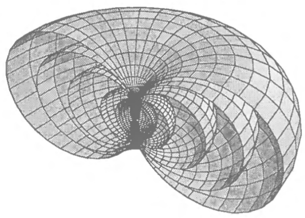
The Study of the Jordan Canonical Forms of Killing Tensor in the frame of General Theory of Relativity



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The implementation of the doctoral thesis was partly co-funded by Greece and the European Union (European Social Fund) through the Operational Programme "Human Resources Development, Education and Lifelong Learning" 2014-2020, within the framework of the Action "Enhancement of human resources through the implementation of doctoral research, Sub-Action 2: IKY Scholarship Program for doctoral candidates of Greek Higher Education Institutions".

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Declaration of Authorship

I, Dionysios Kokkinos, hereby declare that the doctoral dissertation titled as

The Study of the Jordan Canonical Forms of the Killing Tensor in the Frame of General Theory of Relativity,

submitted for the fulfillment of the requirements for the degree of Doctor of Philosophy in Department of Information and Communication Systems Engineering at University of Aegean is entirely my original work. I am the author of this dissertation and have not used any material or sources without proper citation and acknowledgment. Furthermore part of this thesis

- has been submitted to **International Journal of Modern Physics D** and it is under revision arXiv:2309.04203v1 [gr-qc],
- has been presented in the international Conference NEB-20, Athens, September 2023, with title The Study of the Canonical Forms of the Killing Tensor in the Frame of General Theory of Relativity.

I also affirm that:

- All ideas, concepts, and theories presented in this dissertation are my own, except where I have clearly attributed them to others through appropriate citations.
- Where I have quoted from the work of others, the source is always given. With the exception of such quotations, this thesis is entirely my own work.
- I have acknowledged all main sources of help.

I take full responsibility for the content of this dissertation, and I am aware of the potential consequences of academic dishonesty.

Sincerely,

Dionysios Kokkinos

20th September 2023

Abstract

The Study of the Jordan Canonical forms of the Killing Tensor in the frame of the General Theory of Relativity focuses on the extraction of exact solutions which admit the Canonical forms of the Killing Tensor. The existence of a Killing tensor in a Hamiltonian problem is equivalent to the existence of integrals of motion. In this work we seek to uncover analytical solutions of Einstein's Field equations which admit hidden symmetries. Namely, we aim to discover new interesting solutions or discover hidden symmetries of the already known spacetimes correlating them with the existence of canonical forms of Killing tensor. To accomplish this, we confront the field equations of the theory along with the integrability conditions of the Killing equations of the Canonical forms of the Killing Tensor.

The first part of dissertation provides a literature survey of the most known exact solutions. Since the resolving procedure is embodied with the usage of the Newman-Penrose formalism of the null tetrads, we introduce the basic concepts of the formalism and the corresponding notation in terms of the null tetrads.

The next segment concerns the definition of the Killing tensor and therein we acquire the four Jordan canonical forms along with their integrability conditions. Our focus is concentrated on the study of only three of these canonical forms. Besides, the integrability conditions of the Killing equation of each form consist our additional assumption of symmetry in order to solve the field equations. The similarity of two of these forms allows us to handle them simultaneously.

The first result is the Petrov types of each form, which are invariant characterizations of the obtained gravitational fields. We manage to obtain these with the implication of a rotation around the null tetrad frame. The Petrov types that admit K^2 and K^3 forms have in their line a type D solution which we are really interested in. We obtain multiple variations of new exact solutions of this type. Some of them reduce to known spacetimes. One of them emerged to be the Carter's Case [D] which admit the K^2 form apparently.

Finally, along with the analysis of the Carter's Case [D], we correlate the eigenvalues of the tensor with the constants of motion, giving rise to the significance of the entanglement of the Killing tensor.

Περίληψη

Η Μελέτη των Κανονικών Jordan μορφών του τανυστή Killing στο πλαίσιο της Γενικής Θεωρίας της Σχετικότητας εστιάζει στην απόκτηση αναλυτικών λύσεων της Γενικής Σχετικότητας που αποδέχονται τις κανονικές μορφές του τανυστή Killing. Η ύπαρξη του τανυστή Killing σε ένα Χαμιλτονιανό πρόβλημα είναι ισοδύναμο με την ύπαρξη των ολοκληρωμάτων της κίνησης. Σε αυτό το πλαίσιο αναζητούμε αναλυτικές λύσεις των εξισώσεων Einstein που αποδέχονται κρυμμένες συμμετρίες. Συγκεκριμένα, σκοπεύουμε είτε να ανακαλύψουμε νέες λύσεις είτε να συσχετίσουμε τις κρυμμένες συμμετρίες των γνωστών λύσεων με τις κανονικές μορφές του τανυστή. Για να πετύχουμε κάτι τέτοιο πρέπει να λύσουμε ταυτόχρονα τις εξισώσεις πεδίου με τις εξισώσεις ολοκληρωσιμότητας των κανονικών μορφών του τανυστή Killing.

Η διατριβή έχει ως αφετηρία μια σύντομη σύνοψη της Γενικής Θεωρίας της Σχετικότητας και μια βιβλιογραφική έρευνα των ήδη γνωστών ακριβών λύσεων σε κάποιες από τις οποίες αναφερόμαστε συχνά. Η διαδικασία επίλυσης πραγματώνεται με τη χρήση του φορμαλισμού των ισότροπων τετράδων των Newman-Penrose. Σε αυτό το πλαίσιο, έπειτα από την παρουσίαση των βασικών στοιχείων της Γενικής Σχετικότητας και των ακριβών λύσεων, εισάγουμε πολύ συντομα τις βασικές αρχές του φορμαλισμού καθώς και τον αντίστοιχο συμβολισμό συναρτήσει των ισότροπων τετράδων.

Το επόμενο στάδιο αφορά τον ορισμό του τανυστή Killing, σε αυτό το κεφάλαιο αποκτούμε τις τέσσερεις κανονικές Jordan μορφές και τις εξισώσεις ολοκληρωσιμότητάς τους, που αποτελούν την αρχική μας υπόθεση, με σκοπό την επίλυση των εξισώσεων πεδίου. Στην διατριβή αυτή θα ασχοληθούμε μόνο με τις τρεις κανονικές μορφές. Η ομοιότητα των μορφών K^2 και K^3 μας επιτρέπει να τις χειριστούμε ταυτόχρονα.

Το πρώτο μας αποτέλεσμα είναι ο τύπος κατά Petrov της κάθε μορφής, που αποτελεί έναν αναλοίωτο χαρακτηρισμό του βαρυτικού πεδίου. Καταφέραμε να καταλήξουμε σε αυτό εφαρμόζοντας μια στροφή γύρω από το ισότροπο σύστημα τετράδων μιας και έτσι απλοποιούνται οι συντελεστές συνοχής spin. Οι τύποι κατά Petrov που αποδέχονται τις μορφές K^2 και K^3 περιλαμβάνουν μια αρκετά ενδιαφέρουσα λύση τύπου D. Αποκτήσαμε διάφορες εκδοχές αναλυτικών λύσεων αυτού του τύπου. Κάποιες από αυτές καταλήγουν σε γνωστούς χωροχρόνους. Ένας από τους αναδυόμενους χώρους ο οποίος αποδέχεται μόνο την μορφή K^2, είναι η περίπτωση [D] των χώρων του Carter την οποία αναλύσαμε εκτεταμμένα. Παρουσιάζουμε ειδικές περιπτώσεις των ειδικών χώρων του, καθώς και τις τροχιές των γεωδαισιακών του.

Τέλος η χρήση του τανυστή Killing μας παρέχει εκφράσεις που συσχετίζουν τις ιδιοτιμές του με τις σταθερές της κίνησης αναδεικνύοντας τη σημαντικότητά του.

Acknowledgements

The idea to study the subject of this dissertation emerged by insightful discussions with my Professor Taxiarchis Papakostas, which took place at his office on the 2nd floor of the Physics Department in the University of Crete.

Already from my undergraduate years, he tried to show me the appropriate way to think about the geometrized General Relativity and to deal with the 'fiendish' mathematical concept of the Killing Family of Tensors, as he calls them. Moreover, his guidance and his encouragement paved the way for me to make the first steps in the long journey to be a physicist, to challenge myself through these years and to accomplish this dissertation at last. For all these reasons, I am deeply thankful to him.

Also, I am obliged to Associate Professor Georgios Kofinas who supported me and advised me all these years, and to Professor Theocharis Apostolatos for his willingness to participate in my PhD's committee. I am also grateful to my friends and colleagues who have provided all the help and support that I needed throughout my studies. Special thanks to *Anna* who stood by me all these years.

Finally, I am grateful to my parents, who always support me through my choices.

At last, I manage to leave my personal imprint on the following pages. My dissertation is dedicated to the memory of my beloved friend, comrade and colleague **Dionysios Xydias** who sadly did not have the opportunity to grace us presenting his own. I will always remember our discussions about the virtue of science in one hand and the social duty of a scientist during our turbulent times as well. This was the "culture" which inspired me to willingly follow this path.

I will always be grateful to you
my dear friend...

Abbreviations

Latin indices

$$i, j, k, ... = 1, 2, 3$$

Greek indices

$$\alpha, \beta, \kappa, \lambda, ... = 1, 2, 3, 4$$

Derivative in Einstein's Summation Convention

$$v^{\nu}(x)_{,\mu} \equiv \frac{\partial v^{\nu}(x)}{\partial x^{\mu}}$$

Covariant Derivative

$$v^{\mu}{}_{;\alpha} \equiv v^{\mu}{}_{,\alpha} + \Gamma^{\mu}{}_{\kappa\alpha} v^{\kappa}$$

- [**GR**] General Relativity
- [**NP**] Newman & Penrose
- [**NUT**] Newman, Unti and Tamburino
- [**EFEs**] Einstein's Field Equations
- [**BH**] Black Hole
- [**NPEs**] Newman-Penrose Equations
- [**IC**] Integrability Conditions

- [**CR**] Commutation Relations

Contents

Chapter 1

Introduction

The epochs in the annals of science are distinguished by the innovative theories that emerged, adorning the mosaic of the scientific progress. General Theory of Relativity has been demonstrated as the most adequate theory of gravity, providing a unified description of cosmos as a geometric property of spacetime. As "generalization" of Special Theory of Relativity, it establishes the rejection of the system of physical concepts and ideas of absolute space and time in contrast to Newtonian theory of gravity. Relativity creates a new concept of our universe while it encapsulates all physical laws of the previous theory. As every theory, General Relativity had to be tested. The classical tests of Relativity, proposed by Einstein in 1916 [1], would establish the first success of the theory [2]. These tests were

1. The perihelion precession of Mercury's orbit.
2. The deflection of light by the Sun.
3. The gravitational redshift of light.

Most theories in modern physics involve mathematical models which are described by differential equations consisted of purely geometric requirements imposed by the idea that space and time can be represented by a pseudo-Riemannian (Lorentzian) manifold, together with the depiction of the interaction of matter. These mathematical models are characterized by observable quantities, creating a connection between theory and the physical world. Besides, this is the way to connect theory with nature, while if these quantities do not exist theoretically, the reliability of the theory will never be proven. Few decades ago, some of these models described astronomical objects and vicious astronomical phenomena which were difficult to detect.

But ages bring changes. The growth of technology brings to surface multiple alternatives to reach the unreachable, to "see" galaxies far far away. One of the most recent significant proofs of the validation of General Relativity was the detection of gravitational waves resulting from the merging of black holes, by the Laser Interferometer Gravitational-Wave Observatory (LIGO) [3]. However, this waveform signal would be just noise if the exact solution of EFEs, which describe a stationary axially symmetric black hole, had not been found by Roy P. Kerr in

1963 [4].

The interpretation and the analysis of exact solutions in the context of General Relativity consists a whole regime of research. In this scientific branch, a lot of people are in pursuit of spacetimes attempting to obtain every kind of solution that the theory has to offer. Along these lines, the exact solutions of the theory are the hidden trophy behind the fearsome non-linearity of the equations of gravity. However, it is already within our understanding that the Einstein's Field Equations cannot be solved without additional assumptions regarding the nature of a spacetime.

The existence of mathematical assumptions is essential in order to acquire exact solutions. Symmetries have a crucial role during this procedure. As a matter of fact Schwarzschild found one of the most known exact solutions in history, which obtained with the assumption that the spacetime in vacuum admits spherical symmetry [5]. Hence, the assumed symmetries, along with the EFEs, could help one to solve the problem. Our problem though is a bit more complicated since we seek to discover analytical solutions of Einstein's Field Equations which admit hidden symmetries.

State of the Art

It is known that the physical position of any object is hosted in lines called geodesics. In a sense, the existence of these kind of curves indicates the physical substance of a spacetime. A relevant conjecture says that: in a spacetime, which admit a non-trivial Killing tensor, there are closed trajectories [6]. Moreover the Killing-Yano Tensor [7], a generalization of the Killing Tensor, is connected with the conservation of the Laplace-Runge-Lenz vector in the Kepler-Coulomb problem along geodesics as a constant of motion [8],[9].

For these reasons, we aim to find spacetimes which admit Killing Tensors since they are responsible for explicit and hidden symmetries. The study of dynamics of a Hamiltonian system brings to surface this kind of symmetries providing either generators of isometries (Killing Vectors) or differently constants of motion. The direct generalization of a Killing vector is the Killing Tensor encoding hidden symmetries. **In essence, the existence of a Killing tensor equivalently means the existence of constants of motion and the separation of Hamilton-Jacobi equation which rises the integrability of trajectories** [10],[11],[12].

Besides, there are two ways to benefit from the usage of a Killing tensor, either by assuming its existence to find a metric or by revealing the hidden symmetries of a known metric. **The only works in the literature that utilize a Killing tensor to explore new spacetimes include Hauser and Malhiot's work on vacuum [13] and Papakostas' work on interior solutions with perfect fluid [14]. Both of these works serve as the only paradigms for our research, since these are the only works both originate from the assumption of the existence of a Killing tensor with two double eigenvalues.**

$$K_{\mu\nu} = \begin{pmatrix} 0 & \lambda_1 & 0 & 0 \\ \lambda_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \lambda_2 \\ 0 & 0 & \lambda_2 & 0 \end{pmatrix} \qquad \text{(Paradigm)}$$

We choose to take advantage of the canonical forms of a Killing Tensor to discover new metrics with hidden symmetries or to acquire more information in case where we find known metrics. The most desirable result would be solutions of type I according to Petrov classification, admitting only one spacelike Killing vector, since these are quite general metrics. However, this dissertation aims to address fundamental questions concerning solutions that admit the canonical forms of a Killing Tensor.

In this dissertation we will solve the Einstein's Field equations using as additional symmetry the integrability conditions of the Killing equations of our Canonical forms trying to answer the following questions.

- An **open question** is that: what kind of gravitational fields (Petrov types) could someone obtain assuming more general forms of Killing tensor?
- Another **open question** is: May we obtain new solutions assuming the existence of the Canonical forms of Killing tensor?

Actually, the main idea that motivated us to tackle this problem is the following: *we might find new interesting spacetimes in vacuum if we deal with more general forms of a Killing tensor with more than two distinct eigenvalues*, as postulated by Hauser and Malhiot in 1976 [15].

The Canonical forms of a Killing tensor are more general than the Killing tensor $K_{\mu\nu}$ of our paradigm, and we obtained them in Chapter 5 through geometric methods, building upon the work of Churchill [16]. We derived four Canonical forms of the Killing tensor, but for our study, we will focus on three of them. Although, all of them is a generalization of that of our paradigm.

$$K^1{}_{\mu\nu} = \begin{pmatrix} \lambda_0 & \lambda_1 & 0 & 0 \\ \lambda_1 & 0 & 0 & 0 \\ 0 & 0 & \lambda_7 & \lambda_2 \\ 0 & 0 & \lambda_2 & \lambda_7 \end{pmatrix}, K^2{}_{\mu\nu} = \begin{pmatrix} \lambda_0 & \lambda_1 & 0 & 0 \\ \lambda_1 & \lambda_0 & 0 & 0 \\ 0 & 0 & \lambda_7 & \lambda_2 \\ 0 & 0 & \lambda_2 & \lambda_7 \end{pmatrix}, K^3{}_{\mu\nu} = \begin{pmatrix} \lambda_0 & \lambda_1 & 0 & 0 \\ \lambda_1 & -\lambda_0 & 0 & 0 \\ 0 & 0 & \lambda_7 & \lambda_2 \\ 0 & 0 & \lambda_2 & \lambda_7 \end{pmatrix},$$

It is evident that the annihilation of λ_0 and λ_7 make our Canonical forms to coincide with the Killing tensor of our paradigm.

- Along these lines, another **open question arises**: Is there any correlation between the Petrov types that admit the $K_{\mu\nu}$ and those that admit our Canonical forms?

The Study of the 1st Canonical form

In **Chapter 6**, we provided the answer to the third question using two approaches. At first, we annihilated λ_7 and we set λ_0 to be equal to $q = \pm 1$. These two choices yield a new form which is very similar to that of our paradigm. Actually, the only difference is the component q.

$$K^1_{\mu\nu} = \begin{pmatrix} q & \lambda_1 & 0 & 0 \\ \lambda_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \lambda_2 \\ 0 & 0 & \lambda_2 & 0 \end{pmatrix} \qquad \text{(Jordan form)}$$

We applied the general methodology, which is given in p.10, to this new form up to **point 4**. The implied rotation provided us simplifications of spin coefficients, which we referred to as 'Key relations', used by us as a *starter culture* in order to find solutions of the field equations. The solutions we obtained numbered three, all falling into Type N. This conclusion immediately points out that **we cannot attain Petrov types as general as those within our paradigm**. Besides, Hauser-Malhiot spaces encompass Petrov types of Type I and D, which are more general than Type N.

However, we also explored an alternative method to address this question. With this approach, we "spare" some of the arbitrariness of the coordinate system by choosing specific spin coefficients to be equal, namely $\pi = \tau$. This choice is grounded in two invariant relations under rotations that emerge from the work of Debever et. al. [17].

$$\pi\bar{\tau} - \tau\bar{\pi} = 0$$

$$\mu\bar{\rho} - \rho\bar{\mu} = 0$$

Using this choice **we obtained a unique solution which is also of type N**. Consequently, **assuming the existence of the K^1 form we cannot find Petrov types as general as that of our paradigm**. At last, even we can answer to the third open question, we did not manage to find the metric in full detail. More work is needed to be done for this case!!!

The Study of the 2nd and 3rd Canonical form

In **Chapter 7** we answered to the first two **open questions**. Initially, we defined a factor $q = \pm 1$ in order to study the two forms simultaneously. In addition, we decided to set λ_7 to zero because our study of the K^1 form revealed that the application of rotation is only possible when the diagonal elements are absent.

$$K^{2,3}_{\mu\nu} = \begin{pmatrix} \lambda_0 & \lambda_1 & 0 & 0 \\ \lambda_1 & q\lambda_0 & 0 & 0 \\ 0 & 0 & 0 & \lambda_2 \\ 0 & 0 & \lambda_2 & 0 \end{pmatrix} \tag{1.1}$$

The simultaneous study of these two forms was conducted up to **point 5** of the general methodology. We discovered that

- Petrov types III, D, and N admit both K^2 and K^3 forms.
- **Furthermore, we discovered that the combination of spin coefficients characterize a new type D solution that admit both K^2 and K^3 forms**.
- The latter answers affirmatively to the second **open question**.

The combination $\kappa \neq 0 = \sigma$ in vacuum with cosmological constant is not known in the literature [18]. Although, in order to find the metric we need to define a coordinate system using the Frobenius theorem. We obligated to make a suitable choice of spin coefficients in order to do so.

$$\kappa = -\bar{\nu} \qquad \pi = -\bar{\tau}$$

With this choice we are able to imply the Frobenius theorem but only for K^2 since the aforementioned choice yields that $q = +1$ dictated by the Key relation $\kappa\nu = \pi\tau$.

The following paragraphs of this study concerns only K^2 form. Moreover, we know that type D solutions admit (at least) two Killing vectors [19]. Solving the system of equations we found these two Killing vectors. With these two vectors one could imply the separation of the Hamilton-Jacobi equation defining the 4th constant of motion and retrieving valuable simplifications for the metric functions. The general metric that describes our solution is the following, although, we managed to determine the functions below in full detail.

$$ds^2 = 2\left[M^2(x)d\tilde{t}^2 - S^2(x)dx^2\right] - 2\left[P^2(y)d\tilde{z}^2 + R^2(y)dy^2\right]$$

Solving the differential equations with three different choices we obtained eight different metrics and only one of them is recognizable in the literature (Carter's Case [D]) [20].

Generally, this metric describes cosmological models and it is a direct product of 2-product spaces with constant curvature. Moreover it admits a 6-parameter group of motion. Although, all these metrics can be reduced to the general metric of two product spaces with arbitrary function Σ^2 and Ω_1,Ω_2 to be constants of integration.

$$ds^2 = \Omega_1\left[\Sigma^2(u, g_1)dt^2 - du^2\right] - \Omega_2\left[\Sigma^2(w, g_2)dz^2 + dw^2\right]$$

Indeed, all of these metrics can be reduced to the latter with the following transformations

$$du = \int S(x)dx \qquad dw = \int R(y)dy$$

However, the novelty of our dissertation could be summarised as follows: using the Canonical Forms of Killing Tensor as initial assumption to vacuum spacetime with cosmological constant we manage to answer to the three **open questions** and

1. We found that the types D, N, III solutions admits both K^2 and K^3 forms, where type D solution results to be new.

2. A special case of the type D solution yielded new metrics which admit a 6-parameter group of motion.

3. These metrics are direct products of 2-product spaces describing cosmological models with constant curvature.

4. We found the geodesics of all these metrics using the separation of variables of Hamilton-Jacobi.

5. Also, we defined the hidden symmetry (Carter's Constant) for all these spacetimes with the usage of Killing tensor.

Dissertation structure

We attempted to establish a coherent structure for our dissertation. At **Chapter 2**, we provide a brief overview of the main elements of the General Theory of Relativity. This is necessary in order to establish a connection between the standard metric formalism and the Newman-Penrose formalism [21]. Afterwards, in **Chapter 3**, we introduce some of the most noteworthy exact solutions to Einstein's Field Equations. Many of these solutions are revisited throughout the dissertation.

Following this, in **Chapter 4** we introduce the Complex Vectorial Formalism of Newman-Penrose [22], [23]. We explain the fundamental concepts of this formalism and we present its essential components in terms of null tetrads. This chapter contains expressions that are employed in the resolution process, like the Lorentz rotation around a null tetrad frame and the commutation relations of the covariant derivatives.

Moving forward, we detail the methodology employed to obtain the canonical forms of Killing Tensor. **Chapters 6 and 7** are dedicated to the study of the 1st Canonical Forms and the 2nd and 3rd Canonical Forms, respectively. These chapters unveil our primary findings, including the Petrov types accommodated by our canonical forms. Each chapter focuses on different Petrov types, with **Chapter 7** being the centerpiece of this thesis. We present new metrics and potential reductions to known spacetimes in our analysis of the solutions.

Methodology

In this paragraph, we outline the fundamental methodology that we will employ to derive analytical solutions for each canonical form. The process begins with the collection and characterization of the canonical forms of the Killing tensor that takes place in **Chapter 5**. We proceed applying the same methodology for each form. We are going to deal with the 1st canonical form in **Chapter 6** and in **Chapter 7** with the 2nd and 3rd canonical forms simultaneously since their similarities permit us to do so.

The method of extraction of analytical solution with the usage of the canonical forms of Killing tensor contains:

1. We obtain the Killing equations of each form.

2. We derive the Integrability Conditions of the Killing equations.

3. We apply a rotation around the null tetrad frame in order to obtain simplifications of the spin coefficients.

4. We determine the Petrov types of the obtained solutions in order to categorize our solutions geometrically.

5. We choose one of these solutions in order to determine its metric.

6. We define the system of coordinates implying the Frobenius theorem of integrability.

7. We solve the system of equations for this solution in order to determine the metric functions (Newman-Penrose Equations, Bianchi Identities, Integrability Conditions).

In summary

It is known that introducing a Killing tensor as an additional symmetry in Einstein's spacetimes can prove to be fruitful, offering new analytical solutions. This is because Einstein's equations, Bianchi Identities, along with the integrability conditions of Killing equations, create an overdetermined but solvable system of equations. Furthermore, this approach allows us to obtain integrable geodesics and to separate the Hamilton-Jacobi equation.

So far, the only case of a Killing tensor that has been studied is one with two double eigenvalues, this case serve as a paradigm to us. While, this is a special case of the canonical forms of a Killing tensor, its study has yielded new and general exact solutions.

However, it is both interesting and important to explore Einstein's spacetimes that admit more general forms of Killing tensors beyond the case of two double eigenvalues as the initial assumption. These more general forms represent the canonical forms of Killing tensor. It is intriguing to investigate whether these more general forms can lead to new solutions or generalizations of the ones already known.

It's worth noting that, in pursuit of this goal, the use of the standard metric formalism is not practical, as it cannot accommodate the symmetries associated with a Killing tensor. In this context, the most suitable formalism to achieve this is the complex vectorial formalism. The reason for this choice is that during the resolution process, the classification of the gravitational field of a spacetime according to Petrov occurs in the early stages of the process. This allows us to deduce theorems of symmetries for each Petrov type, determining an appropriate coordinate system using the Frobenius theorem of integrability.

Furthermore, the Killing equations results in simplifications between the spin coefficients. These simplifications play a pivotal role in solving the field equations without the need for specific coordinate system. This is possible since the Newman-Penrose field equations are first-order differential equations of spin coefficients.

In fact, the complex vectorial formalism we employ provides insights not only into the essential characteristics of null congruences, which are related to singularity theorems, but also into the essential characteristics of the principal null directions of spacetime (geodesic, expansion, shear).

If we attempt to recapitulate, it is worth mentioning that we have successfully proven that the Petrov types of spacetimes, which admit a special case of canonical forms, are not related to those types involving more general cases. Furthermore, we have achieved the derivation of a new Type D solution and have inherited symmetries into all these newly obtained spacetimes. Additionally, we present a methodology for obtaining analytical solutions to Einstein's equations, assuming an arbitrary Killing tensor and considering its canonical form. It would certainly be of interest to us if this methodology were applied to the K^0 form, which we did not have the time to address.

