psi(p)$ *to give different weight to* $F_X^{\leftarrow}(p)$ *depending on the existing systemic risk of each market and thus it quantifies systemic risk. Also from the above setting we observe that* $\psi(p)$ *grants more weight to the increasing losses of* $F_X^{\leftarrow}(p)$ *as more systemic risk occurs.*

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