Finally, this dissertation is the initial part of a general study that we scope to continue. We foresee that there is a potential of interesting spacetimes that cannot be found during the years of a Philosophical Doctorate. For these reasons we aim to continue this research to gain a broadened view about the hidden symmetries of spacetimes that admit the bespoken canonical forms.

Concluding, the study of the exact solutions of Einstein's equation satisfying answers to the queries of the community, not only about the nature of objects experiencing stronger gravitational fields, but also for their imprints, known as gravitational radiation in violent phenomena. Therefore, we are motivated by the perception that the usage of analytical models, by sectors associated with the simulation of astrophysical objects, would possibly provide sophisticated theories along with significant experimental results about the nature of this "strange" distortion of spacetime, we call gravity.

Chapter 2

General Theory of Relativity

The Special Relativity was born in 1905, and now we have at our disposal an elegant and consistent theory that describes physical phenomena based upon inertial reference frames. Eleven years a new theory would come in 1915 encapsulating the Special Relativity. This new theory is the General Relativity concerning every reference frame as a generalized theory. *Finally all coordinate systems are equivalent...*

2.1 The Principle of Equivalence

In order to achieve the equivalence of any reference frame, Einstein had to immerse them into a curved Riemannian space or into a gravitational field. This is the Principle of Equivalence and simultaneously the cornerstone of Einstein in purpose to construct the theory [1]. The observer, according to this, can't separate his acceleration from a gravitational field that attracts him to the opposite direction.

Actually, this is the Weak Principle of Equivalence because it refers only to gravity. The Strong Principle of Equivalence according to the modern terminology refers to the equivalence of a free fall in a gravitational field with an inertial frame in the Special Theory of Relativity. All the physical experiments must be equivalent in both cases [24].

2.2 Mathematical background

From the geometrical point of view, General Relativity geometrizes the gravitational field creating the need for a quantitative description of the transition from a flat Minkowski spacetime to a curved Riemannian spacetime [25].

2.2.1 Tensor Calculus

Riemannian Manifold

Geometrically, spacetime represents a smooth 4-dimensional curved manifold, denoted as $\mathcal{M}$. A manifold is a set of points described by the values of the 4 components that describe spacetime: t, x^1, x^2, x^3. Our manifold is a spacetime with four coordinates, which can be locally considered as flat or Euclidean.

In general, a manifold is an amorphous collection of points without any specific description of its shape. The geometric object that precisely defines its shape is the metric. The metric is a 2nd-rank symmetric tensor that contains all the necessary information about the manifold. If the metric tensor is not positive-definite, our manifold is a generalization of a Riemannian manifold, also known as a Pseudo-Riemannian manifold. Essentially, a manifold can be characterized as Riemannian manifolds when:

1. It contains a metric tensor, and
2. The manifold is differentiable due to smoothness.

Tensors

A tensor is a geometric object that collectively represents by scalars (0-rank tensor), vectors (1-rank tensor), etc. which describe the manifold via the values of its components. Also, tensors are able to describe linear relations between other tensors. In this point, it should be noted that in flat spacetime there is no essential difference between a covariant and a contravariant vector.

Covariant Derivative & Affine Connection

The Covariant Derivative describes the derivation of a tensor due to its components and due to its coordinates. In the 0-rank tensor, case the derivation is as simple as before

$$\frac{\partial T}{\partial x^\mu} \equiv T_{,\mu} \tag{2.1}$$

In some other coordinate system y^μ, the derivation of a scalar will be

$$\frac{\partial T}{\partial y^\mu} \equiv \frac{\partial T}{\partial x^\nu}\frac{\partial x^\nu}{\partial y^\mu} \tag{2.2}$$

In case that we want to see how the derivation produces tensors, we will take the same example about a covariant vector $a_\lambda(x^\mu)$ where the derivation expands not only in the coordinates but in the components as well,

$$\frac{\partial a_\lambda}{\partial x^\mu} \equiv a_{\lambda,\mu} \tag{2.3}$$

In another coordinate system y^μ the corresponding expression for the covariant vector $\tilde{a}_\lambda(y^\mu)$ is

$$\frac{\partial \tilde{a}_\lambda}{\partial y^\mu} = \frac{\partial^2 x^\kappa}{\partial y^\lambda \partial y^\mu} a_\kappa + \frac{\partial x^\kappa}{\partial y^\lambda}\frac{\partial x^\nu}{\partial y^\mu} a_{\kappa,\nu} \tag{2.4}$$

As we said before, the covariant derivation concerns also the derivation in coordinates. In the next relation, the (affine) *connection* created by the derivation in coordinates while the first term of right part of the equation is just the derivative of the components of vector $a_\lambda(x^\mu)$,

$$\tilde{a}_{\lambda;\mu} = a_{\lambda,\mu} + \Gamma^\kappa{}_{\lambda\mu} a_\kappa \tag{2.5}$$

Hence we see the difference between flat spacetime and curved spacetime. At flat spacetime with the Cartesian coordinates the *connection* or *Christoffel's symbols* would be zero since the coordinates do not change on a single translations.

The Christoffel's symbols $\Gamma^\kappa{}_{\lambda\mu}$

The previous is obvious in the derivation of a basis vector e_λ which is defined as

$$\frac{\partial e_\lambda}{\partial x^\mu} = \Gamma^\kappa{}_{\lambda\mu} e_\kappa \tag{2.6}$$

Generally the meaning of the Christoffel's symbols is of great importance because it declares the existence of curvature as components of ∇e_α. The components of the basis vectors are different for every point in a curved manifold and it is represented via non-zero Christoffel's symbols. The Curvature is a property of manifolds that is acquired by the metric tensor. The Christoffel's symbols are defined by the following relation where the symmetricity in the indices of the metric induces a symmetricity in the lower two indices of $\Gamma^\kappa{}_{\lambda\mu}$.

$$\Gamma^\kappa{}_{\lambda\mu} = \frac{g^{\kappa\nu}}{2}(g_{\lambda\nu,\mu} + g_{\mu\nu,\lambda} - g_{\lambda\mu,\nu}) \tag{2.7}$$

2.2.2 Killing vectors

The procedure for imposing restrictions determines on the form of the metric leads to a non-trivial problem. The imposed symmetries in our case of interests, namely axial symmetry and stationarity, is better implemented through the use of Killing vectors which consists a coordinate independend and covariant method. As mentioned earlier, the transformation from one coordinate system to another indicates the isometry of the metric [26]. So a metric $g_{\mu\nu}$ is form-invariant under a transformation from x^μ to ${x'}^m u$ if $g'_{\mu\nu}$ is the same function of x'^μ as $g_{\mu\nu}$ is of x^μ. Thus

$$g'_{\mu\nu}(x^\mu) = g_{\mu\nu}(x^\mu) \tag{2.8}$$

Now consider the infinitesimal transformation

$$x'^{\mu} = x^{\mu} + a\xi^{\mu}, \qquad |a| \ll 1 \tag{2.9}$$

If we substitute to

$$g_{\mu\nu}(x) = \frac{\partial x'^{\rho}}{\partial x^{\mu}} \frac{\partial x'^{\kappa}}{\partial x^{\nu}} g'_{\rho\kappa}(x') = \frac{\partial x'^{\rho}}{\partial x^{\mu}} \frac{\partial x'^{\kappa}}{\partial x^{\nu}} g_{\rho\kappa}(x') \tag{2.10}$$

We take

$$g_{\mu\kappa} \frac{\partial \xi^{\mu}}{\partial x^{\rho}} + g_{\rho\mu} \frac{\partial \xi^{\mu}}{\partial x^{\kappa}} + \frac{\partial g_{\rho\kappa}}{\partial x^{\mu}} \xi^{\mu} = 0 \tag{2.11}$$

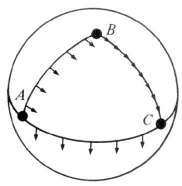
Using the definition of covariant derivative, we obtain the Killing equation with ξ^{μ} to be the Killing vector.

$$\xi_{\kappa;\rho} + \xi_{\rho;\kappa} = 0 \tag{2.12}$$

Thus, if a solution of the Killing equation exists, the corresponding Killing vector represents an infinitesimal isometry of the metric and implies that the metric has a certain symmetry. In any transformed coordinate system the metric has a corresponding isometry. This is important because the form of the metric changes according to the coordinate system.

2.3 Parallel Transport

The curvature of a manifold could be characterized by the parallel transport of a vector along a closed curve. If the manifold is provided by an *(affine) connection* then the vector could be transported parallelly, along a closed curve according to this *connection*. The following sketch was taken by [27].



The procedure of parallel transport, described by Christoffel's symbols or components of the connection, explains the transformation of basis vectors under infinitesimal shifts. Furthermore, parallel transport is referred to as the transfer of information along a curve within the manifold. In this context, "information" specifically pertains to the knowledge about local geometry and its connection with nearby points through parallel transport. This is why it is called a "connection". In essence, the connection is related to the infinitesimal parallel transport, and the parallel transport represents the local implementation of the (affine) connection.

Finally, it's worth noting that parallel transport is a crucial process that allows us to define the connection on a manifold.

2.4 Geodesics

A geodesic curve is the shortest curve between two points. As we know, in flat spacetimes, this curve coincides with a straight line in contrast with curved spaces where geodesics depend from the geometry of the spacetime. Also, the geodesics are curves that host an inertial observer. Someone could conclude that along a geodesic there is only parallel transportation of the vector of the observer which is tangent to the curve. In a few lines we will notice that when a vector, that represents an inertial observer (that means parallel transportation), is tangent along a curve, this curve is a geodesic curve.

A curve in a pseudo-Riemannian manifold, as our spacetime, is defined by

$$x^{\mu} = f^{\mu}(\lambda) \tag{2.13}$$

where the parameter λ is a scalar.

A random vector $y^{\kappa}(x^{\mu})$ which is moving parallel along the curve shouldn't be changed. The tangent vector of a curve at every point is equal to

$$u^{\mu} = \frac{dx^{\mu}}{d\lambda} \tag{2.14}$$

If we consider a curve where the arbirtrary vector which is parallely transported, the conservation law is obtained, since the considered curve is a geodesic,

$$u^{\mu} u^{\kappa}{}_{;\mu} = 0 \tag{2.15}$$

The Geodesic equations is

$$\frac{d^2 x^{\mu}}{d\lambda^2} + \Gamma^{\mu}{}_{\lambda\kappa} \frac{dx^{\lambda}}{d\lambda} \frac{dx^{\kappa}}{d\lambda} = 0 \tag{2.16}$$

Hence a geodesic in a curved spacetime is the curve when a parallel transport of a vector makes the vector to be tangent to the curve in every point.

In the end, a necessary condition for computing the geodesic curves is the knowledge of Christoffel's symbols. The metric tensor for a physical spacetime is obtained by solving the Einstein's Equations.

2.5 Einstein's Field Equations

Generally, Einsten's equations are characterized as non-linear partial differential equations. This characterization noting the level of difficulty met whenever exact solutions are extracted. The Einstein's Field Equations, in accordance to Newtonian physics, corresponds to the newtonian field equations

$$\nabla^2\phi = 4\pi G\rho, \tag{2.17}$$

where ρ is the mass density and ϕ represents the Newtonian gravitational potential which is dimensionless in units where the velocity $c = 1$.

Trying to describe the Newtonian analogue of Einstein's equations, we must start by the causal relation between the mass density and the gravitational field. The mass density produces the gravitational field which distorts simultaneously the spacetime. The relativistic analogue of the previous input is described by the Stress-Energy tensor $T_{\mu\nu}$, therefore the EFE are

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu} \tag{2.18}$$

2.5.1 Stress-Energy Tensor

The concept of field equations in General Relativity (GR) represents the conservation of energy and momentum. In accordance with the Newtonian analogy, the source responsible for the "twisting" of spacetime is the stress-energy tensor, denoted as $T_{\mu\nu}$. This tensor comprises various components, including the energy density T_{00}, the energy flux T_{0i} (equivalent to the momentum density T_{i0}), and due to the tensor's symmetry, the pressure (e.g of the star's fluid), denoted as T_{ii}, as well as the shear stresses, which make up the remaining elements of the tensor. It is important to note that the stress-energy tensor is divergenceless, which signifies the physical principle of energy and momentum conservation.

In a broader context, the stress-energy tensor $T_{\mu\nu}$ represents the flux of the μ component of momentum through a constant ν surface. A physically reasonable energy-momentum tensor must adhere to the dominant energy condition: the local energy density, as measured by an observer with velocity u^μ, is non-negative, and the local energy flow vector q^α is not spacelike. The last condition are expressed as

$$T_{\alpha\beta}u^\alpha u^\beta \geq 0 \tag{2.19}$$

$$q^\alpha q_\alpha \geq 0, \qquad q^\alpha \equiv T^\alpha_\beta u^\beta \tag{2.20}$$

The dominant energy condition should hold for all timelike vectors u^μ and, by continuity, these inequalities must still be true if we replace u^μ by a null vector. For Segré type [111,1] [28] (and its degeneracies), $T_{\alpha\beta}$ can be diagonalized, so that $T_{\alpha\beta} = diag(e, p_x, p_y, p_z)$ and then

$$e \geq 0, \qquad -e \leq p_i \leq e$$

These inequalities hold for a non-null electromagnetic field and impose reasonable restrictions on the energy density e and pressure p ($p = p_x = p_y = p_z$) of a perfect fluid. The dominant energy condition is also satisfied by the energy-momentum tensors of pure radiation field and null electromagnetic fields. Types $[11, Z\bar{Z}]$ and $[1,3]$ (and their degeneracies), violate even the weak energy condition. Therefore these types are not physically significant [29].

2.5.2 The Curvature Tensor

The Curvature tensor or Riemann Curvature tensor measures the degree in which the metric tensor is not locally isometric to that of flat spacetime via the procedure of parallel transport. Indeed, choosing two different infinitesimal displacements δx^{μ} and dx^{μ} at the same point P, considering a parallelogram $PABCP$ and applying the procedure of parallel transport to a vector u^{λ} at P. Then we take an infinitesimal change of the vector which is analogous to the Riemann tensor.

$$\delta u^{\lambda} = -\frac{1}{2} R^{\lambda}{}_{\rho\mu\nu} u^{\rho} (dx^{\mu} \delta x^{\nu} - dx^{\nu} \delta x^{\mu}) \tag{2.21}$$

The previous computation is nothing else but the integration of the infinitesimal element of u^{λ}, along the parallelogram $PABCP$, which is given by

$$\delta u^{\nu} = -\Gamma^{\lambda}{}_{\mu\nu} dx^{\nu} \tag{2.22}$$

As we see from the previous relations, which were taken by [30], the infinitesimal parallel transportations of a vector u^{λ} characterizes the curvature of a surface via the form of the Riemann Tensor. The infinitesimal δu^{λ} is equal to zero, if all the components of Riemann tensor are equals to zero. In this case we take the flat spacetime.

However the definition of this tensor is the following

$$R^{\rho}{}_{\lambda\mu\nu} = -\Gamma^{\rho}{}_{\lambda\mu,\nu} + \Gamma^{\rho}{}_{\lambda\nu,\mu} - \Gamma^{\kappa}{}_{\lambda\mu} \Gamma^{\rho}_{\kappa\nu} + \Gamma^{\kappa}{}_{\lambda\nu} \Gamma^{\rho}{}_{\kappa\mu} \tag{2.23}$$

The previous relation denotes the antisymmetricity of the tensor in the last two indices.

$$R^{\rho}{}_{\lambda\mu\nu} = -R^{\rho}{}_{\lambda\nu\mu} \qquad\qquad R^{\rho}{}_{\lambda[\mu\nu]} = 0 \tag{2.24}$$

Since we have a metric, we can lower the upper index ρ in order to express the Riemann tensor in terms of trace-free tensor quantities. This expression is always known as the **decomposition of Curvature tensor**,

$$R_{\lambda\mu\nu\rho} = g_{\lambda\alpha} R^{\alpha}{}_{\mu\nu\rho} = C_{\lambda\mu\nu\rho} + E_{\lambda\mu\nu\rho} + G_{\lambda\mu\nu\rho} \tag{2.25}$$

$$E_{\lambda\mu\nu\rho} = \frac{1}{2}(g_{\lambda\rho} S_{\mu\nu} + g_{\mu\nu} S_{\lambda\rho} - g_{\lambda\nu} S_{\mu\rho} - g_{\mu\rho} S_{\lambda\nu}) \tag{2.26}$$

$$G_{\lambda\mu\nu\rho} = \frac{R}{12}(g_{\lambda\nu}g_{\mu\rho} - g_{\lambda\rho}g_{\mu\nu}) = \frac{R}{12}g_{\lambda\mu\nu\rho} \tag{2.27}$$

The tensor $C_{\lambda\mu\nu\rho}$ is the **Weyl tensor** or the *conformal curvature tensor* because the conformal transformation doesn't affect it and satisfies the same symmetry identities as Riemann tensor. The Weyl tensor is responsible for the tidal forces due to gravity and its the only tensor that survives in vaccum since the Ricci tensor, and by extension its scalar, are zero.

The tensor $S_{\mu\nu}$ is the traceless part of the Ricci tensor and it is defined as follows.

$$S_{\mu\nu} = R_{\mu\nu} - \frac{1}{4}Rg_{\mu\nu} \tag{2.28}$$

Finally, from the Curvature tensor, because of its order (1,3), can constucted other tensors via contraction like the bespoken Ricci tensor

$$R_{\mu\nu} \equiv R^{\rho}{}_{\mu\nu\rho} \tag{2.29}$$

The Ricci tensor, even if is not divergeless, it is used in order to construct the divergenceless Einstein's tensor $G_{\mu\nu}$. As we can see, the divergence of Ricci is taken by

$$R^{\alpha\beta}{}_{;\beta} = (\frac{1}{2}g^{\alpha\beta}R)_{;\beta} \qquad R \equiv g^{\mu\nu}R_{\mu\nu} \tag{2.30}$$

In the end, the Einstein's tensor is defined by the following relation in order to be divergenceless,

$$G^{\alpha\beta} \equiv R^{\alpha\beta} - \frac{1}{2}g^{\alpha\beta}R \tag{2.31}$$

Finally, we take a flavor about the construction of a consistent relativistic theory of gravity incarnated mathematically by Einstein's Field Equations,

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = \frac{8\pi G}{c^4}T_{\mu\nu} \tag{2.32}$$

These equations construes the gravity as a declaration of spacetime curvature induced by the presence of matter. The curvature, as an inborn property of our spacetime, is embodied in the equations. The resolving of the equations, which incorporate the inborn property of spacetime's curvature, is a difficult task which is shown by the limited number of solutions.

Chapter 3

Exact Solutions of Einstein's Equations

The evolution of mathematical models into physical theories requires the settlement of systems of differential equations, translating their solutions into physical meaningful statements about the physical world. For this reason, this procedure is so pretentious as necessary, specifically in GR, because of the intractability of the equations. The Einstein's field equations are a set of 10 non-linear, hyperbolic-elliptic partial differential equations of second order in respect to the metric tensor.

When considering physical theories, it is important to recognize that they must adhere to certain fundamental principles and symmetries. Consequently, exact solutions of EFE are obligated to meet the corresponding physical criteria. These exact solutions play a crucial role in describing the gravitational fields of both interior and exterior stellar models, encompassing various celestial bodies such as stars, neutron stars, and black holes, as well as contributing to cosmological models.

In order to construct a different exact solution, it is necessary to impose restrictions. These restrictions are different for each solution since their form vary. Some of the restrictions are imposed to the stress-energy-momentum tensor defining the kind of matter that will be used as source of gravity. Specifically the implication of symmetries determines the physical substance of our solution as well as the form of our metrics.

However, the last decades there were developed different techniques which managed to construct new analytical models using the already known exact solutions [31], [32]. The Scalar Field theories is the main player in this game providing us with new results. Along with the known analytical solutions of EFEs there are theories which manage to solve the problems that arise during a "soldering" between solutions, as it happens with exact solutions embedded in cosmological backgrounds [33] [34].

In this chapter[1] we are going to present some of the most known exact solutions. The differences between them are based mainly to symmetries, since most of them are calculated with the stress energy-momentum tensor to be absent.

[1]The sketches in the following segment are taken by [27].

3.1 Static Spacetime – Spherical Symmetric Solutions

The publication of the theory of General Relativity [1] was followed by the first exact solution in the same year by Schwarzschild [5]. It isn't a coincidence that the first exact solution concerns a static spacetime with spherical symmetry since these two assumptions form "approximately" the simplest physical situation of an exterior gravitational field of an isolated stellar model. The first assumption is the spherical symmetry, which is equivalent with lack of rotation, and the second approximation is the lack of variation over time.

3.1.1 Exterior Schwarzschild Solution

"Spatial spherical symmetry is assumed and a corresponding exact solution for Einstein's theory searched for. After a historical outline, we apply the equivalence principle to a freely falling particle and try to implement that on top of the Minkowskian line element. In this way, we heuristically arrive at the Schwarzschild metric.."

"It is quite a wonderful thing that from such an abstract idea the explanation of the Mercury anomaly emerges so inevitably."

Karl Schwarzschild (1915)

In fact, imposing spherical symmetry is equivalent to changing from Cartesian coordinates to spherical coordinates, while preserving the expression $d\theta^2 + \sin^2\theta d\phi^2$ invariant. Therefore, only the g_{tt} and g_{rr} components of the metric will exhibit radial dependence. The static spacetime is attained by requiring that time is independent of these two metric components. These constraints, in conjunction with the Einstein Field Equations (EFE), yield the Schwarzschild solution.

It's worth noting that Schwarzschild initially began with an approximate solution in Einstein's "perihelion paper," published on November 25th. Fortunately, the preliminary field equation in the "perihelion paper" is correct in the vacuum case. This solution specifically describes the gravitational field of a spherically symmetric, stationary body. As we mentioned earlier, this remains an approximation of stellar models still in use today.

Furthermore, it is necessary to impose the condition that the solution approaches flat spacetime as $r \to \infty$, ensuring asymptotic flatness. Schwarzschild, with all these constraints, ultimately succeeded in finding his solution, *Schwarzschild metric*

$$ds^2 = \left(1 - \frac{2M}{r}\right) dt^2 - \frac{1}{1 - \frac{2M}{r}} dr^2 - r^2(d\theta^2 + \sin^2\theta d\phi^2) \tag{3.1}$$

Now we can observe the static nature of the solution, where its geometry remains unchanged under time reversal ($t \to -t$). This unchanging geometry for every moment signifies the unrealistic nature of the solution. However, what happens when we consider non-static spherical geometries?

In 1923, Birkhoff provided an answer, though it referred only to the static case [35]. Birkhoff's theorem states that when we choose spherical symmetry, solving Einstein's Field Equations (EFE) will inevitably lead us to a static and asymptotically flat geometry. The theorem's generalization extends to situations where we choose spherical symmetry and solve Einstein-Maxwell's Field Equations, yielding a unique result: a stationary and asymptotically flat geometry. An example of the static case in this generalization is the Reissner-Nordström solution [36].

This theorem has significant implications, not only for these solutions but for an entire class of solutions representing spherical symmetry in vacuum solutions. Therefore, any spherically symmetric solution that allows for external gravitational fields must be static.

Birkhoff's theorem and its generalization have brought crucial outcomes. One remarkable achievement was the utilization of the Schwarzschild solution to unravel the mystery of the precession of Mercury's perihelion—an observation that required confirmation through the resolution of geodesics. This confirmation was eventually achieved! The calculation of geodesics accurately matched the observed value of $43''$ of arc. This represents the precession of Mercury's perihelion when its cycle time is approximately 0.24 years. This result is significant, as it goes beyond the predictions of Newtonian gravity and showcases the impressive agreement between theoretical predictions and observations in General Relativity.

The Schwarzschild spacetime as a Black Hole

"We are thus driven to consider the consequences of a situation in which a star collapses right down to a state in which the effects of General Relativity become so important that they eventually dominate over all other forces."

R. Penrose (1969)

In General Relativity, there is the concept of gravitational collapse which concerns stars whose masses are so large where the equilibrium disappears and its total mass contracts to a singularity [37], creating a black hole. Penrose in this article in Nuovo Cimento studied the collapsing of a Schwarzschild geometry to a Black Hole. But what is a Black Hole? A better definition is that **it is a region where space is falling faster than light**. However, the exterior field remains the same after the collapsing.

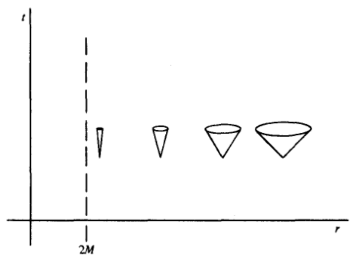
The "sensitive" point of Schwarzschlid's metric, in our case, is when the components of the metric show an abnormal behaviour. This case is the vulnerable point when $r = 2m$ and represents the borders of a black hole [2] where it is widely known as "Event Horizon". In case that we set the radius r to be equal with $2m$ we take the following values for the components.

$$g_{00} = g^{11} = 0 \qquad g^{00} = g_{11} = \infty$$

A more serious anomaly is that of geodesics. The radial geodesics, with constant values of t, θ, ϕ are timelike in the interior region $r < 2m$ spacelike in the exterior $r > 2m$. Because of parallel transport, these two regions must connect smoothly at the event horizon $r = 2m$.

The event horizon is the boundary in spacetime which connects events that can communicate with events that cannot. In other words, it is a boundary in spacetime that separates the events in two regions: trapped events inside the horizon and untrapped events out of it.

The region $r > 2m$ represents the exterior gravitational field while the region $r < 2m$ represent the BH. An observer far away from the BH waits infinite time to get the information from an object crossing the horizon. The object, however, reaches the horizon in a finite time according to its clock. But the observer concludes that the infalling clock is slowing down and eventually stopping. A similar way to see this, is to admit that a photon experiences an infinite infrared. The horizon is a null surface after all, where the paths of the "marginal" null rays lay.



It is also implied that the communication of the falling observer is shrinking approaching the event horizon at $r = 2m$. This behaviour is manifested by solving the equation $ds^2 = 0$ for $\phi = const = \theta$ to find

$$\frac{dt}{dr} = \pm \frac{1}{1 - \frac{2m}{r}}$$

[2]The phrase "Black hole" is commonly associated with Wheeler in [38] and referred about a notorious dungeon in Calcutta in the 18th century, apparently a place of no return.

3.1.2 De-Sitter Solution

The de-Sitter spacetime is a maximally symmetric, conformally flat spacetime like Minkowski. The contraction of the Einstein's Field Equations, in case of vacuum, correlates the Ricci scalar with the appeared cosmological constant ($\Lambda > 0$) as $R = 4\Lambda$. The latter relation is where lies the difference between any other vacuum solution of GR. Although both de-Sitter and Minkowski spacetimes have zero Weyl components. In case where the Cosmological Constant takes negative values ($\Lambda < 0$) the spacetime is called anti-de-Sitter. The Riemann tensor takes the following form,

$$R_{\mu\nu\sigma\rho} = \frac{R}{12}(g_{\mu\sigma}g_{\nu\rho} - g_{\mu\rho}g_{\sigma\nu}) \tag{3.2}$$

The spacetime can be described by the following metric in spherically symmetric coordinates (T, r, θ, ϕ)

$$ds^2 = \left(1 - \frac{\Lambda}{3}r^2\right)dT^2 - \frac{1}{1 - \frac{\Lambda}{3}r^2}dr^2 - r^2(d\theta^2 + \sin^2\theta d\phi^2) \tag{3.3}$$

This metric is also called de-Sitter-Schwarzchild spacetime. With this metric, arranging the values of the cosmological constant we can easily obtain the anti-de-Sitter spacetime for $\Lambda < 0$ and the Minkowski spacetimes for $\Lambda = 0$. The coordinates varies as follows, $T \in (-\infty, +\infty)$, $r \in \left[0, \sqrt{\frac{3}{\Lambda}}\right]$, $\theta \in [0, \pi)$ and $\phi \in (-\infty, +\infty)$.

This spacetime describes a hyperboloid with radius $\sqrt{\frac{3}{\Lambda}}$

$$\frac{3}{\Lambda} = -Z_0^2 + Z_1^2 + Z_2^2 + Z_3^2 + Z_4^2 \tag{3.4}$$

embedded in a 5-dimensional Minkowski spacetime.

$$ds^2 = -dZ_0^2 + dZ_1^2 + dZ_2^2 + dZ_3^2 + dZ_4^2, \tag{3.5}$$

where the Z's are connected with the spherical symmetric coordinates as follows.

$$\begin{aligned} Z_0 &= \sqrt{\frac{3}{\Lambda} - r^2}\sinh\left(T\sqrt{\frac{\Lambda}{3}}\right) \\ Z_1 &= \sqrt{\frac{3}{\Lambda} - r^2}\cosh\left(T\sqrt{\frac{\Lambda}{3}}\right) \\ Z_3 &= r\cos\theta \\ Z_3 &= r\sin\theta\cos\phi \\ Z_4 &= r\sin\theta\sin\phi \end{aligned} \tag{3.6}$$

This is a very interesting spacetime since it is divided in four regions. The interior region, with $r < \sqrt{\frac{3}{\Lambda}}$, where the spacetime is static. The spacetime for $r > \sqrt{\frac{3}{\Lambda}}$ is characterized as non-static where r obtains a timelike character and T becomes spacelike. Thus, the domain of the timelike coordinate r is $\left(\sqrt{\frac{3}{\Lambda}}, +\infty\right)$. This affects also the square roots of Z_0 and Z_1 with a change of a sign describing the other two regions. However the radius of the hyperboloid is considered geometrically as a Killing horizon since the Killing vector ∂_T is becoming null.

3.1.3 General 2-Product Spaces of Constant Curvature

In this section we will array a complete family of direct product of 2-dimensional spaces of constant curvature. This family includes Bertotti-Robinson universe [39], [40], Nariai ($\Lambda > 0$), anti-Nariai ($\Lambda < 0$) universe [41] and Plebański-Hacyan spacetime[42].

For the 2-product spaces we know that they admit a 6-dimensional isometry group and this is the reason that the curvature is constant (Theorem 8.15) at [19]. **Hence the metric of a 2-dimensional constant curvature (K) can be written for any value of K and any metric signature as follows** [43]

$$ds^2 = \frac{dudv}{1 + K\frac{uv}{4}}$$

Consequently any two metrics with the same constant curvature and signature must be locally equivalent[44]. A vector space V_4 with $K \neq 0$ can be considered as a hypersurface. Along these lines, the following metric describes a type D electrovacuum spacetime with cosmological constant in 6-dimensional flat representation [45].

$$ds^2 = dZ_0^2 - \epsilon_1 dZ_1^2 - dZ_2^2 - dZ_3^2 - dZ_4^2 - \epsilon_5 dZ_5^2 \tag{3.7}$$

This is a general metric and describes two submanifolds which are planes, spheres, hyperboloids depended by the sign of ϵ_1, ϵ_2. In this metric the coordinates are constrained by the surfaces of the submanifolds.

$$\epsilon_1(-Z_0^2 + Z_1^2) + Z_2^2 = a^2 \qquad \epsilon_2(Z_3^2 + Z_4^2) + Z_5^2 = b^2 \tag{3.8}$$

Using the follow transformation [46]

$$Z_0 = \frac{v - u}{\sqrt{2}(1 - \frac{\epsilon_1}{2a^2}uv)} \quad Z_1 = \frac{v - u}{\sqrt{2}(1 - \frac{\epsilon_1}{2a^2}uv)} \quad Z_2 = a\frac{1 + \frac{\epsilon_1}{2a^2}uv}{1 - \frac{\epsilon_1}{2a^2}uv} \tag{3.9}$$

$$Z_3 = \frac{\zeta + \bar{\zeta}}{\sqrt{2}(1 + \frac{\epsilon_2}{2b^2}\zeta\bar{\zeta})} \quad Z_4 = -i\frac{\zeta - \bar{\zeta}}{\sqrt{2}(1 + \frac{\epsilon_2}{2b^2}\zeta\bar{\zeta})} \quad Z_5 = b\frac{(1 - \frac{\epsilon_2}{2b^2}\zeta\bar{\zeta})}{(1 + \frac{\epsilon_2}{2b^2}\zeta\bar{\zeta})}$$

we acquire this direct product of spacetimes are depended by the general parameters a,b,ϵ_1, ϵ_2. This is a natural coordinate system to present this family of spacetimes.

$$ds^2 = \frac{2dudv}{(1 - \frac{\epsilon_1}{2a^2}uv)^2} - \frac{2d\zeta d\bar{\zeta}}{(1 + \frac{\epsilon_2}{2b^2}\zeta\bar{\zeta})^2} \tag{3.10}$$

This metric gives multiple spacetimes with appropriate choices of ϵ_1 and ϵ_2 which take values $0, \pm 1$. However, only 6 spacetime geometries are physically reasonable, due to that the energy density of Ψ_{11} has a postitive value [45].

This family is of type D according to Petrov classification. The type D character allows only the Weyl component Ψ_2 to be non-zero

$$C_{1342} = \Psi_2 = -\frac{1}{6}\left(\frac{\epsilon_1}{a^2} + \frac{\epsilon_2}{b^2}\right) \tag{3.11}$$

The Ricci component Φ_{11} and the Ricci scalar for this family are

$$\Phi_{11} = \frac{1}{4}(-\frac{\epsilon_1}{a^2} + \frac{\epsilon_2}{b^2}) \qquad R = 4\Lambda = 2(\frac{\epsilon_1}{a^2} + \frac{\epsilon_2}{b^2}) \tag{3.12}$$

As we mentioned this metric describes two product spaces with constant curvature. The Gaussian curvature of these two 2-dimensional spaces are as follows.

$$K_1 = \frac{\epsilon_1}{a^2} \qquad K_2 = \frac{\epsilon_2}{b^2} \tag{3.13}$$

Regarding these relations, some of the most interesting cases emerge when $a = b$ since the equivalence of ϵ_1 and ϵ_2 gives vacuum or conformally flat spacetimes with the annihilation of Λ. Minkowski spacetime arises with annihilation of ϵ_1, ϵ_2. The choice $\epsilon_1 = -\epsilon_2 = -1$ yields the Bertotti-Robinson cosmological model and Nariai spacetime is obtained with $\epsilon_1 = \epsilon_2 = +1$. If the values $a = b = R_c$, then the final form is the following,

Nariai metric, $\Lambda > 0$

$$ds^2 = \frac{2dudv}{(1 - \frac{uv}{2R_c^2})^2} - \frac{2d\zeta d\bar{\zeta}}{(1 + \frac{\zeta\bar{\zeta}}{2R_c^2})^2} \tag{3.14}$$

An interesting geometric method to find the "generalized Robinson-Bertotti spaces" was developed by Burdet, Papakostas and Perrin using null hypersurfaces (ishyps) [47]. These spaces are characterized by the following combination of spin coefficients and admit an energy-momentum tensor with null electromagnetic field and a six-parameter group of motion. The parameters λ and e are the cosmological constant and the electric charge respectively,

$$\kappa = \nu = \sigma = \lambda = \rho = \mu = \pi = \tau = 0$$

$$\alpha = \beta = imaginary$$

$$ds^2 = \frac{2dudv}{\left[1 + (\frac{\lambda}{2} + e^2)uv)\right]^2} - \frac{2d\zeta d\bar{\zeta}}{\left[1 + (\frac{\lambda}{2} - e^2)\zeta\bar{\zeta})\right]^2}$$

3.2 Stationary Spacetime – Axially Symmetric Solutions

The search for axially symmetric solutions to the equations of General Relativity began in 1917 with static metrics and after 40 years of research, the first stationary and axially symmetric vacuum solution was finally discovered by Roy Kerr in 1963 [4].

Solutions that describe stationary and axisymmetric spacetimes hold great significance, as they allow for the construction of realistic models within the framework of General Relativity. This is in contrast to spherically symmetric solutions, which are considered non-realistic. Constructing a stationary and axially symmetric solution is fundamentally based on symmetries and physical assumptions.

A stationary spacetime is defined as one in which there exists a distinguished family of observers who observe the spacetime geometry remaining unchanged as proper time flows. An axisymmetric spacetime is characterized by a geometry that remains the same under rotations around a specified spacelike line [48]. It is also logical to assume that this rotation is steady, which is equivalent to stationarity. Additionally, we assume that both the star and the surrounding field possess axial symmetry around the axis of rotation (the z-axis), which passes through the origin of coordinates at the center of the star.

To achieve a stationary and axially symmetric spacetime, certain restrictions must be imposed on the metric with respect to symmetry. This process is not as straightforward as it is in spherical symmetry. The symmetries of stationarity and axisymmetry are imposed on the metric components independently of the time coordinate, denoted as $x^0 = t$, and the azimuthal angle coordinate, denoted as $x^3 = \phi$, since the star's field remains invariant when these coordinates are reversed. This, along with symmetry considerations, determines that

$$g_{01} = g_{02} = 0 = g_{23} = g_{13}$$

So a stationary and axially symmetric metric has the form

$$g_{\mu\nu} = \begin{bmatrix} g_{00} & 0 & 0 & g_{03} \\ 0 & g_{11} & 0 & 0 \\ 0 & 0 & g_{22} & 0 \\ g_{30} & 0 & 0 & g_{33} \end{bmatrix} \tag{3.15}$$

The most general solutions characterized by stationarity and axial symmetry belong to type D according to the Petrov Classification[3].

The most general family of solutions that includes a cosmological constant and a non-null electromagnetic field is known as the Debever or Plebański-Demiański solution. However, all type D metrics in a vacuum without a cosmological constant were discovered by Kinnersley [49].

[3]The Petrov classification is an invariant geometric characterization of the gravitational field. In essence, it categorizes the different canonical forms of the Weyl tensor, as we present in the following chapter.

3.2.1 Debever - Plebański - Demiański Solution

This is the most general type D axially symmetric family of solutions with cosmological constant and non-null electromagnetic field, and it was found by Debever in 1971 [50], [51], by Debever, Kamran, McLenaghan [52], and by Plebański and Demiański [53] who presented it in a more convenient way 5 years later [45].

The authors in [17] permit the annihilation of the cosmological constant and the electromagnetic tensor in case where the following conditions hold:

H1. The Weyl tensor is everywhere type D, which is equivalent to the existence of real null vector fields satisfying at every point the relation

$$n^{\nu} n^{\sigma} C_{\mu\nu\sigma[\rho} n_{\kappa]} = l^{\nu} l^{\sigma} C_{\mu\nu\sigma[\rho} l_{\kappa]} = 0, \tag{H1}$$

where the null vectors are basically **the Principal Null Directions (PND)**.

H2. If the Maxwell field tensor is nonzero, it is nonsingular with its PND of the Weyl tensor, namely

$$n^{\mu} F_{\mu[\nu} n_{\sigma]} = l^{\mu} F_{\mu[\nu} l_{\sigma]} = 0 \tag{H2}$$

H3. The invariants of the Weyl tensor and the trace-free Ricci tensor (2.31) must satisfy the following relations

$$C_{\mu\nu\sigma\rho} C^{\mu\nu\sigma\rho} \neq 0 \tag{H3}$$

$$C_{\mu\nu\sigma\rho} C^{\mu\nu\sigma\rho} \neq \frac{4}{3} S_{\mu\nu} S^{\mu\nu}$$

The metric for this general family is described by the following expression.

$$ds^2 = \frac{1}{[1-\alpha pr]^2} \left[\frac{Q(r)[dt - \omega p^2 dz]^2}{r^2 + \omega^2 p^2} - \frac{P(r)[\omega^2 dt + r^2 dz]^2}{r^2 + \omega^2 p^2} - (r^2 + \omega^2 p^2) \left[\frac{dr^2}{Q(r)} + \frac{dp^2}{P(p)} \right] \right] \tag{3.16}$$

$$P(p) = -\left[\alpha(\omega^2 k + e^2 + g^2) + \omega^2 \frac{\Lambda}{3} \right] p^4 + 2\alpha m p^3 - \epsilon p^2 + 2\frac{2n}{\omega} p + k \tag{3.17}$$

$$Q(r) = -\left(\alpha^2 k + \frac{\Lambda}{3} \right) r^4 - \frac{2\alpha n}{\omega} r^3 + \epsilon r^2 - 2mr + \left(\omega^2 k + e^2 + g^2 \right) \tag{3.18}$$

where α, ω was chosen for convenience, also $k, e, g, p, m, n, \epsilon$ are real constants of integration and Λ is the cosmological constant. The type D character of the solution is evident since the only non-zero component of Weyl tensor in terms of Newman-Penrose formalism is Ψ_2,

$$C_{1342} = \Psi_2 = \left(\frac{1 - \alpha pr}{r + i\omega p} \right)^3 \left[(e^2 + g^2) \frac{1 + \alpha pr}{r - i\omega p} - (m + in) \right] \tag{3.19}$$

In order to have Lorentzian signature maintenance, the field $P(p)$ must be always positive while the sign of $Q(r)$ determines the character of the two Killing vectors ∂_t and ∂_z. In case where $Q < 0$, these two vectors are spacelike, while for $Q > 0$, ∂_t is timelike and ∂_z spacelike.

Vacuum is obtained where the constants $m = n = e = g = \Lambda$ are equal to zero. It is worth noting that for this case we have $\kappa = \nu = \sigma = \lambda = 0$, which indicates that the Goldberg-Sachs theorem holds. **Goldberg and Sachs claim that a vacuum space-time is algebraically special if, and only if, it possesses a shear-free geodesic null congruence ($\kappa = \sigma = 0$)**, [45], [54]. Finally the solution with the appropriate choice of constants could be reduced in numerous known metrics.

3.2.2 Carter's Family $[\tilde{\mathcal{A}}]$

"*Brandon Carter's (1968) paper was one of the most significant papers on the Kerr metric during the mid-sixties*".

R. Kerr (2008)

The Carter's Family of metrics $[\tilde{\mathcal{A}}]$ is a general family which was constructed with a specific manner by Carter in order to combine his different metric-cases [A], [B(±)], [C(±)] and [D] in a single formula. Also, it can be characterized by the existence of a second-rank Killing tensor with two double eigenvalues λ_1, λ_2 and a two parameter Abelian isometry group of motion with non-null surfaces of transitivity, and orbits either timelike or spacelike [55], [20].

Indeed, in [56] the Carter's Family was established as one of the most general stationary axisymmetric solution in electrovacuum. As Carter said: "*A new family of solutions of the Einstein-Maxwell equations (with [A] term) is presented, combining many well known but previously unrelated metrics, including the vacuum solutions of de-Sitter, Kasner, Taub-N.U.T. and Kerr, within a single formula*".

The Carter's family $[\tilde{A}]$ of metrics can be written as follows [57]

$$ds^2 = (\Phi+\Psi)\{\frac{fE^2}{(B-A)^2}(dt+Adz)^2 - \frac{H^2}{(B-A)^2}(dt+Bdz)^2 - \frac{f(\Psi_y dy)^2}{4G^2} - \frac{(\Phi_x dx)^2}{4F^2}\} \quad (3.20)$$

Where λ_1, λ_2 are the double eigenvalues of the admitted Killing tensor

$$\lambda_1 = \Phi(x), \qquad \lambda_2 = \Psi(y)$$

$$A = A(x), \qquad H = H(x), \qquad F = F(x)$$

$$B = B(y), \qquad E = E(y), \qquad G = G(y)$$

for $f = +1$, there is one timelike Killing vector $\partial/\partial t$ and one spacelike Killing vector $\partial/\partial z$, and this is the axisymmetric case. For $f = -1$, the Killing vectors are both spacelike. We consider in this paper only the $f = +1$ case. For all plausible energy-momentum tensors, the traceless Ricci components have to be real, the only complex component is Φ_{01} and its imaginary part is equal to

$$\frac{3}{4}\frac{i}{(\Phi+\Psi)}\frac{4GF}{\Phi_x\Psi_y}\left[\ln\left(\frac{\Phi+\Psi}{B-A}\right)\right]_{xy}$$

The vanishing of this expression guides us to define

$$B(y) = \Psi(y), \qquad A(x) = -\Phi(x)$$

Because of the form of the metric in g_{yy}, g_{xx}, it is convenient to redefine our $\Psi(y)$, $\Phi(x)$ as

$$\Psi(y) \equiv \tilde{y}^2 \qquad \Phi(x) \equiv \tilde{x}^2$$

Hence, if we substitute the new relations, we take the **Carter's Family [A]** of metrics

***Carter's Case* [A]**

$$ds^2 = \frac{E^2(y)}{(x^2+y^2)}(dt - x^2dz)^2 - \frac{H^2(x)}{(x^2+y^2)}(dt + y^2dz)^2 - (x^2+y^2)\left[\frac{dy^2}{E^2(y)} + \frac{dx^2}{H^2(x)}\right] \tag{3.21}$$

$$E^2(y) = \frac{\Lambda}{3}y^4 + hy^2 - 2my + p + e^2 \tag{3.22}$$

$$H^2(x) = \frac{\Lambda}{3}x^4 - hx^2 + 2qx + p \tag{3.23}$$

$$F = e\left(\frac{yx(x\cos\gamma + y\sin\gamma)}{x^2+y^2}dt - \frac{y\cos\gamma - x\sin\gamma}{x^2+y^2}dz\right) \tag{3.24}$$

where Λ is the cosmological constant, m is the mass of a particle, q is the NUT parameter, p is the angular momentum and e is the electric charge. The parameter γ is an arbitrary angle determining the complexion of the electromagnetic field, and it has no effect on the metric [55].

***Carter's Case* [D]**

This is a 2 product spacetime with constant curvature. This is a quite general solution that was discovered by Plebański as his Case C [58], by Carter as case [D] [55], by Hauser and Malhiot as Case (0,0) with two Killing Vectors ∂_3, ∂_4 for $\epsilon = +1$ or ∂_1, ∂_2 for $\epsilon = -1$ [13], and by Kasner independently [59],

$$ds^2 = E^2(y)dt^2 - H^2(x)dz^2 - \frac{dy^2}{E^2(y)} - \frac{dx^2}{H^2(x)} \tag{3.25}$$

$$E^2(y) = (\Lambda + e^2)y^2 - 2my + p \tag{3.26}$$

$$H^2(x) = (\Lambda - e^2)x^2 + 2qx + p \tag{3.27}$$

$$F = e(ycos\gamma dt - xsin\gamma dz) \tag{3.28}$$

Solvability of Geodesics

The general family of metrics was introduced by Carter and was used extensively by Debever [50], Carter & McLenaghan [60], Papakostas [61], [62] and others. The family of spaces $[\tilde{\mathcal{A}}]$ is characterized by the fact that the Hamilton-Jacobi equation for the geodesics is solvable by separation of variables (x, y), giving rise to the fourth constant of motion, which is quadratic in the velocities. Equivalently, the Carter's family admits the existence of a Killing tensor.

Carter's family of metrics contains the integrability of geodesics as an innate characteristic, providing the required physical profile into the metric. The separation of Hamilton-Jacobi equation for the geodesics is solvable because it provides the conservation of four integrals of motion[4].

The latter has been imposed into the Killing equation whose vector p^α represents the canonical momentum of an observer,

$$p^\alpha \nabla_\alpha (K_{\mu\nu} p^\mu p^\nu) = 0 \tag{3.29}$$

The covariant derivative of the Killing tensor gives zero since the fourth constant of motion is defined as follows.

$$\mathcal{K} \equiv K_{\mu\nu} p^\mu p^\nu \tag{3.30}$$

Along these lines, the Killing equation takes the following form

$$p^\alpha \nabla_{(\alpha} K_{\mu\nu)} = 0 \Leftrightarrow K_{(\mu\nu;\alpha)} = 0 \tag{3.31}$$

In the context of NP formalism, the Killing tensor is written in terms of the tetrads. This was the assumed characteristic Killing tensor that was used by Hauser-Malhiot [13] giving rise to Carter's family $[\tilde{\mathcal{A}}]$

$$K^{HM}_{\mu\nu} = \lambda_1(n_\mu l_\nu + l_\mu n_\nu) + \lambda_2(\bar{m}_\mu m_\nu + m_\mu \bar{m}_\nu) = \begin{pmatrix} 0 & \lambda_1 & 0 & 0 \\ \lambda_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \lambda_2 \\ 0 & 0 & \lambda_2 & 0 \end{pmatrix} \tag{3.32}$$

[4]Three of these constants are the latitudinal components of the particle's momentum p_θ, the energy E and the axial angular momentum of the particle L, where the particle's rest mass is m and the angular momentum of the black hole is α.

3.2.3 Kerr-NUT Solution

One of the generalizations of Kerr solution [4] is the asymptotically non-flat Kerr-Newman, Unti and Tamburino (NUT) solution, which was introduced by Demiański and Newman in 1966 [63]. Kerr-NUT solution is the result of the combination between Kerr solution and NUT solution [64]. The most general axially symmetric solution is the charged Kerr-NUT solution (Kerr-Newman-NUT) but we know that the electrovac solutions are characterized as unphysical. For this reason, the charged Kerr-NUT solution does not consist a subject of study in this thesis.

However, Kerr-NUT as a stationary and axially symmetric solution is characterized by three parameters. They are the usual mass m, angular momentum per mass α and the NUT parameter b. **This spacetime is not asymptotically flat, and the departure from it, is the measure of the NUT parameter. In case that we impose asymptotic flatness, the reduction that we take corresponds to Kerr family of solutions**. This is therefore a very direct way to establish the uniqueness of the Kerr family as well. The metric of Kerr-NUT solution is given below in Boyer-Lindquist coordinates,

$$\begin{aligned} ds^2 = &\left[\frac{\Delta - \alpha^2 \sin^2\theta}{\rho^2} - \frac{\alpha b \cos\theta}{\rho^2}\right] dt^2 + \left[2\alpha \frac{2mr\sin^2\theta}{\rho^2} + \frac{2b\cos\theta(r^2+\alpha^2)}{\rho^2}\right] dt d\phi \\ &- \left[\frac{(r^2+\alpha^2)^2 - \alpha^2 \Delta sin^2\theta}{\rho^2}\sin^2\theta + \frac{b\cos\theta}{\alpha(r^2+\alpha^2\cos^2\theta)}(r^2+\alpha^2)^2\right] d\phi^2 \\ &- \frac{\rho^2}{\Delta} dr^2 - \rho^2 \frac{\alpha^2\sin^2\theta}{\alpha^2\sin^2\theta + b\alpha\cos\theta} d\theta^2 \end{aligned} \tag{3.33}$$

$$where \quad \Delta \equiv r^2 - 2mr + \alpha^2, \ \ \rho^2 \equiv r^2 + \alpha^2\cos^2\theta \tag{3.34}$$

The introduction of Kerr-NUT seperates the Kerr part from the NUT part, thus in case of $b = 0$ we obtain the Kerr metric in Boyer-Lindquists coordinates. Generally, speaking the asymptoticaly flat solutions are considered non-physical. From one point of view, this is correct since the gravitational field become negligible in magnitude at large distances.

If we try to defence the physical substance of non-asymptotically flat solutions we could say that the requirement of the asymptotical flatness isn't a necessary condition in order to consider an astronomical object, since **the region far away from the star is not described by an asymptotically flat spacetime. In reverse, an isolated star is embedded in a homogeneous, isotropic and expanded universe which is known as Friedmann–Lemaître–Robertson–Walker solution (FLRW)**. Concluding, the study of non-asymptotically flat spacetimes embedded in FLRW geometry, may gives insight into the modern theories about the spacetime around an isolated stellar object.

Chapter 4

Newman & Penrose Formalism

In recent decades, the use of tetrads has been proven highly advantageous, not only for the study of gravitational radiation but also for addressing problems related to exact solutions within the framework of General Theory of Relativity. The selection of tetrads depends on the inherent symmetries of the spacetime, which are constraints imposed by our specific problem.

By carefully choosing a suitable tetrad basis consisting of four linearly independent vector-fields, we can project the relevant quantities into this basis and ensure that the equations conform to it. This formalism is closely related to the Newman & Penrose formalism, which was initially proposed by Ezra T. Newman in 1962 [21].

4.1 The Concept of the Formalism

Einstein was sure about the existence of gravitational radiation and his equations confirmed that. This sureness may came from the fact that EFE are characterized hyperbolic, as well as the Maxwell equations.

Newman & Penrose formalism or Spin Coefficients formalism is a tetrad formalism based on the novelty of the use of null tetrads. The formalism was created by Newman & Penrose in order to study the gravitation radiation. The usage of null tetrads wasn't a random choice since the null tetrads represent isotropic light-like vectors.

"The underlying motivation for the choice of a null basis was Penrose's strong belief that the essential element of a spacetime is its light-cone structure, which firstly makes possible the introduction of a spinor basis, and on the second hand it will appear that the light-cone structure of the black-hole solutions of General Relativity is exactly of the kind that makes NP formalism most effective for grasp-

ing the inherent symmetries of these space-times and revealing their analytical richness."[65]

S. Chandrasekhar

The main concept of the formalism could be briefly described as follows. **The need to interpret the gravitational radiation more conveniently forces us to associate the Riemann tensor with isotropic null tetrads (light-like vectors)**. The latter could happen in a 3-dimensional complex bivector space (C_3) spanned by self-dual 2-forms.

The antisymmetricity of the electromagnetic tensor $F_{\mu\nu}$ provides us with six independent components $\mu\nu = 12, 13, 14, 23, 24, 34$ which are considered as a bivector basis in the 6-dimensional linear space (M_6) and also as elements of the orthochronous[1] Lorentz group $SO^+(1,3)$ since they generate a Lie algebra. Moreover the 6-dimensional Lie algebra SO(1,3) is isomorphic to a complex 3-dimensional Lie algebra SL(2,C). Hence, the irreducible representation of the complex Lie algebra of the Lorentz group is embodied by the self-dual bivector basis.

In this direction we are permitted to describe the Riemann tensor with respect to the isotropic null tetrads. This geometric analysis of the formalism was made by Cahen, Debever and Defrise in 1967 [22], [23].

4.2 Coordinate system

In Riemannian geometry any smooth manifold $\mathcal{M}$ could "locally" be approximated as Euclidean, what is called local linearization and it is modeled with the usage of tangent T_xM and Cotagent T_x^*M space. The cotagent space is the dual space of the tangent space and both are vector spaces which are defined at point x.

The notion "locally" actually referred to a neighborhood $\mathcal{U}$ of a point $x \in \mathcal{M}$. In this point a natural basis of the tangent space is ω_μ and the dual basis are the 1-forms ω^μ of the cotagent space, which are orthonormal tetrads of the Lorentz frame and the greek indices take values $\mu = 1, 2, 3, 4$. One knows that the tangent spaces of a smooth manifold are Minkowski spaces, and in our case we choose the signature as $(1, -1, -1, -1)$,

$$ds^2 = (\omega^1)^2 - (\omega^2)^2 - (\omega^3)^2 - (\omega^4)^2 = \eta_{\mu\nu}\omega^\mu\omega^\nu$$

, or equivalently the spacetime can be written as

$$ds^2 = \left[\omega^1 + \omega^2\right]\left[\omega^1 - \omega^2\right] - \left[\omega^3 + i\omega^4\right]\left[\omega^3 - i\omega^4\right] \tag{4.1}$$

We can easily describe our metric using a pseudo-orthonormal basis.

$$ds^2 = 2\left(dudv - d\zeta d\bar{\zeta}\right) \tag{4.2}$$

This basis is related to the orthonomal basis via the matrix w_μ^α.

[1]The orthochronous Lorentz Group does not have mirroring in the timelike direction.

$$dx^{\alpha} = w^{\alpha}{}_{\mu}\omega^{\mu} \quad \rightarrow \quad \begin{pmatrix} du \\ dv \\ d\zeta \\ d\bar{\zeta} \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 & 0 \\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} & 0 & 0 \\ 0 & 0 & \frac{1}{\sqrt{2}} & \frac{i}{\sqrt{2}} \\ 0 & 0 & \frac{1}{\sqrt{2}} & \frac{-i}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} \omega^1 \\ \omega^2 \\ \omega^3 \\ \omega^4 \end{pmatrix} \tag{4.3}$$

The corresponding matrix for the dual basis

$$\partial_{\alpha} = \tilde{w}^{\mu}_{\alpha}\omega_{\mu} \quad \rightarrow \quad \begin{pmatrix} \frac{\partial}{\partial u} & \frac{\partial}{\partial v} & \frac{\partial}{\partial \zeta} & \frac{\partial}{\partial \bar{\zeta}} \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 & 0 \\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} & 0 & 0 \\ 0 & 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 0 & 0 & \frac{-i}{\sqrt{2}} & \frac{i}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \\ \omega_4 \end{pmatrix} \tag{4.4}$$

where the two matrices are connected with the Kronecker $\delta^{\alpha}{}_{\beta}$.

$$w^{\alpha}{}_{\mu}\tilde{w}^{\mu}_{\beta} = \delta^{\alpha}{}_{\beta} \tag{4.5}$$

Indeed, the Minkowski space can be easily described in the new pseudo-orthonormal non-coordinate basis in the Newman-Penrose frame.

$$\boldsymbol{\theta}^1 \equiv n_{\mu}dx^{\mu} \qquad \boldsymbol{\theta}^2 \equiv l_{\mu}dx^{\mu} \qquad \boldsymbol{\theta}^3 \equiv -\bar{m}_{\mu}dx^{\mu} \qquad \boldsymbol{\theta}^4 \equiv -m_{\mu}dx^{\mu} \tag{4.6}$$

The directional derivatives (dual basis) of the formalism are given by

$$\boldsymbol{D} = l^{\mu}\partial_{\mu} \qquad \boldsymbol{\Delta} = n^{\mu}\partial_{\mu} \qquad \boldsymbol{\delta} = m^{\mu}\partial_{\mu} \qquad \bar{\boldsymbol{\delta}} = \bar{m}^{\mu}\partial_{\mu}$$

with the components of the vector

$$n_{\mu} = \begin{pmatrix} 1 & 0 & 0 & 0 \end{pmatrix} \qquad n^{\mu} = \begin{pmatrix} 0 & 1 & 0 & 0 \end{pmatrix} \tag{4.7}$$

$$l_{\mu} = \begin{pmatrix} 0 & 1 & 0 & 0 \end{pmatrix} \qquad l^{\mu} = \begin{pmatrix} 1 & 0 & 0 & 0 \end{pmatrix} \tag{4.8}$$

$$m_{\mu} = \begin{pmatrix} 0 & 0 & 1 & 0 \end{pmatrix} \qquad m^{\mu} = \begin{pmatrix} 0 & 0 & 0 & -1 \end{pmatrix} \tag{4.9}$$

$$\bar{m}_{\mu} = \begin{pmatrix} 0 & 0 & 0 & 1 \end{pmatrix} \qquad \bar{m}^{\mu} = \begin{pmatrix} 0 & 0 & -1 & 0 \end{pmatrix} \tag{4.10}$$

Thus the metric takes the form

$$ds^2 = 2(\boldsymbol{\theta}^1\boldsymbol{\theta}^2 - \boldsymbol{\theta}^3\boldsymbol{\theta}^4) \tag{4.11}$$

where the general metric $g_{\mu\nu}$ is the following and equal to its inverse $g^{\mu\nu}$.

$$g_{\mu\nu} = l_{\mu}n_{\nu} + n_{\mu}l_{\nu} - m_{\mu}\bar{m}_{\nu} - \bar{m}_{\mu}m_{\nu} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{pmatrix} \tag{4.12}$$

The orthogonality properties of the vector components are

$$
\begin{aligned}
l_\mu l^\mu = m_\mu m^\mu = \overline{m}_\mu \bar{m}^\mu = n_\mu n^\mu = 0 \\
l_\mu n^\mu = 1 = -m_\mu \bar{m}^\mu \\
l_\mu m^\mu = l_\mu \bar{m}^\mu = n_\mu m^\mu = n_\mu \bar{m}^\mu = 0
\end{aligned}
\tag{4.13}
$$

To introduce the space of bivectors, it would be useful to define the volume element of the Levi-Civita tensor

$$
\boldsymbol{\epsilon} = 4!\; dx^1 \wedge dx^2 \wedge dx^3 \wedge dx^4 \tag{4.14}
$$

The 4-form of the Levi-Civita tensor is

$$
\epsilon_{\mu\nu\sigma\rho} = \begin{cases} +1, for\ even\ permutations \\ -1, for\ odd\ permutations \\ 0,\ \ for\ two\ same\ indices \end{cases} \tag{4.15}
$$

In a same fashion we can define the 4-volume element

$$
\eta_{\mu\nu\sigma\rho} = \sqrt{-det(g_{\mu\nu})}\epsilon_{\mu\nu\sigma\rho} = i\epsilon_{\mu\nu\sigma\rho} \tag{4.16}
$$

$$
\eta^{\mu\nu\sigma\rho} = -\frac{1}{\sqrt{-det(g_{\mu\nu})}}\epsilon^{\mu\nu\sigma\rho} = i\epsilon^{\mu\nu\sigma\rho} \tag{4.17}
$$

thus the 4-volume element in an isotropic frame takes the downward values

$$
\eta_{\mu\nu\sigma\rho} = \begin{cases} +i, for\ even\ permutations \\ -i, for\ odd\ permutations \\ 0,\ \ for\ two\ same\ indices \end{cases} \tag{4.18}
$$

Using the Cartan's method we can calculate the connection 1-forms ω^α_β. From the geometric point of view, this calculation actually could be described as the parallel transport of a vector in the tangent space T_xM at point x to another neighboring point $x + dx$ with tangent space $T_{x+dx}M$. This translation actually connects these two tangent spaces due to the isomorphic relation between them.

Along these lines, the parallel transport allows us to define the covariant derivative which is an operation that gives birth to the connection forms, since the derivative acts upon not only on the components of a vector but also to the basis.

Let's consider now a general dual vector basis with their components

$$
\boldsymbol{e}_\alpha = e_\alpha{}^\mu \partial_\mu \tag{4.19}
$$

$$
\boldsymbol{\theta}^\alpha = h^\alpha{}_\mu dx^\mu, \tag{4.20}
$$

where $\delta^\alpha{}_\beta = h^\alpha{}_\mu e^\mu{}_\beta$ [2]. The action of the derivative upon to $\boldsymbol{e_\alpha \theta^\alpha}$ is

[2] Generally speaking, the tensor indices indicate the kind of "morphisms" applied to a vector under the action of a tensor, since the tensor e is embodied by tensor products of α tangent spaces and μ cotangent spaces at a point [66]. Along these lines, the matrix $e_\alpha{}^\mu$ is not equal to the matrix $e^\alpha{}_\mu$ in any case, since their transformations differ. However, in our case $e_\alpha{}^\mu$ and $e^\alpha{}_\mu$ are equal.

$$\nabla_\nu \boldsymbol{e}_\alpha = e_\nu{}^\gamma e^\mu_{\alpha;\gamma}\partial_\mu = (e^\gamma_\nu e^\mu_{\alpha;\gamma} h^\beta_\mu)(e^\mu_\beta \partial_\mu) \tag{4.21}$$

$$d\boldsymbol{\theta}^\alpha = h^\alpha_{\mu;\nu} dx^\nu \wedge dx^\mu \tag{4.22}$$

Equivalently, these equations can be written as follows

$$\nabla_\nu \boldsymbol{e}_\alpha = \Gamma^\mu{}_{\nu\alpha} \boldsymbol{e}_\mu \tag{4.23}$$

$$d\boldsymbol{\theta}^\alpha = -\Gamma^\alpha{}_{\mu\nu} \boldsymbol{\theta}^\mu \wedge \boldsymbol{\theta}^\nu \tag{4.24}$$

where $\Gamma^\alpha{}_{\mu\nu} \equiv e^\kappa_\mu h^\alpha_{\kappa;\rho} e^\rho_\nu$. Introducing the connection 1-forms $\boldsymbol{\Gamma}^\alpha{}_\nu \equiv \Gamma^\alpha{}_{\mu\nu}\boldsymbol{\theta}^\mu$, the last relation takes the form

$$d\boldsymbol{\theta}^\alpha = -\boldsymbol{\Gamma}^\alpha{}_\nu \wedge \boldsymbol{\theta}^\nu \tag{4.25}$$

which is explicitly written as follows

$$d\theta^1 = (\gamma+\bar\gamma)\theta^1\wedge\theta^2 + (\bar\alpha+\beta-\bar\pi)\theta^1\wedge\theta^3 + (\alpha+\bar\beta-\pi)\theta^1\wedge\theta^4 - \bar\nu\theta^2\wedge\theta^3 - \nu\theta^2\wedge\theta^4 - (\mu-\bar\mu)\theta^3\wedge\theta^4 \tag{4.26}$$

$$d\theta^2 = (\epsilon+\bar\epsilon)\theta^1\wedge\theta^2 + \kappa\theta^1\wedge\theta^3 + \bar\kappa\theta^1\wedge\theta^4 - (\bar\alpha+\beta-\tau)\theta^2\wedge\theta^3 - (\alpha+\bar\beta-\bar\tau)\theta^2\wedge\theta^4 - (\rho-\bar\rho)\theta^3\wedge\theta^4 \tag{4.27}$$

$$d\theta^3 = -(\bar\tau+\pi)\theta^1\wedge\theta^2 - (\bar\rho+\epsilon-\bar\epsilon)\theta^1\wedge\theta^3 - \bar\sigma\theta^1\wedge\theta^4 + (\mu-\gamma+\bar\gamma)\theta^2\wedge\theta^3 + \lambda\theta^2\wedge\theta^4 + (\alpha-\bar\beta)\theta^3\wedge\theta^4 \tag{4.28}$$

$$d\theta^4 = -(\tau+\bar\pi)\theta^1\wedge\theta^2 - \sigma\theta^1\wedge\theta^3 - (\rho\epsilon+\bar\epsilon)\theta^1\wedge\theta^4 + \bar\lambda\theta^2\wedge\theta^3 + (\bar\mu+\gamma-\bar\gamma)\theta^2\wedge\theta^4 - (\bar\alpha-\beta)\theta^3\wedge\theta^4 \tag{4.29}$$

the greek letters represent the 12 complex spin coefficients. In Newman-Penrose formalism the Christoffel symbols are represented by the spin coefficients or spin connections. The following forms denote the relations between spin connections and the components of the tetrads.

$$\lambda = -n_{\mu;\nu}\bar m^\mu \bar m^\nu = -\Gamma_{232} \qquad \sigma = l_{\mu;\nu} m^\mu m^\nu = \Gamma_{141}$$

$$\rho = l_{\mu;\nu} m^\mu \bar m^\nu = \Gamma_{142} \qquad \mu = -n_{\mu;\nu}\bar m^\mu m^\nu = -\Gamma_{231}$$

$$\pi = -n_{\mu;\nu}\bar m^\mu l^\nu = -\Gamma_{234} \qquad \tau = l_{\mu;\nu} m^\mu n^\nu = \Gamma_{143}$$

$$\kappa = l_{\mu;\nu} m^\mu l^\nu = \Gamma_{144} \qquad \nu = -n_{\mu;\nu}\bar m^\mu n^\nu = -\Gamma_{233}$$

$$\beta = \frac{1}{2}(l_{\mu;\nu} n^\mu m^\nu - m_{\mu;\nu}\bar m^\mu m^\nu) = \frac{1}{2}(\Gamma_{341} - \Gamma_{211}) \qquad \alpha = \frac{1}{2}(l_{\mu;\nu} n^\mu \bar m^\nu - m_{\mu;\nu}\bar m^\mu \bar m^\nu) = \frac{1}{2}(\Gamma_{432} - \Gamma_{122})$$

$$\epsilon = \frac{1}{2}(l_{\mu;\nu} n^\mu l^\nu - m_{\mu;\nu}\bar m^\mu l^\nu) = \frac{1}{2}(\Gamma_{344} - \Gamma_{214}) \qquad \gamma = \frac{1}{2}(l_{\mu;\nu} n^\mu n^\nu - m_{\mu;\nu}\bar m^\mu n^\nu) = \frac{1}{2}(\Gamma_{433} - \Gamma_{123})$$

The relations (4.26)-(4.28) are taken with the usage of the the covariant derivatives of the null tetrads

$$\begin{aligned} n_{\mu;\alpha} = &-(\epsilon+\bar\epsilon)n_\alpha n_\mu - (\gamma+\bar\gamma)l_\alpha n_\mu + (\alpha+\bar\beta)m_\alpha n_\mu + (\bar\alpha+\beta)\bar m_\alpha n_\mu + \pi n_\alpha m_\mu \\ &+ \nu l_\alpha m_\mu - \lambda m_\alpha m\mu - \mu \bar m_\alpha m_\mu + \bar\pi n_\alpha \bar m_\mu + \bar\nu l_\alpha \bar m_\mu - \bar\mu m_\alpha \bar m_\mu - \bar\lambda \bar m_\mu \bar m \nu \end{aligned} \tag{4.30}$$

$$\begin{aligned} l_{\mu;\alpha} = &(\epsilon+\bar\epsilon)n_\alpha l_\mu + (\gamma+\bar\gamma)l_\alpha l_\mu - (\alpha+\bar\beta)m_\alpha l_\mu - (\bar\alpha+\beta)\bar m_\alpha l_\mu - \bar\kappa n_\alpha m_\mu \\ &- \bar\tau l_\alpha m_\mu + \bar\sigma m_\alpha m\mu + \bar\rho \bar m_\alpha m_\mu - \kappa n_\alpha \bar m_\mu - \tau l_\alpha \bar m_\mu + \rho m_\alpha \bar m_\mu + \sigma \bar m_\mu \bar m \nu \end{aligned} \tag{4.31}$$

$$\begin{aligned} m_{\mu;\alpha} = &-\kappa n_\alpha n_\mu - \tau l_\alpha n_\mu + \rho m_\alpha n_\mu + \sigma \bar m_\alpha n_\mu + \bar\pi n_\alpha l_\mu + \bar\nu l_\alpha l_\mu - \bar\mu m_\alpha l_\mu \\ &- \bar\lambda \bar m_\alpha l_\mu + (\epsilon-\bar\epsilon)n_\alpha m_\mu + (\gamma-\bar\gamma)l_\alpha m_\mu - (\alpha-\bar\beta)m_\alpha m_\mu + (\bar\alpha-\beta)\bar m_\alpha m_\mu \end{aligned} \tag{4.32}$$

Next, aiming to acquire the curvature 2-form from the connection, we are going to apply a second derivation upon the relations (4.26)-(4.29) resulting to the second Cartan equation

$$d\boldsymbol{\Gamma}^{\alpha}{}_{\mu} + \boldsymbol{\Gamma}^{\alpha}{}_{\nu} \wedge \boldsymbol{\Gamma}^{\nu}{}_{\mu} = \boldsymbol{\Theta}^{\alpha}{}_{\mu} \tag{4.33}$$

The curvature 2-forms are defined by the following relation.

$$\boldsymbol{\Theta}^{\alpha}{}_{\mu} \equiv \frac{1}{2} R^{\alpha}{}_{\mu\sigma\rho} \boldsymbol{\theta}^{\sigma} \wedge \boldsymbol{\theta}^{\rho} \tag{4.34}$$

In the same fashion, we can obtain the Bianchi identities applying a second derivation on the connection 1-forms,

$$d\boldsymbol{\Theta}^{\alpha}{}_{\mu} = \boldsymbol{\Theta}^{\alpha}{}_{\nu} \wedge \boldsymbol{\Gamma}^{\nu}{}_{\mu} - \boldsymbol{\Gamma}^{\alpha}{}_{\nu} \wedge \boldsymbol{\Theta}^{\nu}{}_{\mu} \tag{4.35}$$

4.3 Bivector Space

In the concept of General Relativity, there are anti-symmetric objects that can easily be described with pairs of indices. These objects are defined as anti-symmetric self-dual 2-forms or self-dual bivectors. As we know, the Electromagnetic tensor $F_{\mu\nu}$ is a 2-rank anti-symmetric tensor, hence, the electromagnetic field could be defined as a bivector in M_4.

Regarding that, the electromagnetic tensor in Minkowski space is given by the following

$$F^{M}_{\mu\nu} = \partial_{\mu} A_{\nu} - \partial_{\mu} A_{\nu} = \begin{pmatrix} 0 & E_x & E_y & E_z \\ -E_x & 0 & -B_z & B_y \\ -E_y & B_z & 0 & -B_x \\ -E_z & -B_y & B_x & 0 \end{pmatrix} \tag{4.36}$$

In our Newman-Penrose frame, the electromagnetic tensor is a self-adjoint tensor. Using the matrix of transformation relation (4.3) and its transpose, we are going to transform the tensor from the orthonormal frame to our pseudo-orthonormal frame,

$$F_{\mu\nu} = F^{M}_{\alpha\beta} w^{\alpha}{}_{\mu} w^{\beta}{}_{\nu}$$

considering that $(w^{\alpha}{}_{\mu})^{T} = w_{\mu}{}^{\alpha}$, the relation above can be written as

$$F_{\mu\nu} = w_{\mu}{}^{\alpha} F^{M}_{\alpha\beta} w^{\beta}{}_{\nu} = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 & 0 \\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} & 0 & 0 \\ 0 & 0 & \frac{1}{\sqrt{2}} & \frac{i}{\sqrt{2}} \\ 0 & 0 & \frac{1}{\sqrt{2}} & \frac{-i}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} 0 & E_x & E_y & E_z \\ -E_x & 0 & -B_z & B_y \\ -E_y & B_z & 0 & -B_x \\ -E_z & -B_y & B_x & 0 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 & 0 \\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} & 0 & 0 \\ 0 & 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 0 & 0 & \frac{i}{\sqrt{2}} & \frac{-i}{\sqrt{2}} \end{pmatrix} \tag{4.37}$$

The latter calculation results to the following expansion

$$F_{\mu\nu} = \begin{pmatrix} 0 & F_{12} & F_{13} & F_{14} \\ -F_{12} & 0 & F_{23} & F_{24} \\ -F_{13} & -F_{23} & 0 & F_{34} \\ -F_{14} & -F_{24} & -F_{34} & 0 \end{pmatrix} = \begin{pmatrix} 0 & -E_x & \frac{\mathbb{F}_y + i\mathbb{F}_z}{2} & \frac{\bar{\mathbb{F}}_y - i\bar{\mathbb{F}}_z}{2} \\ E_x & 0 & \frac{\bar{\mathbb{F}}_y + i\bar{\mathbb{F}}_z}{2} & \frac{\mathbb{F}_y - i\mathbb{F}_z}{2} \\ -\frac{\mathbb{F}_y + i\mathbb{F}_z}{2} & -\frac{\bar{\mathbb{F}}_y + i\bar{\mathbb{F}}_z}{2} & 0 & iB_x \\ -\frac{\bar{\mathbb{F}}_y - i\bar{\mathbb{F}}_z}{2} & -\frac{\mathbb{F}_y - i\mathbb{F}_z}{2} & -iB_x & 0 \end{pmatrix} \tag{4.38}$$

Where the vector $\mathbb{F}$ represents the complex Riemann-Silberstein vector [67] which is depended from the electric and magnetic vectors and is defined as $\mathbb{F} = \boldsymbol{E} + i\boldsymbol{B}$, in SI units [68]. We ought to notice that $F_{13} = \bar{F}_{14}$ and $F_{23} = \bar{F}_{24}$. In the next section, the latter properties would be proved useful in order to acquire the Lorentz invariants.

4.3.1 Lorentz invariants

In this section we scope to represent the two invariants under the Lorentz transformations, in bivector space. The indispensable operation which has to be applied is the passage from M_6 to C_3 with the usage of bivectors. In bivector space the corresponding Lorentz invariants are the same with the two Lorentz invariants of the Minkowski space, which actually are the real and imaginary parts of the squared $\mathbb{F}^2$ of the Riemann-Silberstein vector,

$$\Lambda_1 = \boldsymbol{E}^2 - \boldsymbol{B}^2 \qquad \Lambda_2 = \boldsymbol{E} \cdot \boldsymbol{B} \tag{4.39}$$

Let us proceed with the definition of the bivector and the dual-bivector. A bivector could be defined as any 2-form, which is characterized by the property of anti-symmetricity. Hence, a definition of an arbitrary bivector is

$$x^{\mu\nu} \equiv x^{\mu} \wedge x^{\nu}, \tag{4.40}$$

where the property of anti-symmetricity is embedded in the wedge product.

The definition of the electromagnetic field in the bivector space takes the following form,

$$\boldsymbol{F} = \frac{1}{2} F_{\mu\nu} dx^{\mu} \wedge dx^{\nu} \tag{4.41}$$

where we are already aware of the following property of the electomagnetic tensor.

$$F_{\mu\nu} = -F_{\nu\mu} \tag{4.42}$$

On the other hand, the dual electromagnetic tensor in Newman-Penrose formalism is given as follows,

$$\tilde{F}_{\mu\nu} = \frac{\eta_{\mu\nu\sigma\rho}}{2} F^{\sigma\rho} \tag{4.43}$$

Hence, the dual-electromagnetic tensor (pseudotensor) has the following form

$$\tilde{F}_{\mu\nu} = i \begin{pmatrix} 0 & F_{34} & -F_{13} & F_{14} \\ -F_{34} & 0 & F_{23} & -F_{24} \\ F_{13} & -F_{23} & 0 & F_{12} \\ -F_{14} & F_{24} & -F_{12} & 0 \end{pmatrix} \tag{4.44}$$

The complex electromagnetic bivector $\mathcal{F}_{\mu\nu}$ is composed with the aid of the two bivectors $F_{\mu\nu}$ and $\tilde{F}_{\mu\nu}$.

$$\mathcal{F}_{\mu\nu} \equiv F_{\mu\nu} + i\tilde{F}_{\mu\nu} \tag{4.45}$$

,

The self-dual character of the new tensor is evident since the following property holds

$$\bar{\mathcal{F}}_{\mu\nu} = -i\mathcal{F}_{\mu\nu} \tag{4.46}$$

The Lorentz invariants are actually the quadratic form of the complex bivector.

$$\mathcal{F}_{\mu\nu}\mathcal{F}^{\mu\nu} = 2F_{\mu\nu}F^{\mu\nu} + 2i\tilde{F}_{\mu\nu}F^{\mu\nu} = -16\left[\Lambda_1 - i\Lambda_2\right] \tag{4.47}$$

So, the electromagnetic field is defined with respect to the complex bivector and the vector basis of the Newman-Penrose Formalism,

$$\boldsymbol{F} = F_{\mu\nu}\boldsymbol{\theta}^{\mu} \wedge \boldsymbol{\theta}^{\nu} = \frac{1}{4}\mathcal{F}_{\mu\nu}dx^{\mu} \wedge dx^{\nu} \tag{4.48}$$

Actually, Debever [23] postulated that *"with any non-singular bivector there is a single (or a pair) associated isotropic characteristic vector that coincides."* Indeed, the non-singularity property of our bivectors holds if $\Lambda_2 = 0$ ($\tilde{F}_{\mu\nu}F^{\mu\nu} = 0$) and the Lorentz group transforms simple bivectors to simple bivectors. Simultaneously, the invariance of Λ_1 guarantees the invariance of the metric up to a sign [22].

4.3.2 Complex vectorial basis - Curvature 2-forms

Let us proceed with the expression of the electromagnetic field with respect to the bivector basis. A similar derivation of electromagnetic tensor in bivector space was also operated by Santos [69].

$$\boldsymbol{F} = F_a\boldsymbol{Z}^a + \bar{F}_a\bar{\boldsymbol{Z}}^a \tag{4.49}$$

Equivalently, the latter relation could be written as follows since a = 1,2,3,

$$\boldsymbol{F} = -F_{14}\boldsymbol{Z}^1 + \frac{F_{12}+F_{34}}{2}\boldsymbol{Z}^2 + F_{23}\boldsymbol{Z}^3 - F_{13}\bar{\boldsymbol{Z}}^1 + \frac{F_{12}-F_{34}}{2}\bar{\boldsymbol{Z}}^2 + F_{24}\bar{\boldsymbol{Z}}^3 \tag{4.50}$$

A suitable basis for the space (C_3) of complex self-dual bivector (2-forms) is given by the downward relations.

$$\boldsymbol{Z}^1 = \boldsymbol{\theta}^1 \wedge \boldsymbol{\theta}^3 = Z^1_{\alpha\beta}dx^{\alpha} \otimes dx^{\beta}\ ;\ \ Z^1_{\alpha\beta} = -n_{\alpha}\bar{m}_{\beta} + n_{\beta}\bar{m}_{\alpha} \tag{4.51}$$

$$\boldsymbol{Z}^2 = \boldsymbol{\theta}^1 \wedge \boldsymbol{\theta}^2 - \boldsymbol{\theta}^3 \wedge \boldsymbol{\theta}^4 = Z^2_{\alpha\beta}dx^{\alpha} \otimes dx^{\beta}\ ;\ \ Z^2_{\alpha\beta} = n_{\alpha}l_{\beta} - n_{\beta}l_{\alpha} - \bar{m}_{\alpha}m_{\beta} + m_{\alpha}\bar{m}_{\beta} \tag{4.52}$$

$$\boldsymbol{Z}^3 = \boldsymbol{\theta}^4 \wedge \boldsymbol{\theta}^2 = Z^3_{\alpha\beta}dx^{\alpha} \otimes dx^{\beta}\ ;\ \ Z^3_{\alpha\beta} = -m_{\alpha}l_{\beta} + m_{\beta}l_{\alpha} \tag{4.53}$$

The composite of the metric in this base are

$$\gamma^{ab} = 4\left[\delta^a{}_{(1}\delta^b{}_{3)} - \delta^a{}_2\delta^b{}_2\right] = \begin{pmatrix} 0 & 0 & 2 \\ 0 & -4 & 0 \\ 2 & 0 & 0 \end{pmatrix} \tag{4.54}$$

The complex connection 1-forms $\boldsymbol{\sigma}^a_b$ is produced by the derivation of the basis $\boldsymbol{Z}^a$, i.e

$$d\boldsymbol{Z}^a = -\boldsymbol{\sigma}^a_b \wedge \boldsymbol{Z}^b \tag{4.55}$$

the vectorial connection 1-form $\boldsymbol{\sigma}_a$ is defined by

$$\boldsymbol{\sigma}^a_b = 8\epsilon^{kac}\boldsymbol{\sigma}_k\gamma_{cb} \tag{4.56}$$

Expanding $\boldsymbol{\sigma}_k$ in the basis of $\boldsymbol{\theta}^\mu$, we obtain

$$\boldsymbol{\sigma}_k = \frac{1}{8}\epsilon_{kac}\gamma^{cb}\boldsymbol{\sigma}^a_b = \kappa_{k\mu}\boldsymbol{\theta}^\mu, \tag{4.57}$$

where ϵ_{abc} is the Levi-Civita tensor and the tetrad components $\kappa_{k\mu}$ contain the 12 complex spin coefficients

$$\kappa_{k\mu} = \begin{bmatrix} \kappa & \tau & \sigma & \rho \\ \epsilon & \gamma & \beta & \alpha \\ \pi & \nu & \mu & \lambda \end{bmatrix} \tag{4.58}$$

The complex curvature 2-forms $\boldsymbol{\Sigma}^b_d$ are defined by

$$\boldsymbol{\Sigma}^b_d = d\boldsymbol{\sigma}^b_d + \boldsymbol{\sigma}^b_g \wedge \boldsymbol{\sigma}^g_d \tag{4.59}$$

and the vectorial curvature 2-form by

$$\boldsymbol{\Sigma}_a = \frac{1}{8}e_{abg}\gamma^{gd}\boldsymbol{\Sigma}^b{}_d \tag{4.60}$$

The corresponding expanding of $\boldsymbol{\Sigma}_a$ with respect to the basis of $\left(\boldsymbol{Z}^a, \bar{\boldsymbol{Z}}^a\right)$ is given

$$\boldsymbol{\Sigma}_a = (C_{ab} - \frac{1}{6}R\gamma_{ab})\boldsymbol{Z}^b + E_{a\bar{b}}\bar{\boldsymbol{Z}}^b, \tag{4.61}$$

where these quantities are related with the curvature components of the formalism

$$C_{ab} = \begin{bmatrix} \Psi_0 & \Psi_1 & \Psi_2 \\ \Psi_1 & \Psi_2 & \Psi_3 \\ \Psi_2 & \Psi_3 & \Psi_4 \end{bmatrix}, \qquad E_{a\bar{b}} = \begin{bmatrix} \Phi_{00} & \Phi_{01} & \Phi_{02} \\ \Phi_{10} & \Phi_{11} & \Phi_{12} \\ \Phi_{20} & \Phi_{21} & \Phi_{22} \end{bmatrix} \tag{4.62}$$

In this formalism, the 10 Weyl's components are represented by the 5 complex scalar functions.

$$\Psi_0 = C_{\kappa\lambda\mu\nu}l^\kappa m^\lambda l^\mu m^\nu = C_{1313}$$
$$\Psi_1 = C_{\kappa\lambda\mu\nu}l^\kappa n^\lambda l^\mu m^\nu = C_{1213}$$

$$\Psi_2 = \frac{1}{2} C_{\kappa\lambda\mu\nu} l^\kappa n^\lambda \left[l^\mu n^\nu - m^\mu \bar{m}^\nu \right] = C_{1342} \quad (4.63)$$
$$\Psi_3 = C_{\kappa\lambda\mu\nu} n^\kappa l^\lambda n^\mu \bar{m}^\nu = C_{1242}$$
$$\Psi_4 = C_{\kappa\lambda\mu\nu} n^\kappa \bar{m}^\lambda n^\mu \bar{m}^\nu = C_{4242}$$

The Ricci tensor components are represented by $E_{a\bar{b}}$, and its elements are expressed with respect to the traceless Ricci tensor $S_{\mu\nu}$ which was referred in (2.28), and are divided in to real and complex components. The real components have the same index and the complex components are constrained by $\Phi_{ab} = \bar{\Phi}_{ba}$.

$$\Phi_{00} = \frac{1}{2} S_{\mu\nu} l^\mu l^\nu = \frac{1}{2} R_{44} \qquad \Phi_{01} = \bar{\Phi}_{10} = \frac{1}{2} S_{\mu\nu} l^\mu m^\nu = \frac{1}{2} R_{41}$$
$$\Phi_{11} = \frac{1}{4} S_{\mu\nu} (l^\mu n^\nu + m^\mu \bar{m}^\nu) = \frac{1}{4} (R_{43} + R_{12}) \qquad \Phi_{02} = \bar{\Phi}_{20} = \frac{1}{2} S_{\mu\nu} m^\mu m^\nu = \frac{1}{2} R_{11} \quad (4.64)$$
$$\Phi_{22} = \frac{1}{2} S_{\mu\nu} n^\mu n^\nu = \frac{1}{2} R_{33} \qquad \Phi_{12} = \bar{\Phi}_{21} = \frac{1}{2} S_{\mu\nu} n^\mu m^\nu = \frac{1}{2} R_{33}$$

All these quantities describe the main parts of the Einstein's Field Equations. The EFE in this formalism are represented by the corresponding field equations which are known either as Newman-Penrose Field Equations or as Ricci identities [70]. The Newman-Penrose field equations relate the spin-coefficients to the derivatives of the tetrad components, the spin-coefficient equations describe the relationship of the curvature tensor with the derivatives of the connection (the spin-coefficients). A novelty here is that these equations are integrated not one set at a time, but together, i.e., by going back and forth between the sets. The following relations are presented without the spin coefficients σ and λ since in our case they are annihilated since the beginning.

The Newman-Penrose Field Equations are:

$$D\rho - \bar{\delta}\kappa = \rho^2 + \rho(\epsilon + \bar{\epsilon}) - \bar{\kappa}\tau - \kappa \left[2(\alpha + \bar{\beta}) + (\alpha - \bar{\beta}) - \pi \right] \quad \text{(a)}$$
$$\delta\kappa = \kappa \left[\tau - \bar{\pi} + 2(\bar{\alpha} + \beta) - (\bar{\alpha} - \beta) \right] - \Psi_o \quad \text{(b)}$$
$$D\tau = \Delta\kappa + \rho(\tau + \bar{\pi}) + \tau(\epsilon - \bar{\epsilon}) - 2\kappa\gamma - \kappa(\gamma + \bar{\gamma}) + \Psi_1 \quad \text{(c)}$$
$$D\nu - \Delta\pi = \mu(\pi + \bar{\tau}) + \pi(\gamma - \bar{\gamma}) - 2\nu\epsilon - \nu(\epsilon + \bar{\epsilon}) + \Psi_3 \quad \text{(i)}$$
$$\bar{\delta}\pi = -\pi(\pi + \alpha - \bar{\beta}) + \nu\bar{\kappa} \quad \text{(g)}$$
$$\delta\tau = \tau(\tau - \bar{\alpha} + \beta) - \bar{\nu}\kappa \quad \text{(p)}$$
$$D\mu - \delta\pi = \mu\bar{\rho} + \pi(\bar{\pi} - \bar{\alpha} + \beta) - \mu(\epsilon + \bar{\epsilon}) - \kappa\nu + \Psi_2 + 2\Lambda \quad \text{(h)}$$
$$\delta\nu - \Delta\mu = \mu(\mu + \gamma + \bar{\gamma}) - \bar{\nu}\pi + \nu(\tau - 2(\bar{\alpha} + \beta) + (\bar{\alpha} - \beta)) \quad \text{(n)}$$
$$\Delta\rho - \bar{\delta}\tau = -\bar{\mu}\rho - \tau(\bar{\tau} + \alpha - \bar{\beta}) + \nu\kappa + \rho(\gamma + \bar{\gamma}) - \Psi_2 - 2\Lambda \quad \text{(q)}$$
$$\delta\rho = \rho(\bar{\alpha} + \beta) + \tau(\rho - \bar{\rho}) + \kappa(\mu - \bar{\mu}) - \Psi_1 \quad \text{(k)}$$
$$\bar{\delta}\mu = -\mu(\alpha + \bar{\beta}) - \pi(\mu - \bar{\mu}) - \nu(\rho - \bar{\rho}) + \Psi_3 \quad \text{(m)}$$

$$D\alpha - \bar{\delta}\epsilon = \alpha(\rho + \bar{\epsilon} - 2\epsilon) - \bar{\beta}\epsilon - \bar{\kappa}\gamma + \pi(\epsilon + \rho) \tag{d}$$

$$D\beta - \delta\epsilon = \beta(\bar{\rho} - \bar{\epsilon}) - \kappa(\mu + \gamma) - \epsilon(\bar{\alpha} - \bar{\pi}) + \Psi_1 \tag{e}$$

$$\Delta\alpha - \bar{\delta}\gamma = \nu(\epsilon + \rho) + \alpha(\bar{\gamma} - \bar{\mu}) + \gamma(\bar{\beta} - \bar{\tau}) - \Psi_3 \tag{r}$$

$$-\Delta\beta + \delta\gamma = \gamma(\tau - \bar{\alpha} - \beta) + \mu\tau - \epsilon\bar{\nu} - \beta(\gamma - \bar{\gamma} - \mu) \tag{o}$$

$$\delta\alpha - \bar{\delta}\beta = \mu\rho + \alpha(\bar{\alpha} - \beta) - \beta(\alpha - \bar{\beta}) + \gamma(\rho - \bar{\rho}) + \epsilon(\mu - \bar{\mu}) - \Psi_2 + \Lambda \tag{l}$$

$$D\gamma - \Delta\epsilon = \alpha(\tau + \bar{\pi}) + \beta(\bar{\tau} + \pi) - \gamma(\epsilon + \bar{\epsilon}) - \epsilon(\gamma + \bar{\gamma}) + \Psi_2 - \Lambda - \kappa\nu + \tau\pi \tag{f}$$

$$\bar{\delta}\nu = -\nu\left[2(\alpha + \bar{\beta}) + (\alpha - \bar{\beta}) + \pi - \bar{\tau}\right] + \Psi_4 \tag{j}$$

The Bianchi Identities without the presence of the electromagnetic field are:

$$\bar{\delta}\Psi_0 - D\Psi_1 = (4\alpha - \pi)\Psi_0 - 2(2\rho + \epsilon)\Psi_1 + 3\kappa\Psi_2 \tag{I}$$

$$\bar{\delta}\Psi_1 - D\Psi_2 = 2(\alpha - \pi)\Psi_1 - 3\rho\Psi_2 + 2\kappa\Psi_3 \tag{II}$$

$$\bar{\delta}\Psi_2 - D\Psi_3 = -3\pi\Psi_2 + 2(\epsilon - \rho)\Psi_3 + \kappa\Psi_4 \tag{III}$$

$$\bar{\delta}\Psi_3 - D\Psi_4 = -2(\alpha + 2\pi)\Psi_3 + (4\epsilon - \rho)\Psi_4 \tag{IV}$$

$$\Delta\Psi_0 - \delta\Psi_1 = (4\gamma - \mu)\Psi_0 - 2(2\tau + \beta)\Psi_1 \tag{V}$$

$$\Delta\Psi_1 - \delta\Psi_2 = \nu\Psi_0 + 2(\gamma - \mu)\Psi_1 - 3\tau\Psi_2 \tag{VI}$$

$$\Delta\Psi_2 - \delta\Psi_3 = 2\nu\Psi_1 - 3\mu\Psi_2 + 2(\beta - \tau)\Psi_3 \tag{VII}$$

$$\Delta\Psi_3 - \delta\Psi_4 = 3\nu\Psi_2 - 2(\gamma + 2\mu)\Psi_3 + (4\beta - \tau)\Psi_4 \tag{VIII}$$

Also, the Lie bracket plays an important role to the theory in order to obtain the commutation relations of the NP formalism. The commutation relations emerged by using the Lie brackets of the vectors $n^\mu, l^\mu, m^\mu, \bar{m}^\mu$. The proper definition reads as follows for an arbitrary vector basis.

$$[\boldsymbol{e}_\mu, \boldsymbol{e}_\nu] = -2\Gamma^\sigma{}_{[\mu\nu]}\boldsymbol{e}_\sigma \tag{4.65}$$

With the usage of the null tetrad basis the commutation relations result to the following,

$$[n^\mu, l^\mu] = [D, \Delta] = (\gamma + \bar{\gamma})D + (\epsilon + \bar{\epsilon})\Delta - (\pi + \bar{\tau})\delta - (\bar{\pi} + \tau)\bar{\delta} \tag{CR1}$$

Hence, we have four commutations relations (CR) in every possible combination. We present it here with every detail divided into real and imaginary parts implying that $\sigma = \lambda = 0$,

$$[(\delta+\bar{\delta}), D] = (\alpha+\bar{\alpha}+\beta+\bar{\beta}-\pi-\bar{\pi})D+(\kappa+\bar{\kappa})\Delta-(\bar{\rho}+\epsilon-\bar{\epsilon})\delta-(\rho-\epsilon+\bar{\epsilon})\bar{\delta} \tag{$CR2_+$}$$

$$[(\delta-\bar{\delta}), D] = (-\alpha+\bar{\alpha}+\beta-\bar{\beta}+\pi-\bar{\pi})D+(\kappa-\bar{\kappa})\Delta-(\bar{\rho}+\epsilon-\bar{\epsilon})\delta+(\rho-\epsilon+\bar{\epsilon})\bar{\delta} \tag{$CR2_-$}$$

$$[(\delta+\bar{\delta}), \Delta] = -(\nu+\bar{\nu})D+(\tau+\bar{\tau}-\alpha-\bar{\alpha}-\beta-\bar{\beta})\Delta+(\mu-\gamma+\bar{\gamma})\delta+(\bar{\mu}+\gamma-\bar{\gamma})\bar{\delta} \tag{$CR3_+$}$$

$$[(\delta-\bar{\delta}), \Delta] = -(\nu-\bar{\nu})D+(\tau-\bar{\tau}+\alpha-\bar{\alpha}-\beta+\bar{\beta})\Delta+(\mu-\gamma+\bar{\gamma})\delta-(\bar{\mu}+\gamma-\bar{\gamma})\bar{\delta} \tag{$CR3_-$}$$

$$[\delta,\bar{\delta}] = -(\mu-\bar{\mu})D - (\rho-\bar{\rho})\Delta + (\alpha-\bar{\beta})\delta - (\bar{\alpha}-\beta)\bar{\delta} \qquad (CR4)$$

All the above sets of equations contribute to the Newman-Penrose Field Equations, the Bianchi Identities and the commutation relations of the basis vectors with $\sigma = \lambda = 0$. NPEs is a set of 18 linear equations in comparison with the non-linear 10 equations of 3-1 formalism. Despite the fact that we have to solve a considerably larger number of equations in comparison with 3-1 formalism, this formalism has great advantages. All differential equations here are of first order. Also, gauge transformations of the tetrad can be used to simplify the field equations. However, one can easily extract invariant properties of the gravitational field (Petrov types) without using a coordinate basis [19].

The usage of this formalism allows the concentration of the field equations on individual 'scalar' equations with particular physical or geometric significance, thus, a natural hierarchical structure is evident. Also, it allows us to search for solutions with specific special features, such as the presence of one or two null directions that might be singled out by physical or geometric considerations. In conclusion, Newman-Penrose formalism, at first sight, looks like a complicated tool but it is proved to be sophisticated and convenient at last.

4.4 Null Congruences

The conformal symmetry of a Lorentz rotation around one of the null tetrads is presented. During the resolving procedure of EFEs, the conformal transformation could be proved really helpful providing simplifications between spins themselves. In the following case, l^μ is fixed. In order to achieve this, we define the complex rotation parameters $t \equiv a + ib$ and $p \equiv c + id$,

$$\tilde{\theta}^1 = e^{-a}(\theta^1 + p\bar{p}\theta^2 + \bar{p}\theta^3 + p\theta^4) = \tilde{n}_\mu dx^\mu$$
$$\tilde{\theta}^2 = e^{a}\theta^2 = \tilde{l}_\mu dx^\mu$$
$$\tilde{\theta}^3 = e^{-ib}(\theta^3 + p\theta^2) = -\tilde{\bar{m}}_\mu dx^\mu$$
$$\tilde{\theta}^4 = e^{ib}(\theta^4 + \bar{p}\theta^2) = -\tilde{m}_\mu dx^\mu$$

The rotation is applied also to spin coefficients,

$$\tilde{\kappa} = e^{2a+ib}\kappa$$
$$\tilde{\rho} = e^{a}\left[\rho - p\kappa\right]$$
$$\tilde{\sigma} = e^{a+2ib}\left[\sigma - \bar{p}\kappa\right]$$
$$\tilde{\epsilon} = e^{a}\left[\epsilon - p\kappa\frac{1}{2}Dt\right]$$
$$\tilde{\pi} = e^{-ib}\left[\pi + p^2\kappa - 2p\epsilon - Dp\right]$$
$$\tilde{\tau} = e^{ib}\left[\tau - p\sigma - \bar{p}\rho + p\bar{p}\kappa\right]$$
$$\tilde{\alpha} = e^{-ib}\left[\alpha - p\rho - p\epsilon + p^2\kappa - \frac{1}{2}pDt + \frac{1}{2}\bar{\delta}t\right]$$

$$\tilde{\beta} = e^{ib}\left[\beta - p\sigma - \bar{p}\epsilon + p\bar{p}\kappa - \frac{1}{2}\bar{p}Dt + \frac{1}{2}\delta t\right]$$

$$\tilde{\lambda} = e^{-a-2ib}\left[\lambda - p^3\kappa + 2p^2\epsilon + p^2\rho - p\pi - p\alpha + pDp - \bar{\delta}p\right]$$

$$\tilde{\mu} = e^{-a}\left[\mu - p^2\bar{p}\kappa + 2p\bar{p}\epsilon + p^2\sigma - \bar{p}\pi - 2p\beta + \bar{p}Dp - \delta p\right]$$

$$\begin{aligned}\tilde{\nu} = e^{-2a-ib}[\nu + p^3\bar{p}\kappa - 2p^2\bar{p}\epsilon - p^3\sigma + 2p^2\beta + p^2\tau + p\bar{p}\pi + 2p\bar{p}\sigma \\ - p\mu - \bar{p}\lambda - 2p\gamma - p\bar{p}Dp + p\delta p + \bar{p}\bar{\delta}p - \Delta p]\end{aligned}$$

$$\tilde{\gamma} = e^{-a}\left[\gamma + p\bar{p}\rho + p\bar{p}\epsilon - p^2\bar{p}\kappa + p^2\sigma - p\beta - \bar{p}\alpha - p\tau + \frac{1}{2}p\bar{p}Dt - \frac{1}{2}p\delta t - \frac{1}{2}\bar{p}\delta t + \frac{1}{2}\Delta t\right]$$

The implication of a rotation to null tetrad frames affects the Weyl tensor components as well. The following relations describe the impact due to the rotation with l^μ fixed,

$$\begin{aligned}\Psi_4' &= e^{-2t}(\Psi_4 - 4p\Psi_3 + 6p^2\Psi_2 - 4p^3\Psi_1 + p^4\Psi_0) \\ \Psi_3' &= e^{-t}(\Psi_3 - 3p\Psi_2 + 3p^2\Psi_1 - p^3\Psi_0) \\ \Psi_2' &= \Psi_2 - 2p\Psi_1 + p^2\Psi_0 \\ \Psi_1' &= e^{t}(\Psi_1 - p\Psi_0) \\ \Psi_0' &= e^{2t}\Psi_0\end{aligned} \tag{4.66}$$

4.5 Petrov Classification

The Petrov classification is an invariant characterization of gravitational field. Actually, Petrov [71] found the canonical forms of the Weyl tensor C_{ab} and classified them as we can see below. The starting point is the eigenvalue equation for Weyl tensor in bivector space [19].

$$\frac{1}{2}C^a_b Z^b = \lambda Z^a \tag{4.67}$$

As we mentioned the group $SO(3,\mathbb{C})$ in our formalism is isomorphic to the orthochronous Lorentz group. Hence, the Lorentz transformations hold and also they can be applied in order to find the canonical forms.

The five canonical forms are presented in the following table where the relation $\lambda_1 + \lambda_2 + \lambda_3 = 0$ is the constraint. The absent Weyl components are zero.

Canonical Forms	Eigenvalues	Weyl components	Petrov Types
$\begin{pmatrix}\lambda_1 & 0 & 0\\ 0 & \lambda_2 & 0\\ 0 & 0 & \lambda_3\end{pmatrix}$	$\lambda_1 \neq \lambda_2 \neq \lambda_3$	$\Psi_0 = \Psi_4 = \frac{\lambda_2-\lambda_1}{2}; \Psi_2 = -\frac{\lambda_3}{2}$	Type I
$\begin{pmatrix}\lambda_1 & 0 & 0\\ 0 & \lambda_1 & 0\\ 0 & 0 & \lambda_3\end{pmatrix}$	$\lambda_1 = \lambda_2 \neq \lambda_3$	$\Psi_2 = -\frac{\lambda_3}{2}$	Type D
$\begin{pmatrix}1-\frac{\lambda}{2} & -i & 0\\ -i & -\frac{\lambda}{2}-1 & 0\\ 0 & 0 & \lambda\end{pmatrix}$	$\lambda_1 = \lambda_2 = -\frac{\lambda}{2}; \lambda_3 = \lambda$	$\Psi_2 = -\frac{\lambda}{2}; \Psi_4 = -2$	Type II
$\begin{pmatrix}1 & -i & 0\\ -i & -1 & 0\\ 0 & 0 & 0\end{pmatrix}$	$\lambda_1 = \lambda_2 = \lambda_3 = 0$	$\Psi_4 = -2$	Type N
$\begin{pmatrix}0 & 0 & i\\ 0 & 0 & 1\\ i & 1 & 0\end{pmatrix}$	$\lambda_1 = \lambda_2 = \lambda_3 = 0$	$\Psi_3 = -i$	Type III

A general comment about the Petrov types on vacuum would be that the most general spacetimes are of type I. Also, we know that the stationary and axially symmetric spacetimes that admit (at least) two Killing vectors and they are type D. Type N and type III describe spacetimes with gravitational waves.

Chapter 5

Canonical Forms of the Killing Tensor

The consideration of symmetries in the resolution process of the Einstein's equations is indispensable. The non-linearity character of the equations of gravity are obligate us to introduce further information through symmetries to obtain a solvable-overdetermined system. The preservation of geometry of spacetime through a transformation reveals the existence of symmetries. Hence, these symmetries must leave invariant the elements which characterize the geometry, that is referred to the metric tensor and to the action (Energy conservation) by extension.

5.1 Killing Tensor

Considering that, the geodesic flow is a Hamiltonian system on the cotangent bundle

$$H = \frac{1}{2} g^{\mu\nu} p_\mu p_\nu \tag{5.1}$$

where p_μ are the coordinates on the cotangent spaces or equivalently the canonical momenta of an observer. Then, an integral of motion could be defined as follows [72], [73].

$$\{\mathcal{K}, H\} \equiv 0 \qquad \rightarrow \qquad \frac{\partial H}{\partial x^\mu}\frac{\partial \mathcal{K}}{\partial p_\mu} - \frac{\partial \mathcal{K}}{\partial x^\mu}\frac{\partial H}{\partial p_\mu} \equiv 0 \tag{5.2}$$

The function $\mathcal{K}$: $T^*M \rightarrow \mathbb{R}$ is called polynomial of momenta and is defined as

$$\mathcal{K}(x,p) \equiv K^\mu p_\mu + K^{\mu\nu} p_\mu p_\nu + K^{\mu\nu\sigma} p_\mu p_\nu p_\sigma + K^{\mu\nu\sigma\rho} p_\mu p_\nu p_\sigma p_\rho \tag{5.3}$$

moreover, the components of the object $\mathcal{K}$ called Stäckel-Killing tensors and satisfy the Killing equation [74], [75]. The arguments in the parenthesis in relation (5.3) defines an endomorphism on tangent and on cotangent bundles of a smooth manifold M.

$$K_{(\mu;\nu)} = 0$$

$$K_{(\mu\nu;\alpha)} = 0$$

$$K_{(\mu\nu\sigma;\alpha)} = 0$$

$$K_{(\mu\nu\sigma\rho;\alpha)} = 0$$

Indeed, Killing tensors of rank r give rise to a homogeneous constant of motion of degree r in momenta. The inhomogeneous polynomial integrals of geodesic motion can be decomposed to their homogeneous parts and also to the corresponding parts that are associated with the Killing tensors with the equivalent rank. The Killing tensor of rank 1 equals to Killing vector and generates continuous symmetry transformations, these symmetries are called explicit [76].

The symmetries that correspond to higher-order ranks of momenta associated with Killing tensors of rank $(r > 1)$ are called hidden symmetries [77]. In this work we aim to investigate spacetimes whose integrals of motion are characterized by polynomial of momenta, the existence of which indicates the existence of a special geometric structure.

The investigation of special objects such Killing tensor or Killing-Yano is basically a devilish way to peep into the phase space searching for hidden symmetries [78]. Indeed, the explicit symmetries in a Hamiltonian system always could be "dragged up" instead of hidden symmetries. The assumption of existence of these kind of tensors could be proved fruitful, providing spacetimes with both explicit and hidden symmetries. These kind of symmetries emerge during the study of the dynamics of a system featuring the conserved quantities of the system or one-parameter isometries which is equivalent with the admission of existence of Killing vectors.

As Eisenhart [75] and Kalnins-Miller [79], [80], [81] showed, the geodesic separation is correlated with the existence of Killing vectors and Killing tensors of order two [82]. Indeed, there is a *bizarre* relation between the structure of separated metrics with the structure of its characteristic Killing tensor. Benenti and Francaviglia [83] present a certain example where the additional information about the metric tensor serve as a catalyst in order to obtain the structure of the Killing tensor.

Let us proceed with the definition of Killing tensor: **Any symmetric tensor of order 2 whose the symmetric part of his covariant derivative vanishes is called Killing tensor**.

$$K_{(\mu\nu;\alpha)} = 0$$

The *trivial Killing tensor* is the metric tensor $g_{\mu\nu}$ where its existence indicates the conservation of the rest mass of a moving particle in Hamiltonian systems.

$$\mathcal{H} = \frac{\bar{m}^2}{2} = \frac{1}{2} g_{\mu\nu} u^\mu u^\nu \tag{5.4}$$

The Hamiltonian is a conserved quantity of the problem since it is correlated with the conserved rest mass.

Redundant Killing Tensor is called a Killing tensor which is equal to the metric tensor multiplied by a constant coefficient

$$K_{\mu\nu} = Cg_{\mu\nu} \tag{5.5}$$

or linear combinations of the symmetric part of two Killing vectors.

$$K_{\mu\nu} = CA_{(\mu}B_{\nu)} \tag{5.6}$$

One more interesting case of Killing tensor is the *conformal Killing tensor* which is used widely in the literature. This tensor provides us with integrals of motion that are referred only to isotropic geodesics describing radiation (null orbits). In the following relations the factor v is a tensor with one rank lower than the Killing object. Regarding the latter the conformal Killing equation for a Killing vector is,

$$\nabla_{(\mu}X^C_{\nu)} = vg_{\mu\nu} \tag{5.7}$$

while the v is a constant. Furthermore, the Conformal Killing tensor satisfies the following equation

$$\nabla_{(\alpha}K_{\mu\nu)} = g_{(\alpha\mu}v_{\nu)} \tag{5.8}$$

where in this case v_ν is a covector component.

In a 4-dimensional Lorentzian spacetime that possesses a Killing tensor, our focus lies in seeking two Killing vectors. The geodesic Hamilton-Jacobi equation is separable if and only if there exist two commuting Killing vectors, and the associated Killing tensors possess two shared eigenvectors, denoted as W_μ and V_ν, satisfying the following condition that

$$[W_\mu, V_\nu] = 0, \;\; [W_\mu, W_\nu] = 0, \;\; g(W_\mu, V_\nu) = 0 \tag{5.9}$$

where the $[\cdot,\cdot]_{SN}$ is the Schouten-Nijenhuis bracket which is a generalization of the Lie bracket and allows one to construct from two Killing Tensors a new one [8], [77].

At last, the usage of canonical forms of a Killing tensor could be proved fruitful since can be used as a *starter culture* in order to discover spacetimes with hidden symmetries. Hauser-Malhiot's spacetimes [15] assumed the existence of a Killing tensor of Segré type [(11),(11)] [84], they proved that this assumption admitted by one of the most general family of stationary axially symmetric electro-vacuum spacetimes that found independently by Carter [55]. Regarding the topic just mentioned, the Killing Tensor that has been studied by Hauser-Malhiot is a special case of our canonical forms.

5.2 Canonical forms

Obtaining the canonical forms of a symmetric 2nd-rank tensor can be a challenging task when approached algebraically. However, in a symmetric matrix the additional symmetries could aid us to find its canonical forms geometrically.

The canonical forms contain the minimum number of independent scalars which compose the eigenvalues during the diagonalization [16].

The equation for diagonalization in any tensor takes the following form.

$$(K^{\mu}{}_{\nu} - \lambda\delta^{\mu}_{\nu})z^{\nu} = 0 \tag{5.10}$$

which equivalently it could be written as follows.

$$K^{\mu}{}_{\nu}z^{\nu} - \lambda z^{\mu} = 0 \tag{5.11}$$

In this manner, it becomes clear that the operation of our tensor or our linear vector function on a vector leaves it unchanged. In this context, it is known that every 2nd-order symmetric tensor defines a linear mapping that transforms a vector **k** into another vector **v** unless the vector is an eigenvector. We are searching for these types of directions, which, of course, do not alter the quadratic form of the metric. In our formalism, the norm of a vector must remain invariant.

$$\mathbf{x}\cdot\mathbf{x} = x^{\mu}g_{\mu\nu}x^{\nu} = 2(x^1x^2 - x^3x^4) \tag{5.12}$$

Algebraically the result is that a symmetric matrix K is reducible under an orthogonal transformation with a matrix P to a canonical form PKP^{-1} in which all non-diagonal elements are zero. Besides, a positive definitive matrix can always be diagonalized through a real orthogonal transformation.

A similar analysis can also be found in Landau's book on page 271 regarding the stress-energy-momentum tensor [85]. In this section, he mentions that this procedure is, in fact, the application of the "Principal Axis Theorem" or "Spectral Theorem" for a matrix. However, it appears that the author applies diagonalization to a symmetric tensor in a covariant form. This approach is not correct in general. Diagonalization should be applied to a tensor in a mixed tensor form, as shown in equation (5.7). This is correct because a term proportional to the metric merely shifts all eigenvalues by the same amount [19].

5.2.1 The presence of null vectors within planes

In line with the work of Churchill we obtained the Canonical forms of Killing tensor [16]. The study of Churchill was operated in pseudo-Euclidean spacetime with signature $(-,+,+,+)$ and the object of his study was a linear symmetric vector function whose components represented by a symmetric tensor of valence 2. In this reference Churchill based on the work of Rainich who studied the antisymmetricity of the electromagnetic tensor [86] and he remarks that: *"It is known that every linear vector function in four-dimensional space has at least one invariable plane"*.

Based on the last statement he operated his calculation with a specific manner. He lied his k, l vectors (the corresponding $m, \bar{m}$ of our formalism) in the invariable plane which does not contain any null vectors. With this choice all the canonical forms has the same downward-left block where there are two district eigenvalues. Hence, all forms have different eigenvalues only in the upward-left block where the existence of null vectors takes place.

In the following segment we will present the classification of canonical forms, therein one could observe that in our case the statement of the previous paragraph is evident since all the canonical forms proved to have the same eigenvalues $-(\lambda_2 \pm \lambda_7)$. Although, the K^0 is an exception in this matter since it has one triple eigenvalue. Due to this necessary condition one can prove that the triple eigenvalue satisfies either the first relation

$$\lambda_1 = -(\lambda_2 + \lambda_7)$$

or the second one.

$$\lambda_1 = -(\lambda_2 - \lambda_7)$$

After this introduction let us proceed to the following classification. Our pseudo-Euclidean spacetime comprises three types of planes [87]. It was helpful to mention that our framework is described by a null tetrad frame, hence, our forms have a totally different shape. Besides, we categorize our cases differently. In our **Case 0**, there is a plane with one null vector and it is characterized as a singular case. In **Case 1 and Case 3**, there are two null vectors lie within the plane, and in **Case 2**, there are no null vectors within our plane[1] [88].

In **Case 0** there is only one null vector lies within the plane, using this we obtain the K^0 form. Using the diagonalization procedure one can easily find that it has one triple eigenvalue and it is a Jordan canonical form. The diagonalization reveals that in order to have one timelike and three spacelike eigenvectors as a necessary conidition for our canonical form there are two cases for the triple eigenvalue as we mentioned few lines above. Thus, the canonical form for K^0 results to

$$K^0{}_{\mu\nu} = \begin{pmatrix} 0 & \lambda_1 & -p & -\bar{p} \\ \lambda_1 & 0 & 0 & 0 \\ -p & 0 & \lambda_7 & \lambda_2 \\ -\bar{p} & 0 & \lambda_2 & \lambda_7 \end{pmatrix} \quad p, \bar{p} = \pm 1 \tag{5.13}$$

In **Case 1** we have two canonical forms with one double eigenvalue called as K^1. This case produces two forms but considered as one by us. The reason is based on the symmetry of the symmetrical null tetrad frame. This symmetry referred to the interchanges between the tetrads $n^\mu \leftrightarrow l^\mu$ and $m^\mu \leftrightarrow \bar{m}^\mu$. Regarding this, even if the spin coefficients interchange the result remains the same.

$$K^{1a}{}_{\mu\nu} = \begin{pmatrix} 0 & \lambda_1 & 0 & 0 \\ \lambda_1 & \lambda_0 & 0 & 0 \\ 0 & 0 & \lambda_7 & \lambda_2 \\ 0 & 0 & \lambda_2 & \lambda_7 \end{pmatrix} \quad K^{1b}{}_{\mu\nu} = \begin{pmatrix} \lambda_0 & \lambda_1 & 0 & 0 \\ \lambda_1 & 0 & 0 & 0 \\ 0 & 0 & \lambda_7 & \lambda_2 \\ 0 & 0 & \lambda_2 & \lambda_7 \end{pmatrix} \tag{5.14}$$

Case 2 contains also a canonical form with 4 distinct real eigenvalues called as K^2.

[1]For this case we expect that the two blocks upward-left and downward-right have the exact same form with different components.

$$K^2{}_{\mu\nu} = \begin{pmatrix} \lambda_0 & \lambda_1 & 0 & 0 \\ \lambda_1 & \lambda_0 & 0 & 0 \\ 0 & 0 & \lambda_7 & \lambda_2 \\ 0 & 0 & \lambda_2 & \lambda_7 \end{pmatrix} \tag{5.15}$$

Finally, when considering the K^3 form, it possesses a pair of eigenvalues that are complex conjugates. In a more general scenario, if the form have complex eigenvalues, the tensor can be diagonalized in terms of complex pairs, where each pair consists of eigenvalues that are complex conjugates of each other. However, to maintain the requirement of having one timelike and three spacelike eigenvalues, our tensor can only admit one pair of complex conjugate eigenvalues [85].

$$K^3{}_{\mu\nu} = \begin{pmatrix} \lambda_0 & \lambda_1 & 0 & 0 \\ \lambda_1 & -\lambda_0 & 0 & 0 \\ 0 & 0 & \lambda_7 & \lambda_2 \\ 0 & 0 & \lambda_2 & \lambda_7 \end{pmatrix} \tag{5.16}$$

It should be noted that the only difference between K^2 and K^3 could be described via a factor q. We choose to deal simultaneously with forms K^2 and K^3 with the usage of the parameter $q = \pm 1$ that gives us the 2nd and 3rd forms for $+1$ and -1 accordingly.

5.2.2 The diagonalized form of the Canonical forms

In this section we present the diagronalized canonical forms of the Killing tensor.

$$K^{0\mu}{}_{\nu} = \begin{cases} \lambda_1 = -(\lambda_2 + \lambda_7) \ ; & \begin{pmatrix} \lambda_1 & 1 & 0 & 0 \\ 0 & \lambda_1 & 1 & 0 \\ 0 & 0 & \lambda_1 & 0 \\ 0 & 0 & 0 & -(\lambda_2 - \lambda_7) \end{pmatrix} \\ \lambda_1 = -(\lambda_2 - \lambda_7) \ ; & \begin{pmatrix} \lambda_1 & 1 & 0 & 0 \\ 0 & \lambda_1 & 1 & 0 \\ 0 & 0 & \lambda_1 & 0 \\ 0 & 0 & 0 & -(\lambda_2 + \lambda_7) \end{pmatrix} \end{cases} \tag{Case 0}$$

$$K^{1\mu}{}_{\nu} = \begin{pmatrix} \lambda_1 & 1 & 0 & 0 \\ 0 & \lambda_1 & 0 & 0 \\ 0 & 0 & -(\lambda_7 + \lambda_2) & 0 \\ 0 & 0 & 0 & -(\lambda_2 - \lambda_7) \end{pmatrix} \tag{Case 1}$$

$$K^{2\mu}{}_{\nu} = \begin{pmatrix} \lambda_0 + \lambda_1 & 0 & 0 \\ 0 & \lambda_0 - \lambda_1 & 0 & 0 \\ 0 & 0 & -(\lambda_7 + \lambda_2) & 0 \\ 0 & 0 & 0 & -(\lambda_2 - \lambda_7) \end{pmatrix} \tag{Case 2}$$

$$K^{3\mu}{}_{\nu} = \begin{pmatrix} \lambda_0 + i\lambda_1 & 0 & 0 \\ 0 & \lambda_0 - i\lambda_1 & 0 & 0 \\ 0 & 0 & -(\lambda_7 + \lambda_2) & 0 \\ 0 & 0 & 0 & -(\lambda_2 - \lambda_7) \end{pmatrix} \tag{Case 3}$$

In conclusion, the canonical forms encompass the entire information of arbitrary Killing tensors categorized by their eigenvalues. It is worth to note that the four canonical forms are more general cases of the diagonalized form with two double eigenvalues (*Paradigm*).

Chapter 6

The Study of the 1st Canonical form

6.1 Problem Setup

The K^1 form has been handled in a unique manner. We nullify λ_7 and set λ_0 to be a constant equal to a value represented as $q = \pm 1$. This approach was chosen to achieve a similar form to that of Hauser-Malhiot's, with the only difference being the constant q.

With these simplifications, we obtain a Jordan form of Killing tensor, as opposed to a diagonalized form with two double eigenvalues. Besides, it is widely known that the Jordan canonical form of a matrix embodies all the similar matrices of the family of matrices with the same eigenvalues except the "unique" member of this family, the diagonalized member of the family [89]. We aim thus to find if a Jordan form, which is a more general case than the diagonalized case, encompasses more general Petrov types.

Consequently, this choice facilitates a direct examination of the correlation between the Canonical forms of the Weyl tensor and the Canonical forms of the Killing tensor. We aim to investigate whether the generality of the Canonical forms of the Killing tensor leads to more generalized forms of the Weyl tensor.

Let us begin with the Killing equation as a starting point,

$$K^1_{\mu\nu} = qn_\mu n_\nu + \lambda_1(l_\mu n_\nu + n_\mu l_\nu) + \lambda_2(m_\mu \bar{m}_\nu + \bar{m}_\mu m_\nu) \tag{6.1}$$

Moving forward, we develop the methodology that inserted at the introduction based on our *Paradigm*.

In order to take possible solutions or EFEs, first, we have to consider additional mathematical conditions. In our case the additional conditions are the Integrability Conditions (IC) of the eigenvalues λ_1, λ_2 of our Killing tensor which come by the Killing equation,

$$K_{(\mu\nu;\alpha)} = 0 \tag{6.2}$$

The Killing equation provides us with the following relations. They will be used along with the CR in order to give us the IC of the eigenvalues,

$$\nu = \sigma = \lambda = 0 \tag{6.3}$$

$$q(\epsilon + \bar{\epsilon}) = 0 \qquad q \neq 0 \tag{6.4}$$

$$(\lambda_1 + \lambda_2)\kappa = q(\bar{\alpha} + \beta + \bar{\pi}) \tag{6.5}$$

The directional derivatives for eigenvalues λ_1, λ_2 turned out to be the following

$$D\lambda_1 = q(\gamma + \bar{\gamma}) \tag{6.6}$$

$$\Delta\lambda_1 = 0 \tag{6.7}$$

$$\delta\lambda_1 = (\lambda_1 + \lambda_2)(\bar{\pi} - \tau) \tag{6.8}$$

$$D\lambda_2 = q(\mu + \bar{\mu}) - (\lambda_1 + \lambda_2)(\rho + \bar{\rho}) \tag{6.9}$$

$$\Delta\lambda_2 = (\lambda_1 + \lambda_2)(\mu + \bar{\mu}) \tag{6.10}$$

$$\delta\lambda_2 = 0 \tag{6.11}$$

The relation (6.4) could be used defining a factor Q with its directional derivatives

$$Q \equiv \frac{q}{\lambda_1 + \lambda_2} = \frac{\kappa}{\bar{\alpha} + \beta + \bar{\pi}} \tag{6.12}$$

$$DQ = Q[\rho + \bar{\rho} - Q(\gamma + \bar{\gamma} + \mu + \bar{\mu})] \tag{6.13}$$

$$\Delta Q = -Q(\mu + \bar{\mu}) \tag{6.14}$$

$$\delta Q = -Q(\bar{\pi} - \tau) \tag{6.15}$$

The factor Q was proved helpful in the treatment of the IC and it is a real function since it depends only from real eigenvalues.

6.1.1 Integrabillity Conditions of the 1st Canonical Form

As we mentioned before, the Integrability conditions come to surface by acting of the commutation relations upon to the eigenvalues. Additionally, the commutation relation is the Lie bracket of the basis vectors. We choose to separate the Integrability Conditions of the Eigenvalues in two parts using the factor Q.

Integrability Conditions of Eigenvalue λ_1

$$Q[\delta(\gamma+\bar{\gamma}) - (\gamma+\bar{\gamma})(\bar{\alpha}+\beta-\tau) - (\mu+\bar{\mu})(\bar{\pi}-\tau)] = D(\bar{\pi}-\tau) - (\bar{\pi}-\tau)(\rho+\bar{\rho}) - (\bar{\pi}-\tau)(2\epsilon+\bar{\rho}) \tag{$CR1 : \lambda_1$}$$

$$Q[\Delta(\gamma + \bar{\gamma}) - (\gamma + \bar{\gamma})^2] = 2(\tau\bar{\tau} - \pi\bar{\pi}) \tag{$CR2 : \lambda_1$}$$

$$\Delta(\bar{\pi}-\tau)-(\bar{\pi}-\tau)[(\gamma-\bar{\gamma})-(2\mu+\bar{\mu})]=0 \qquad (CR3:\lambda_1)$$

$$\delta(\pi-\bar{\tau})-\bar{\delta}(\bar{\pi}-\tau)+(\bar{\pi}-\tau)(\alpha-\bar{\beta})-(\pi-\bar{\tau})(\bar{\alpha}-\beta)=Q(\gamma+\bar{\gamma})(\mu-\bar{\mu}) \qquad (CR4:\lambda_1)$$

Integrability Conditions of Eigenvalue λ_2

$$Q[\delta(\mu+\bar{\mu})-2(\mu+\bar{\mu})(\bar{\alpha}+\beta)]=\delta(\rho+\bar{\rho})-(\rho+\bar{\rho})[(\bar{\alpha}+\beta-\bar{\pi})-(\bar{\pi}-\tau)] \qquad (CR1:\lambda_2)$$

$$Q[\Delta(\mu+\bar{\mu})-(\mu+\bar{\mu})[(\mu+\bar{\mu})+2(\gamma+\bar{\gamma})]]=D(\mu+\bar{\mu})+\Delta(\rho+\bar{\rho})-(\rho+\bar{\rho})(\gamma+\bar{\gamma}) \qquad (CR2:\lambda_2)$$

$$\delta(\mu+\bar{\mu})+(\mu+\bar{\mu})[(\bar{\alpha}+\beta-\tau)+(\bar{\pi}-\tau)]=0 \qquad (CR3:\lambda_2)$$

$$2(\mu\bar{\rho}-\bar{\mu}\rho)=Q(\mu+\bar{\mu})(\mu-\bar{\mu}) \qquad (CR4:\lambda_2)$$

6.2 Simplifications: Rotation transformations or a suitable choice

There are various approaches to obtain simplifications for our problem. The most used method involves implementing a null congruence by applying a rotation within the null tetrad frame or exploring different options among the spin coefficients. We have the audacity to consider both of these methods in our pursuit of the most general solution.

6.2.1 Rotation around the null tetrad frame

The IC along with the NPEs end up to be a fearsome system of equations. In this point, we choose to take advantage of the conformal symmetry of a rotation around one of the null tetrads n^μ, l^μ. In our case l is fixed, differently, in a rotation where n was fixed we would get the same results,

$$\tilde{\theta}^1=e^{-a}(\theta^1+p\bar{p}\theta^2+\bar{p}\theta^3+p\theta^4)=\tilde{n}_\mu dx^\mu$$

$$\tilde{\theta}^2=e^{a}\theta^2=\tilde{l}_\mu dx^\mu$$

$$\tilde{\theta}^3=e^{-ib}(\theta^3+p\theta^2)=\tilde{m}_\mu dx^\mu$$

$$\tilde{\theta}^4=e^{ib}(\theta^4+\bar{p}\theta^2)=\tilde{\bar{m}}_\mu dx^\mu$$

$$p\equiv c+id$$

$$K=q\tilde{\theta}^1\otimes\tilde{\theta}^1+\lambda_1(\tilde{\theta}^1\otimes\tilde{\theta}^2+\tilde{\theta}^2\otimes\tilde{\theta}^1)+\lambda_2(\tilde{\theta}^3\otimes\tilde{\theta}^4+\tilde{\theta}^4\otimes\tilde{\theta}^3); \qquad q=\pm1$$

It is easy for someone to prove that only the factor b remains. It does not have any contribution due to the cross terms with $\tilde{\theta}^3, \tilde{\theta}^4$. The rotation applied also in the spin coefficients,

$$\nu=\sigma=\lambda=\epsilon+\bar{\epsilon}=0$$

$$\tilde{\rho}=\rho \qquad \tilde{\mu}=\mu$$

$$\tilde{\kappa}=e^{ib}\kappa$$

$$\tilde{\pi} = e^{-ib}\pi \qquad \tilde{\tau} = e^{ib}\tau$$

$$\tilde{\alpha} = e^{-ib}(\alpha + \frac{\bar{\delta}(ib)}{2}) \qquad \tilde{\beta} = e^{ib}(\beta + \frac{\delta(ib)}{2})$$

$$\tilde{\epsilon} = \epsilon + \frac{D(ib)}{2} \qquad \tilde{\gamma} = \gamma + \frac{\Delta(ib)}{2}$$

Now, one can see that the vanishing of $\tilde{\epsilon}$ satisfies $\epsilon + \bar{\epsilon} = 0$, that was obtained by the Killing equations. It is worth noting that at this point we have multiple options for simplifications using the tilded spin coefficients.

Annihilation of the new spin coefficients

One could take advantage of the previous relations annihilating all the tilded spin coefficients and gain expressions for the non-tilded spin coefficients in terms of the covariant derivatives of the surviving rotation parameter. This could be proved quite helpful yielding the **key relations** which correlate the spin coefficients themselves and with the Weyl components,

$$\Psi_2 - \Lambda = \tau\pi \tag{i}$$

$$\Psi_1 = \kappa\mu \tag{ii}$$

$$\Psi_2 - \Lambda = \mu\rho \tag{iii}$$

$$\mu\tau = 0 \tag{iv}$$

The rotation provides us with these useful relations which connect the spin coefficients along with the Weyl components. This result depends mainly on the form of the Killing Tensor since we demand by the rotation to preserve the Killing tensor invariant. Essentially, the lack either of λ_0 or λ_7 allows us to obtain simplifications such as the **key relations** [1].

The key relations could help us to obtain the Petrov types of this solution. Let us proceed with the cases provided by key relation (iv). The possible cases are three $\mu = 0 \neq \tau$, $\mu = 0 = \tau$ and $\mu \neq 0 = \tau$.

Case 1

The annihilation of μ has an immediate impact at key relations in the first place. Beginning with the condition $\tau \neq 0$, the first relation yields the annihilation of π since $\Psi_2 = \Lambda$ due to relation (iii),

$$0 = \pi \tag{i}$$

$$\Psi_1 = 0 \tag{ii}$$

$$\Psi_2 - \Lambda = 0 \tag{iii}$$

$$\mu = 0 \neq \tau \tag{iv}$$

[1] The annihilation of λ_0 does not serve as a possible reduction to this canonical form since the absence of λ_0 in K^1, K^2, K^3 makes them to coincide. We avoid thus to make such a choice.

Afterwards, we will try to plug these into Newman-Penrose equations and into the integrability conditions. We take the annihilation of every Weyl component except Ψ_0. We already know that $\nu = 0$ nullifies Ψ_4. Next, the simultaneous nullification of π and μ along with the NPEs (m) and (h) clarifies that the solution is of type N since the only survivor is the Ψ_0.

Case 2
The next case is determined by

$$0 = \tau \quad \text{(i)}$$

$$\Psi_1 = 0 \quad \text{(ii)}$$

$$\Psi_2 - \Lambda = 0 \quad \text{(iii)}$$

$$\mu = 0 = \tau \quad \text{(iv)}$$

This combination does not make any difference. Again, we have nullification of Ψ_4 due to NPE (j). Additionally, the IC ($CR3 : \lambda_1$) along with NPEs (i) nullifies Ψ_3 and the $CR2 : \lambda_2$ along with NPE (q) nullifies Ψ_2. Hence, also this case is characterized as Type N.

Case 3
Finally, we get the third case which provides us with

$$\Psi_2 - \Lambda = 0 \quad \text{(i)}$$

$$\Psi_1 = \kappa\mu \quad \text{(ii)}$$

$$0 = \rho \quad \text{(iii)}$$

$$\mu \neq 0 = \tau \quad \text{(iv)}$$

This case is a little trickier but it is proved to have the same Petrov type. Again, the annihilation of ν gives Ψ_4. The IC ($CR4 : \lambda_2$) provides with two possibilities,

$$(\mu + \bar{\mu})(\mu - \bar{\mu}) = 0 \tag{6.16}$$

The only option is $\mu - \bar{\mu} = 0$ since the simultaneous annihilation of ρ and $\mu + \bar{\mu}$ provides us with $\lambda_2 = const$, the latter results to a canonical form without two double eigenvalues. The latter along with NPE (k) provides $\Psi_1 = 0$. The NPE (q) nullifies the Ψ_2 eventually so the only survivors are Ψ_0 and Ψ_3.
We begin with NPE (i) and IC ($CR3 : \lambda_1$) which give

$$\Psi_3 = 2\pi\mu. \tag{6.17}$$

From ($CR1 : \lambda_2$) and ($CR3 : \lambda_2$) we take

$$\pi = \alpha + \bar{\beta} \tag{6.18}$$

Now it is time to introduce Bianchi identity (II) which gives two options,

$$\kappa\Psi_3 = 0 \tag{6.19}$$

It is obvious that the annihilation of Ψ_3 gives the bespoken result. The Petrov type is determined as type N. But the other option, where $\kappa = 0$ along with (6.23) and (6.17) gives $\pi = \alpha + \bar{\beta} = 0$. Then the space results to be conformally flat since the last relation annihilates Ψ_3 as well.

Table 6.1: Rotation around θ^2

Type N	Type N	Type N
$\mu = 0 \neq \tau$ $\Psi_0 \neq 0$	$\mu = 0 = \tau$ $\Psi_0 \neq 0$	$\mu \neq 0 = \tau$ $\Psi_0 \neq 0$
$d\lambda_2 \neq 0$	$d\lambda_2 \neq 0$	$d\lambda_2 \neq 0$

At last, the implication of the rotation provides us with simplifications paving the way to the following theorem.

Theorem: *Petrov type N solution admits $K^1_{\mu\nu}$ Canonical form of Killing tensor.*

The application of rotation is a transformative process that yields valuable relationships connecting not only the spin coefficients amongst themselves but also with the Weyl components through the commutation relations, once the tilded spin coefficients have been annihilated. The outcome critically hinges on the preservation of the structure of the Killing tensor and the form of the spin coefficients. Importantly, the absence of either λ_0 or λ_7 is the catalyst for these relationships. **Employing the complete form of the Killing tensor, conversely, provides no insight into this implied symmetry.**

Remark: Consequently, we infer that additional insights into the conformal symmetry of the rotation can only be gained **when the non-diagonal elements of the Killing tensor are absent.** Thus, we refrain from further exploring the products of the rotational symmetry transformation.

Our next step involves implementing the condition $\pi = \tau$ to the Newman-Penrose equations (NPEs), the Integrability Conditions (IC), and the Bianchi Identities (BI).

6.2.2 Special choice of the rotation parameters

The rotation transformation is a very useful transformation simplifying the system of equations, abolishing all the arbitrariness of our coordinate system provided by the invariant character of Killing tensor during a rotation transformation [13]. In this section we make a special choice of the rotation parameter b which gives us the relation.

$$\pi = \tau \tag{6.20}$$

This choice was used also by Debever et.al in [17]. Actually, Debever obligated to make this choice since the following relations are invariants under the continuous group of transformations.

$$\pi\bar{\tau} - \bar{\pi}\tau = 0 = \mu\bar{\rho} - \bar{\mu}\rho$$

Hence, our choice is an attempt to take advantage of these relations [2]. With this manner we "spare" some of the arbitrariness of the coordinate system. Plugging this relation into $(CR3 : \lambda_1)$ along with the imaginary part of NPE (i), we get

$$\mu - \bar{\mu} = 2(\gamma - \bar{\gamma}) \tag{6.21}$$

$$\Psi_3 - \Psi_3^* = 3(\tau\bar{\mu} - \bar{\tau}\mu) \tag{6.22}$$

Setting (6.22) into (CR3: Ψ_3) or equivalently take the imaginary part of BI (VIII) along with NPEs (i) and (n) we take

$$\Psi_3 = 0 = \tau\bar{\mu} - \bar{\tau}\mu \tag{6.23}$$

Also, the equation $(CR4 : Q)$ will give us the following key relation that lets us unfold the branches of the solution

$$(\mu + \bar{\mu})(\mu - \bar{\mu}) = 0 \tag{6.24}$$

6.2.3 *Case I* : $\mu + \bar{\mu} = 0 \neq \mu - \bar{\mu}$

This case is impossible. The annihilation of the Weyl component Ψ_3, eq. (6.23), along with the relation $(CR4 : \lambda_2)$ leads to

$$\tau + \bar{\tau} = 0 = \rho + \bar{\rho} \tag{6.25}$$

The last relation, and the initial condition which determines the Case I implies $d\lambda_2 = 0$. If we try to take advantage of the latter via NPEs (i) or (n), we take the following result which is impossible for this case

$$\mu - \bar{\mu} = 0 \tag{6.26}$$

6.2.4 *Case II* : $\mu + \bar{\mu} \neq 0 = \mu - \bar{\mu}$

This case is impossible too, as we are going to show it. For this case we already know from $(CR4 : \lambda_2)$ and from the annihilation of Ψ_3 that

$$\mu\bar{\rho} - \bar{\mu}\rho = 0 \Rightarrow \rho - \bar{\rho} = 0 \tag{6.27}$$

$$\mu\bar{\tau} - \bar{\mu}\tau = 0 \Rightarrow \tau - \bar{\tau} = 0 \tag{6.28}$$

Using the above, the imaginary part of NPEs (h) and (g) provide us with

$$\Psi_2 - \Psi_2^* = (\tau + \bar{\tau})\left[(\bar{\alpha} - \beta) - (\alpha - \bar{\beta})\right] \tag{6.29}$$

[2] Regarding these invariant relations, we note that the choice $\pi = \tau$ is not a unique choice. Other possible choices are $\pi = -\tau$ or $\bar{\pi} = \pm\tau$ or $\mu = \pm\rho$ or $\bar{\mu} = \pm\rho$.

For reasons of simplifications we are going to take advantage of $\kappa = Q(\bar{\alpha}+\beta+\bar{\tau})$. If we combine the imaginary parts of NPEs (b), (l), we take the following

$$\delta\bar{\kappa} - \bar{\delta}\kappa = (\kappa-\bar{\kappa})(\tau+\bar{\tau}) - \kappa\left[2(\alpha+\bar{\beta})+(\alpha-\bar{\beta})\right] + \bar{\kappa}\left[2(\bar{\alpha}+\beta)+(\bar{\alpha}-\beta)\right] \quad (a_-)$$

The imaginary part of (l) along with relation (6.26) gives

$$\delta(\alpha+\bar{\beta}) - \bar{\delta}(\bar{\alpha}+\beta) = (\tau+\bar{\tau})\left[(\alpha-\bar{\beta})-(\bar{\alpha}-\beta)\right] + (\alpha+\bar{\beta})(\bar{\alpha}-\beta) - (\bar{\alpha}+\beta)(\alpha-\bar{\beta}) \quad (l_-)$$

Also, from relation (6.15) we know that the factor Q does not have any contribution in the derivation since we know that $\delta Q = 0$. Now, the substitution of $\kappa = Q(\bar{\alpha}+\beta+\bar{\tau})$ in relation (a_-) provide us with

$$\tau(\alpha-\bar{\alpha}) = 0 \quad (6.30)$$

In Appendix I we prove that the Case II is impossible since we cannot reach a solution where the following relation holds,

$$\mu+\bar{\mu} \neq 0 = \mu-\bar{\mu}$$

6.2.5 *Case III* : $\mu+\bar{\mu} = 0 = \mu-\bar{\mu}$

To initiate the treatment of this case, we keep in mind that the relations (6.27) and (6.28) do not contribute here. However, we manage to find key relations which determine our solutions. Considering the last results $\mu = 0$, $\gamma-\bar{\gamma} = 0$ and $\pi = \tau$, let us array the relations that would play a crucial role. We substitute (p) in (h) and the new (h), along with (g), into (q). Then we take

$$\bar{\delta}\tau = -\tau(\tau+\alpha-\bar{\beta}) \quad (g)$$

$$\delta\tau = \tau(\tau-\bar{\alpha}+\beta) \quad (p)$$

$$\Psi_2 + 2\Lambda = -\tau\left[\tau+\bar{\tau}-2(\bar{\alpha}-\beta)\right] \quad (h)$$

$$\Delta\rho - \rho(\gamma+\bar{\gamma}) = -2\tau(\alpha+\bar{\alpha}-\beta-\bar{\beta}) \quad (q)$$

$$(\delta+\bar{\delta})(\tau-\bar{\tau}) = (\tau-\bar{\tau})(\alpha+\bar{\alpha}-\beta-\bar{\beta}) \quad (CR4:\lambda_1)$$

Using the relation (p) with the complex conjugate of (g) we take

$$(\delta+\bar{\delta})(\tau-\bar{\tau}) = -(\tau-\bar{\tau})(\alpha+\bar{\alpha}-\beta-\bar{\beta}) \quad (6.31)$$

The comparison of the latter with the $(CR4:\lambda_1)$ yields

$$(\delta+\bar{\delta})(\tau-\bar{\tau}) = 0 = (\tau-\bar{\tau})(\alpha+\bar{\alpha}-\beta-\bar{\beta}) \quad (6.32)$$

Also the comparison of the real part of (q) with $(CR2:\lambda_2)$ yields

$$(\tau+\bar{\tau})(\alpha+\bar{\alpha}-\beta-\bar{\beta}) = 0 \quad (6.33)$$

As a matter of fact, the equation that we have to deal with is

$$\tau(\alpha+\bar{\alpha}-\beta-\bar{\beta}) = 0 \quad (6.34)$$

The upward relations give birth to three subcases. The only valid solutions of these subcases come to surface with the following choice

$$\tau = 0 = \alpha + \bar{\alpha} - \beta - \bar{\beta}, \tag{6.35}$$

The other two choices, where $\tau = 0 \neq \alpha + \bar{\alpha} - \beta - \bar{\beta}$ or $\tau \neq 0 = \alpha + \bar{\alpha} - \beta - \bar{\beta}$, are not acceptable since they annihilate either the Weyl components, giving us a conformally flat spacetime or the eigenvalues of the Killing Tensor.

6.3 Considerations and Type N Solution

Our problem is now dictated by the following sets of equations.

The Newman-Penrose Field equations

$$D\rho - \bar{\delta}\kappa = \rho^2 - \kappa\left[2(\alpha + \bar{\beta}) + (\alpha - \bar{\beta})\right] \tag{a}$$

$$\delta\kappa = \kappa\left[2(\bar{\alpha} + \beta) - (\bar{\alpha} - \beta)\right] - \Psi_o \tag{b}$$

$$\Delta\kappa - 4\kappa\gamma = \Psi_1 = 0 \tag{c}$$

$$\Psi_2 + 2\Lambda = 0 \tag{h}$$

$$\Delta\rho = \rho(\gamma + \bar{\gamma}) \tag{q}$$

$$\delta\rho = \rho(\bar{\alpha} + \beta) \tag{k}$$

$$\Psi_3 = 0 \tag{m}$$

$$D\alpha - \bar{\delta}\epsilon = \alpha(\rho - 3\epsilon) - \bar{\beta}\epsilon - \bar{\kappa}\gamma \tag{d}$$

$$D\beta - \delta\epsilon = \beta(\bar{\rho} - \bar{\epsilon}) - \kappa\gamma - \epsilon\bar{\alpha} \tag{e}$$

$$\Delta\alpha = \bar{\delta}\gamma + \gamma(\alpha + \bar{\beta}) \tag{r}$$

$$\Delta\beta = \delta\gamma + \gamma(\bar{\alpha} + \beta) \tag{o}$$

$$\delta\alpha - \bar{\delta}\beta = \alpha(\bar{\alpha} - \beta) - \beta(\alpha - \bar{\beta}) + \gamma(\rho - \bar{\rho}) - 3\Lambda \tag{l}$$

$$D\gamma = \Delta\epsilon + \epsilon(\gamma + \bar{\gamma}) = 0 \tag{f}$$

$$\Psi_4 = 0 \tag{j}$$

The Bianchi Identities

$$\bar{\delta}\Psi_0 = 4\alpha\Psi_0 + 3\kappa\Psi_2 \tag{I}$$

$$\rho\Psi_2 = 0 \tag{II}$$

$$\Delta\Psi_0 = 2(\gamma + \bar{\gamma})\Psi_0 \tag{V}$$

Integrability Conditions of Eigenvalue λ_1

$$\delta(\gamma + \bar{\gamma}) = (\gamma + \bar{\gamma})(\bar{\alpha} + \beta) \tag{$CR1 : \lambda_1$}$$

$$\Delta(\gamma + \bar{\gamma}) = (\gamma + \bar{\gamma})^2 \tag{$CR2 : \lambda_1$}$$

Integrability Conditions of Eigenvalue λ_2

$$\delta(\rho + \bar{\rho}) = (\rho + \bar{\rho})(\bar{\alpha} + \beta) \tag{$CR1 : \lambda_2$}$$

$$\Delta(\rho+\bar{\rho}) = (\rho+\bar{\rho})(\gamma+\bar{\gamma}) \qquad (CR2:\lambda_2)$$

Basically our solution is constructed with two constraints. The first one prohibits the annihilation of Weyl tensor and the second maintains the derivatives of the eigenvalues of the Killing tensor. Along these lines, we take the following constraints

$$\gamma+\bar{\gamma} \neq 0 \neq \rho+\bar{\rho}$$

Constrained by the last expression, the BI (II) reduces to the following

$$(\rho-\bar{\rho}) = 0 = \Psi_2 \tag{6.36}$$

The relation (6.36) is the only option since $\rho+\bar{\rho}\neq 0$ [3].

6.3.1 Frobenius Theorem

We are now ready to apply all the previous simplifications of the spin coefficients in order to define our local coordinate system. We proceed with the implication of the Frobenius Integrability theorem.

The Cartan's structure equation are

$$d\theta^1 = (\gamma+\bar{\gamma})\theta^1\wedge\theta^2 + (\bar{\alpha}+\beta)\theta^1\wedge\theta^3 + (\alpha+\bar{\beta})\theta^1\wedge\theta^4 \tag{6.37}$$

$$d\theta^2 = Q(\bar{\alpha}+\beta)\theta^1\wedge\theta^3 + Q(\alpha+\bar{\beta})\theta^1\wedge\theta^4 - (\bar{\alpha}+\beta)\theta^2\wedge\theta^3 - (\alpha+\bar{\beta})\theta^2\wedge\theta^4 \tag{6.38}$$

$$d\theta^3 = -(2\epsilon+\rho)\theta^1\wedge\theta^3 + (\alpha-\bar{\beta})\theta^3\wedge\theta^4 \tag{6.39}$$

$$d\theta^4 = (2\epsilon-\rho)\theta^1\wedge\theta^4 - (\bar{\alpha}-\beta)\theta^3\wedge\theta^4 \tag{6.40}$$

It follows that

$$d\theta^1\wedge\theta^1\wedge\theta^2 = 0 \tag{6.41}$$

$$d\theta^2\wedge\theta^1\wedge\theta^2 = 0 \tag{6.42}$$

$$d(\theta^3-\theta^4)\wedge(\theta^3-\theta^4)\wedge(\theta^3+\theta^4) = 0 \tag{6.43}$$

$$d(\theta^3+\theta^4)\wedge(\theta^3-\theta^4)\wedge(\theta^3+\theta^4) = 0 \tag{6.44}$$

On account of Frobenius Integrability theorem, we define a local coordinate system (t,z,x,y) such that

$$\theta^1 = (L-N)dt + (M-P)dz \tag{6.45}$$

$$\theta^2 = (L+N)dt + (M+P)dz \tag{6.46}$$

$$\theta^3 = Sdx + iRdy \tag{6.47}$$

$$\theta^4 = Sdx - iRdy \tag{6.48}$$

[3] We consider this as the only option because the nullification of $\rho-\bar{\rho}$ serve as a catalyst in order to define our local coordinate system with the most simple away. Differently, the eq. (6.38) would contain the term $(\rho-\bar{\rho})\theta^3\wedge\theta^4$, which adds more metric functions, increasing the complexity of the problem.

where L,N,M,P,S,R are real valued functions of (t, z, x, y) [4]. The corresponding directional derivatives are the following

$$D = \frac{(M+P)\partial_t - (L+N)\partial_z}{2Z} \tag{6.49}$$

$$\Delta = -\frac{(M-P)\partial_t - (L-N)\partial_z}{2Z} \tag{6.50}$$

$$\delta + \bar{\delta} = \frac{\partial_x}{S} \tag{6.51}$$

$$\delta - \bar{\delta} = -i\frac{\partial_y}{R}, \tag{6.52}$$

where the function Z is defined by

$$Z \equiv PL - MN \tag{6.53}$$

The next step concerns a derivation of relations (6.45)-(6.48). Then, we express $d\theta^\mu$ in terms of the two-forms $\theta^\nu \wedge \theta^\sigma$. Afterwards, we are going to equate the new relations with the relations (6.37)-(6.40). In this fashion we can obtain relations of spin coefficients in terms of the metric functions. An immediate impact of the latter is the nullifying of the imaginary part of ϵ. The remains relations are presented below

$$(L-N) = A(M-P) \qquad A = A(t,z) \tag{6.54}$$

$$\Delta S = \Delta R = (\delta + \bar{\delta})R = 0 \tag{6.55}$$

$$(M+P)_t = (L+N)_z \tag{6.56}$$

$$Z_x = Z_y = 0 \tag{6.57}$$

$$\gamma + \bar{\gamma} = \frac{(M-P)_t - (L-N)_z}{2Z} \tag{6.58}$$

$$\alpha - \bar{\beta} = \frac{1}{2}\frac{(\delta - \bar{\delta})S}{S} \tag{6.59}$$

$$\bar{\alpha} + \beta = -\frac{\delta(M-P)}{M-P} = -\frac{\delta(L-N)}{L-N} = \frac{\delta[A(M+P) - (L+N)]}{A(M+P) - (L+N)} \tag{6.60}$$

$$\kappa = Q(\bar{\alpha} + \beta) = -\frac{\delta[(L-N)(L+N)]}{(L-N)^2} = -Q\frac{\delta(M-P)}{M-P} \tag{6.61}$$

$$\rho = -\frac{DS}{S} = -\frac{DR}{R} \tag{6.62}$$

The above relations provide us with information that can be used to reshape the spin coefficients and to simplify the metric functions as well. Moreover, the relation (6.58)-(6.62) could help us to specify the relation between the different parts of equations. Regarding all these we get

[4] In this point it should be noted that the downward indices denote the derivation with respect to coordinates.

$$(L-N)=A(M-P) \qquad A=A(t,z) \tag{6.63}$$

$$M_t-L_z+P_t-N_z=0 \tag{6.64}$$

$$\epsilon=\Delta S=\Delta R=R_x=0 \tag{6.65}$$

$$Z_x=Z_y=0 \quad \rightarrow \quad Z=(M-P)(AM-L) \tag{6.66}$$

$$\gamma+\bar{\gamma}=\frac{1}{(M-P)}\frac{M_t-L_z}{AM-L} \tag{6.67}$$

$$\alpha-\bar{\beta}=\frac{1}{2}\frac{(\delta-\bar{\delta})S}{S} \tag{6.68}$$

$$\bar{\alpha}+\beta=-\frac{\delta(M-P)}{M-P}=\frac{\delta(AM-L)}{AM-L} \tag{6.69}$$

$$\kappa=Q(\bar{\alpha}+\beta)=-\frac{\delta[(L-N)(L+N)]}{(L-N)^2}=-Q\frac{\delta(M-P)}{M-P} \tag{6.70}$$

$$\rho=-\frac{DS}{S}=-\frac{DR}{R} \tag{6.71}$$

Considering that $\kappa=Q(\bar{\alpha}+\beta)$ and the equation (6.71), we take

$$(L-N)[(L+N)-\frac{Q}{2}(L-N)]=\tilde{Z}(t,z) \tag{6.72}$$

$$S=\Sigma(x,y)R \tag{6.73}$$

6.3.2 Type N solution

In this segment we will give the method of solution in detail. The starting point is the information that emerged from the implication of Frobenius theorem.

Lets array once again our equations using the simplifications that we get from the implication of the Frobenius theorem. The integrability conditions for λ_2 coincide with NPEs (q) and (k) since $\rho-\bar{\rho}=0$. Along these lines we abolish by (r) and (o) the derivatives of γ using the integrability condition $(CR1:\lambda_1)$.

$$D\rho-\bar{\delta}\kappa=\rho^2-\kappa\left[2(\alpha+\bar{\beta})+(\alpha-\bar{\beta})\right] \tag{a}$$

$$\delta\kappa=\kappa\left[2(\bar{\alpha}+\beta)-(\bar{\alpha}-\beta)\right]-\Psi_o \tag{b}$$

$$\Delta\rho=\rho(\gamma+\bar{\gamma}) \tag{q}$$

$$\delta\rho=\rho(\bar{\alpha}+\beta) \tag{k}$$

$$D(\bar{\alpha}+\beta)=(\bar{\alpha}+\beta)[\rho-Q(\gamma+\bar{\gamma})] \tag{d)+(e}$$

$$D(\alpha-\bar{\beta})=\rho(\alpha-\bar{\beta}) \tag{d)-(e}$$

$$\Delta(\bar{\alpha}+\beta)=2(\gamma+\bar{\gamma})(\bar{\alpha}+\beta) \tag{r)+(o}$$

$$\Delta(\bar{\alpha}-\beta)=0 \tag{r)-(o}$$

$$\delta(\alpha+\bar{\beta})-\bar{\delta}(\bar{\alpha}+\beta)=-(\alpha-\bar{\beta})[\bar{\alpha}+\beta+\alpha+\bar{\beta}] \qquad (l_-)$$

$$(\delta-\bar{\delta})(\alpha-\bar{\beta})=-2(\alpha-\bar{\beta})^2 \qquad (l_+)$$

$$D\gamma=0 \qquad \text{(f)}$$

The Bianchi Identities become

$$\bar{\delta}\Psi_0=2\Psi_0[(\alpha+\bar{\beta})+(\alpha-\bar{\beta})] \qquad \text{(I)}$$

$$\Delta\Psi_0=2(\gamma+\bar{\gamma})\Psi_0 \qquad \text{(V)}$$

The Integrability Conditions of Eigenvalue λ_1 take the form

$$\delta(\gamma+\bar{\gamma})=(\gamma+\bar{\gamma})(\bar{\alpha}+\beta) \qquad (CR1:\lambda_1)$$

$$\Delta(\gamma+\bar{\gamma})=(\gamma+\bar{\gamma})^2 \qquad (CR2:\lambda_1)$$

We start our proof with equations $(d)-(e)$ and eq. (6.71). The function of integration V is not a function of t,z since $DV=\Delta V=0$, thus

$$\frac{D(\alpha-\bar{\beta})}{\alpha-\bar{\beta}}=\rho=-\frac{DS}{S}\Rightarrow \qquad \alpha-\bar{\beta}=\frac{-i}{2}\frac{V}{S} \qquad (6.74)$$

Additionally, trying to satisfy the relation $(l)_+$ along with (6.73), we get that $V=V(x)$ and it satisfies the following

$$V(x)=\frac{S_y}{R} \qquad \Leftrightarrow \qquad S(t,z,x,y)=V(x)\int R(t,z,y)\,dy \qquad (6.75)$$

The next step is to determine the form of $\bar{\alpha}+\beta$. In order to achieve it, we start with the following equations

$$DQ=Q[2\rho-Q(\gamma+\bar{\gamma})] \qquad (6.13)$$

$$D(\bar{\alpha}+\beta)=(\bar{\alpha}+\beta)[\rho-Q(\gamma+\bar{\gamma})] \qquad \text{(d)+(e)}$$

If we try to substitute the factor $[\rho-Q(\gamma+\bar{\gamma})]$ using the (6.13) we get

$$\frac{D(\bar{\alpha}+\beta)}{\bar{\alpha}+\beta}=\frac{DQ}{Q}+\frac{DS}{S} \qquad (6.76)$$

The latter results to $\bar{\alpha}+\beta=\tilde{W}QS$, where $D\tilde{W}=0$ and W is a complex function. Using now equation (r)-(o) we get

$$(\bar{\alpha}+\beta)=WQS(\gamma+\bar{\gamma})^2; \qquad W(x,y)=W_R(x,y)+iW_I(x,y) \qquad (6.77)$$

Furthermore, trying to combine equations (q), (k) and $(CR2:\lambda_1)$, we get a relation which correlates ρ with γ with the integration function r. Along these lines it worths saying that the annihilation due to the action of Δ gives $\partial_t-A\partial_z$,

$$(\rho+\bar{\rho})=r(t,z)(\gamma+\bar{\gamma}); \qquad r_t-Ar_z=0 \qquad (6.78)$$

Another useful step is the integration of Bianchi Identity (V) using of cource the equation $(CR2:\lambda_1)$ again. The latter yields the following

$$\Psi_0 = \Psi(\gamma+\bar{\gamma})^2 \qquad \Delta\Psi = 0 \quad \& \quad \Psi = \Psi_R + i\Psi_I \tag{6.79}$$

We have managed thus to express all the equations (6.77)-(6.79) in terms of $\gamma+\bar{\gamma}$. This is a necessary step in order to acquire a necessary simplification. This simplification concerns the NPE (a) and is represented by $D\rho-\rho^2=0$. In order to obtain such an indispensable result, we must solve the equations NPEs $(a),(b)$ and BI (I) simultaneously. This would not be possible without the three aforementioned equations.

If we make the necessary derivations for NPE (a) and (b) taking into account that $\kappa = Q(\bar{\alpha}+\beta)$ and $(CR1:\lambda_1)$, we get the corresponding real and imaginary parts. Moreover, for reasons of simplification, we define the left part of (a_+) as Ω, which does not depend from x, y [5],

$$\Omega = \frac{1}{SR}\left[[W_R SR]_x - [W_I S^2]_y\right] \tag{a_+}$$

$$0 = [W_I SR]_x + [W_R S^2]_y \tag{a_-}$$

$$\Psi_R = -\frac{Q^2}{2SR}\left[[W_R SR]_x + W_{I,y}S^2\right] \quad \xrightarrow{(a_+)} \quad \Psi_R = Q^2\left[\frac{\Omega}{2} - \frac{[W_I S]_y}{R}\right] \tag{b_+}$$

$$\Psi_I = -\frac{Q^2}{2SR}\left[[W_I SR]_x - W_{R,y}S^2\right] \quad \overrightarrow{(a_-)} \quad \Psi_I = Q^2\left[\frac{[W_R S]_y}{R}\right] \tag{b_-}$$

If we consider now the form $\Psi_0 = \Psi(\gamma+\bar{\gamma})^2$, the BI (I) takes the form

$$\bar{\delta}\Psi = 2\Psi(\alpha-\bar{\beta}) \tag{6.80}$$

If we substitute our new relations for the real and imaginary part of Ψ, and try to abolish the function R with the usage of (6.75), we get

$$SR\Psi_{R,x} - [\Psi_I S^2]_y = 0 \quad \rightarrow \quad 2\frac{S_y}{S} = -\frac{\Psi_{I,y}}{\Psi_I - \frac{\Psi_{R,x}}{2V}} \tag{I_+}$$

$$SR\Psi_{I,x} + [\Psi_R S^2]_y = 0 \quad \rightarrow \quad 2\frac{S_y}{S} = -\frac{\Psi_{R,y}}{\Psi_R + \frac{\Psi_{I,x}}{2V}} \tag{I_-}$$

At this point we use the relations $(b_\pm)$ in relation (I_-),

$$2\frac{S_y}{S} = -\frac{\left[\frac{[W_I S]_y}{R}\right]_y}{\left[\frac{\Omega}{2} - \frac{[W_I S]_y}{R}\right] + \frac{[W_R S]_{xy}}{2VR}} \tag{6.81}$$

In order to obtain our result, we have to deal with $\frac{[W_R S]_{xy}}{2VR}$. Using the relation (a_+) we derive the final result

[5] In appendix II we prove that $\Omega = 2\frac{D\rho-\rho^2}{Q^2(\gamma+\bar{\gamma})^2} \not\ni x, y$.

$$2\frac{S_y}{S} = -\frac{\left[\frac{[W_I S]_y}{R}\right]_y}{\Omega + \frac{S}{2S_y}\left[\frac{[W_R S]_y}{R}\right]_y} \qquad \Rightarrow \qquad \Omega = 0 = D\rho - \rho^2 \tag{6.82}$$

Up to this point, the metric takes the following form and the functions have to be determined in full detail.

$$ds^2 = (M^2 - P^2)\left[A^2(t,z)dt^2 + dz^2\right] + 2(M-P)\left[A(t,z)M + L\right]dtdz - R^2\left[\Sigma(x,y)dx^2 + dy^2\right] \tag{6.83}$$

Hence, more work is needed to be done. **Nevertheless, the key conclusion drawn from the study of the $K^1_{\mu\nu}$ form is that, despite being a more general Killing tensor, it leads to entirely different Petrov types.**

Chapter 7

The Study of the 2nd and the 3rd Canonical forms

By the study of the K^1 form we can conclude that the assumption of existence of more general Killing tensor forms does not guarantee more general Petrov types. Additionally, based on our previous study, we can conclude that the annihilation of λ_7 leads to further simplifications, facilitated by the rotation, as indicated in the remark on p. 56.

7.1 Problem Setup

In order to find exact solutions of Einstein's field equations (in the context of Newman-Penrose formalism) which are restricted by the Bianchi identities we impose additional symmetries (Killing vectors, Killing tensors). The commutation relations of the tetrads along with the simultaneous use of the integrability conditions of Killing tensor permit us to simplify the system of equations. Along these lines, we begin with the Killing equation

$$K_{(\mu\nu;\alpha)} = 0 \tag{7.1}$$

Defining the factor $q = \pm 1$, we consider a unified approach for both canonical forms K^2 and K^3. The only difference in the K^2 and K^3 forms is the -1 in the K^3_{22} component of the tensor. Obviously, we get the $K^2_{\mu\nu}$ for $q = +1$ and the $K^3_{\mu\nu}$ for $q = -1$. This modification allows us to solve the problem

$$K^{2,3}_{\mu\nu} = \lambda_0(n_\mu n_\nu + q l_\mu l_\nu) + \lambda_1(l_\mu n_\nu + n_\mu l_\nu) - \lambda_2(m_\mu \bar{m}_\nu + \bar{m}_\mu m_\nu) \tag{7.2}$$

The Killing equation along with the commutation relations results to the integrability conditions

$$\sigma = \lambda = 0 \tag{7.3}$$

The directional derivatives of λ_0, λ_1, λ_2 can be seen to be the following

$$D\lambda_0 = 2\lambda_0(\epsilon + \bar{\epsilon}) \tag{7.4}$$

$$\Delta\lambda_0 = -2\lambda_0(\gamma + \bar{\gamma}) \tag{7.5}$$

$$\delta\lambda_0 = 2(\lambda_0(\bar{\alpha} + \beta + \bar{\pi}) - \kappa(\lambda_1 + \lambda_2)) \tag{7.6}$$

$$\delta\lambda_0 = 2(-\lambda_0(\bar{\alpha} + \beta + \tau) + q\bar{\nu}(\lambda_1 + \lambda_2)) \tag{7.7}$$

$$\delta\lambda_0 = \lambda_0(\bar{\pi} - \tau) - (\kappa - q\bar{\nu})(\lambda_1 + \lambda_2) \tag{7.8}$$

$$D\lambda_1 = 2\lambda_0(\gamma + \bar{\gamma}) \tag{7.9}$$

$$\Delta\lambda_1 = -2q\lambda_0(\epsilon + \bar{\epsilon}) \tag{7.10}$$

$$\delta\lambda_1 = -q\lambda_0(\kappa - q\bar{\nu}) + (\lambda_1 + \lambda_2)(\bar{\pi} - \tau) \tag{7.11}$$

$$D\lambda_2 = \lambda_0(\mu + \bar{\mu}) - (\lambda_1 + \lambda_2)(\rho + \bar{\rho}) \tag{7.12}$$

$$\Delta\lambda_2 = -q\lambda_0(\rho + \bar{\rho}) - (\lambda_1 + \lambda_2)(\mu + \bar{\mu}) \tag{7.13}$$

$$\delta\lambda_2 = 0 \tag{7.14}$$

The above relations (7.6) and (7.7) indicate that we can define a factor Q.

$$Q \equiv \frac{\lambda_0}{\lambda_1 + \lambda_2} = \frac{\kappa + q\bar{\nu}}{2(\bar{\alpha} + \beta) + \bar{\pi} + \tau} \tag{7.15}$$

and then,

$$DQ = Q(2(\epsilon + \bar{\epsilon}) + (\rho + \bar{\rho})) - Q^2(2(\gamma + \bar{\gamma}) + (\mu + \bar{\mu})) \tag{7.16}$$

$$\Delta Q = -Q(2(\gamma + \bar{\gamma}) + (\mu + \bar{\mu})) + qQ^2(2(\epsilon + \bar{\epsilon}) + (\rho + \bar{\rho})) \tag{7.17}$$

$$\delta Q = (qQ^2 - 1)(\kappa - q\bar{\nu}) \tag{7.18}$$

The factor Q is proved helpful during the treatment of the IC and it is a real scalar function since it depends solely on real scalars.

7.1.1 Integrabillity Conditions of the 2nd and 3rd Canonical form

We use the commutators of the tetrads in order to obtain the integrability conditions of Killing tensor. As we mentioned in **Chapter 5**, the commutation relations are equivalent with the Lie bracket of the null tetrads. We choose to divide the integrability conditions using the factor Q.

Integrability Conditions of λ_0

$$2Q[D(\gamma+\bar{\gamma})+\Delta(\epsilon+\bar{\epsilon})+\pi\bar{\pi}-\tau\bar{\tau}] = -[(\pi+\bar{\tau})(q\bar{\nu}-\kappa)+(\bar{\pi}+\tau)(q\nu-\bar{\kappa})] \quad (CR1:\lambda_0)$$

$$Q[2[\delta(\epsilon+\bar{\epsilon})-(\epsilon+\bar{\epsilon})(\bar{\alpha}+\beta-\bar{\pi})]-[D(\bar{\pi}-\tau)-(\bar{\pi}-\tau)(\bar{\rho}+\epsilon-\bar{\epsilon})]+2\kappa(\gamma+\bar{\gamma})-(q\bar{\nu}-\kappa)[2(\gamma+\bar{\gamma}) + (\mu+\bar{\mu})]] = D(q\bar{\nu}-\kappa)-(q\bar{\nu}-\kappa)[2\epsilon+\bar{\rho}+\epsilon+\bar{\epsilon}+\rho+\bar{\rho}] \quad (CR2:\lambda_0)$$

$$Q[2[\delta(\gamma+\bar{\gamma})+(\gamma+\bar{\gamma})(\bar{\alpha}+\beta-\tau)]+[\Delta(\bar{\pi}-\tau)+(\bar{\pi}-\tau)(\mu-\gamma+\bar{\gamma})]-2\bar{\nu}(\epsilon+\bar{\epsilon})-q(q\bar{\nu}-\kappa)[2(\epsilon+\bar{\epsilon}) + \rho+\bar{\rho}]] = \Delta(\kappa-q\bar{\nu})+(\kappa-q\bar{\nu})[2(\gamma+\bar{\gamma})+(\mu+\bar{\mu})+\mu-\gamma+\bar{\gamma}] \quad (CR3:\lambda_0)$$

$$Q[\bar{\delta}(\bar{\pi}-\tau)-\delta(\pi-\bar{\tau})-(\bar{\pi}-\tau)(\alpha-\bar{\beta})+(\pi-\bar{\tau})(\bar{\alpha}-\beta)+2[(\epsilon+\bar{\epsilon})(\mu-\bar{\mu})-(\gamma+\bar{\gamma})(\rho-\bar{\rho})]] = \delta(q\nu-\bar{\kappa})-\bar{\delta}(q\bar{\nu}-\kappa)+(q\bar{\nu}-\kappa)(\alpha-\bar{\beta})-(q\nu-\bar{\kappa})(\bar{\alpha}-\beta) \quad (CR4:\lambda_0)$$

Integrability Conditions of λ_1

$$Q[\Delta(\gamma+\bar{\gamma})-3(\gamma+\bar{\gamma})^2+q[D(\epsilon+\bar{\epsilon})+3(\epsilon+\bar{\epsilon})^2]+\frac{q}{2}[(\pi+\bar{\tau})(q\bar{\nu}-\kappa)+(\bar{\pi}+\tau)(q\nu-\bar{\kappa})]] = -(\pi\bar{\pi}-\tau\bar{\tau}) \quad (CR1:\lambda_1)$$

$$Q[2[\delta(\gamma+\bar{\gamma})-(\gamma+\bar{\gamma})(\bar{\alpha}+\beta-\bar{\pi})]-q[D(q\bar{\nu}-\kappa)+(q\bar{\nu}-\kappa)(\epsilon+3\bar{\epsilon}+\bar{\rho})-2\kappa(\epsilon+\bar{\epsilon})]] = D(\bar{\pi}-\tau)-(\bar{\pi}-\tau)(\rho+2\bar{\rho}+\epsilon-\bar{\epsilon})-2(\gamma+\bar{\gamma})(q\bar{\nu}-\kappa) \quad (CR2:\lambda_1)$$

$$Q[2q[\delta(\epsilon+\bar{\epsilon})+(\epsilon+\bar{\epsilon})(\bar{\alpha}+\beta-\tau)]+q[\Delta(q\bar{\nu}-\kappa)-(q\bar{\nu}-\kappa)(3\gamma+\bar{\gamma}-\mu)]-2\bar{\nu}(\gamma+\bar{\gamma})] = -[\Delta(\bar{\pi}-\tau)+(\bar{\pi}-\tau)(2\mu+\bar{\mu}-\gamma+\bar{\gamma})+2q(q\bar{\nu}-\kappa)(\epsilon+\bar{\epsilon})] \quad (CR3:\lambda_1)$$

$$Q[q[\delta(q\nu-\bar{\kappa})-\bar{\delta}(q\bar{\nu}-\kappa)+(q\bar{\nu}-\kappa)(\alpha-\bar{\beta})-(q\nu-\bar{\kappa})(\bar{\alpha}-\beta)]+2[q(\epsilon+\bar{\epsilon})(\rho-\bar{\rho})-(\gamma+\bar{\gamma})(\mu-\bar{\mu})] = \bar{\delta}(\bar{\pi}-\tau)-\delta(\pi-\bar{\tau})-(\bar{\pi}-\tau)(\alpha-\bar{\beta})+(\pi-\bar{\tau})(\bar{\alpha}-\beta) \quad (CR4:\lambda_1)$$

Integrability Conditions of λ_2

$$Q[[\Delta(\mu+\bar{\mu})-(\mu+\bar{\mu})-5(\gamma+\bar{\gamma})]+q[D(\rho+\bar{\rho})+(\rho+\bar{\rho})[(\rho+\bar{\rho})-5(\epsilon+\bar{\epsilon})]]] = \Delta(\rho+\bar{\rho})-(\rho+\bar{\rho})(\gamma+\bar{\gamma})+D(\mu+\bar{\mu})+(\mu+\bar{\mu})(\epsilon+\bar{\epsilon}) \quad (CR1:\lambda_2)$$

$$Q[\delta(\mu+\bar{\mu})-(\mu+\bar{\mu})[(\bar{\alpha}+\beta+\tau)-2\bar{\pi}]+q(\rho+\bar{\rho})(2\kappa-q\bar{\nu})] = \delta(\rho+\bar{\rho})-(\rho+\bar{\rho})[\bar{\alpha}+\beta+\tau-2\bar{\pi}]+(\mu+\bar{\mu})(2\kappa-q\bar{\nu}) \quad (CR2:\lambda_2)$$

$$qQ[\delta(\rho+\bar{\rho})+(\rho+\bar{\rho})[\bar{\alpha}+\beta+\bar{\pi}-2\tau]+(\mu+\bar{\mu})(\kappa-2q\bar{\nu})] = \delta(\mu+\bar{\mu})+(\mu+\bar{\mu})[\bar{\alpha}+\beta+\bar{\pi}-2\tau]+q(\rho+\bar{\rho})(\kappa-2q\bar{\nu}) \quad (CR3:\lambda_2)$$

$$Q[(\mu+\bar{\mu})(\mu-\bar{\mu})-q(\rho+\bar{\rho})(\rho-\bar{\rho})] = (\mu-\bar{\mu})(\rho+\bar{\rho})-(\rho-\bar{\rho})(\mu+\bar{\mu}) \quad (CR4:\lambda_2)$$

7.1.2 Rotation around the null tetrad frame

The IC along with the NPEs end up to be a cumbersome system of equations. We choose to take advantage of the conformal symmetry of a rotation around one of the null vectors n^μ, l^μ. We choose l^μ to be fixed as in the previous chapter. Furthermore, the Killing tensor remains invariant under the rotation,

$$K = \lambda_0(\tilde{\theta}^1 \otimes \tilde{\theta}^1 + q\tilde{\theta}^2 \otimes \tilde{\theta}^2) + \lambda_1(\tilde{\theta}^1 \otimes \tilde{\theta}^2 + \tilde{\theta}^2 \otimes \tilde{\theta}^1) + \lambda_2(\tilde{\theta}^3 \otimes \tilde{\theta}^4 + \tilde{\theta}^4 \otimes \tilde{\theta}^3); \qquad q = \pm 1$$

It is easy for someone to prove that the only non-zero rotation parameter is b. This is valid due to the existence of the cross terms with $\tilde{\theta}^3, \tilde{\theta}^4$. The rotation is also applied to spin coefficients yielding the relations below:

$$\sigma = 0 = \lambda$$

$$\begin{aligned} \tilde{\rho} &= \rho & \tilde{\mu} &= \mu \\ \tilde{\kappa} &= e^{ib}\kappa & \tilde{\tau} &= e^{ib}\tau \\ \tilde{\pi} &= e^{-ib}\pi & \tilde{\nu} &= e^{-ib}\nu \\ \tilde{\alpha} &= e^{-ib}(\alpha + \frac{\bar{\delta}(ib)}{2}) & \tilde{\beta} &= e^{ib}(\beta + \frac{\delta(ib)}{2}) \\ \tilde{\epsilon} &= \epsilon + \frac{D(ib)}{2} & \tilde{\gamma} &= \gamma + \frac{\Delta(ib)}{2} \end{aligned}$$

The above relations can be used in order to simplify the spin coefficients. We can set the four last tilded spin coefficients equal to zero. Then, we substitute the four last relations into the CR, resulting the key relations which unfold the branches of the solution,

$$\Psi_2 - \Lambda = \kappa\nu - \tau\pi \tag{i}$$

$$\Psi_1 = \kappa\mu \tag{ii}$$

$$\Psi_2 - \Lambda = \mu\rho \tag{iii}$$

$$\mu\tau = 0 \tag{iv}$$

The rotation provided us with these useful relations connecting the spin coefficients with the Weyl tensor components. The result depends mainly on the form of the Killing tensor since we demand the invariance of the Killing tensor under the rotation. Essentially, the lack either of λ_0 or λ_7 supplies us with key relations that simplifie the problem. In case we choose to deal with the full expression of Killing tensor, we wouldn't get any further information based on the implying symmetry. The latter is a remarkable assistance in the pursuit of the solution.

We present the classes of solutions are classified by the usage of key relations. Initiating by $\mu\tau = 0$ the NPEs, IC, BI, provides us with the three main classes of our solution. These classes emerged by the three branches of the latter relation. Also, we aim to study solutions in which only one of the eigenvalues of Killing tensor, is allowed to be constant. It would be helpful for the reader if at this point we array the most useful NPEs and the $(CR4 : \lambda_2)$ along with the key relations.

Besides, we mainly use these relations in order to determine the classes of the solution. So, we have

$$\delta\kappa = \kappa(\tau - \bar{\pi} + 2(\bar{\alpha} + \beta) - (\bar{\alpha} - \beta)) - \Psi_o \quad \text{(b)}$$

$$\bar{\delta}\pi = -\pi(\pi + \alpha - \bar{\beta}) + \nu\bar{\kappa} \quad \text{(g)}$$

$$\delta\tau = \tau(\tau - \bar{\alpha} + \beta) - \bar{\nu}\kappa \quad \text{(p)}$$

$$\delta\rho = \rho(\bar{\alpha} + \beta) + \tau(\rho - \bar{\rho}) + \kappa(\mu - \bar{\mu}) - \Psi_1 \quad \text{(k)}$$

$$\bar{\delta}\mu = -\mu(\alpha + \bar{\beta}) - \pi(\mu - \bar{\mu}) - \nu(\rho - \bar{\rho}) + \Psi_3 \quad \text{(m)}$$

$$\bar{\delta}\nu = -\nu(2(\alpha + \bar{\beta}) + (\alpha - \bar{\beta}) + \pi - \bar{\tau}) + \Psi_4 \quad \text{(j)}$$

$$Q[(\mu + \bar{\mu})(\mu - \bar{\mu}) - q(\rho + \bar{\rho})(\rho - \bar{\rho})] = (\mu - \bar{\mu})(\rho + \bar{\rho}) - (\rho - \bar{\rho})(\mu + \bar{\mu}) \quad (CR4 : \lambda_2)$$

7.1.3 *Class I* : $\mu = 0$

The annihilation of μ permit us to obtain the first class of solution. The key relations take the form

$$\Psi_2 - \Lambda = 0 = \kappa\nu - \tau\pi \quad \text{(i)}$$

$$\Psi_1 = 0 \quad \text{(ii)}$$

$$\Psi_2 - \Lambda = 0 \quad \text{(iii)}$$

$$\mu = 0 \quad \text{(iv)}$$

Considering that $\mu = 0$ the relation $(CR4 : \lambda_2)$ gives $(\rho + \bar{\rho})(\rho - \bar{\rho}) = 0$. According to IC (7.12)-(7.14) the derivative of the eigenvalue λ_2 depends only from the real parts of μ and ρ, so our priority is to avoid the extinction of $\rho + \bar{\rho} = 0$. Unavoidably, the BI (II) implies that the derivative of λ_2 vanishes. However, from NPE (m) we take $\Psi_3 = \nu(\rho - \bar{\rho}) = 0$. Hence, the Bianchi Identities take the form

$$\bar{\delta}\Psi_0 = (4\alpha - \pi)\Psi_0 + 3\kappa\Psi_2 \quad \text{(I)}$$

$$0 = (\rho + \bar{\rho})\Psi_2 \quad \text{(II)}$$

$$0 = -3\pi\Psi_2 + \kappa\Psi_4 \quad \text{(III)}$$

$$D\Psi_4 = -4\epsilon\Psi_4 \quad \text{(IV)}$$

$$\Delta\Psi_0 = 4\gamma\Psi_0 \quad \text{(V)}$$

$$\Delta\Psi_1 = \nu\Psi_0 - 3\tau\Psi_2 \quad \text{(VI)}$$

$$-\delta\Psi_4 = 3\nu\Psi_2 + (4\beta - \tau)\Psi_4 \quad \text{(VIII)}$$

In the following Tables 7.1, 7.2, every column represents different solutions according to different choices which are ordered by the key relations and specifically by the BI (II). The columns of the tables contain the main charasteristics of our solutions. The second and third column of Table 7.1 is distinguished by the different choices that take place due to the BI (III) and BI (VI).

Table 7.1: $\rho - \bar{\rho} = 0 = \rho + \bar{\rho}$

$\rho = 0 \neq \Psi_2$ Type D	$\rho + \bar{\rho} = 0 = \Psi_2$ Type N	$\rho + \bar{\rho} = 0 = \Psi_2$ Type N
$\kappa\nu = \pi\tau$ $\Psi_0\Psi_4 = 9{\Psi_2}^2$ $\Psi_1 = \Psi_3 = 0$	$\nu = 0 = \pi\tau$ $\Psi_0 \neq 0$ $\Psi_1 = \Psi_2 = \Psi_3 = \Psi_4 = 0$	$\kappa = 0 = \pi\tau$ $\Psi_4 \neq 0$ $\Psi_0 = \Psi_1 = \Psi_2 = \Psi_3 = 0$
$d\lambda_2 = 0$	$d\lambda_2 = 0$	$d\lambda_2 = 0$

The other choice where $\rho + \bar{\rho} \neq 0 = \Psi_2$ yields type N solutions where the eigenvalue λ_2 is not a constant. The combination of BI (III) with BI (VI) determines the non-zero Weyl component.

Table 7.2: $\rho - \bar{\rho} = 0 \neq \rho + \bar{\rho}$

Type N	Type N	Type N	Type N	Type N	Type N
$\nu = 0 = \tau$ $\Psi_0 \neq 0$	$\nu = 0 = \pi$ $\Psi_4 \neq 0$	$\kappa = 0 = \tau$ $\Psi_0 \neq 0$	$\kappa = 0 = \pi$ $\Psi_4 \neq 0$	$\nu = \pi = \tau = 0$ $\Psi_0 \neq 0$	$\kappa = \pi = \tau = 0$ $\Psi_4 \neq 0$
$d\lambda_2 \neq 0$	$d\lambda_2 \neq 0$	$d\lambda_2 \neq 0$	$d\lambda_2 \neq 0$	$d\lambda_2 \neq 0$	$d\lambda_2 \neq 0$

7.1.4 *Class II:* $\mu = 0 = \tau$

As before, the NPE (m) yields $\Psi_3 = \nu(\rho - \bar{\rho}) = 0$. Considering the key relation (ii) we obtain $\Psi_1 = 0$, similarly considering NPE (i) we obtain $\kappa\nu = 0$. Thus, it is obvious that the Class II consists a subset of Class I. These solutions are the type N solutions of the previous class with $\tau = 0$.

7.1.5 *Class III:* $\tau = 0$

In this class we encountered new algebraically special solutions. NPE (p) for $\tau = 0$ yields $\bar{\kappa}\nu = 0$. As a result, the key relations are reformulated as follows,

$$\Psi_2 - \Lambda = 0 = \kappa\nu \tag{i}$$

$$\Psi_1 = \kappa\mu \tag{ii}$$

$$\Psi_2 - \Lambda = 0 = \mu\rho \tag{iii}$$

$$\tau = 0 \tag{iv}$$

The branch where $\mu = 0$ is already known from the Class I, II, which are type N solutions with a further simplification $\tau = 0$. On the other hand, the case of $\mu \neq 0 = \rho$ yields solutions which are worth to be mentioned. Furthermore, the relation $(CR4 : \lambda_2)$ plays a crucial role since the annihilation of ρ implies $(\mu + \bar{\mu})(\mu - \bar{\mu}) = 0$. The constraint $\mu \neq 0$ leads us to two separate solutions for case $\kappa = 0 \neq \nu$, which are both of Type III. The other case, where both κ and ν are zero, concerns two solutions where only Ψ_3 is not equal to zero determining that the type of the solutions are Type III.

The last branch of solutions contains the case $\kappa \neq 0 = \nu$. It should be noted that NPE (k) with $\rho = 0$ gives us the form of Ψ_1, which results to the annihilation of the real part of μ and at the same time it dictates that $d\lambda_2 = 0$, since we know that the derivatives of λ_2 depends only on the real part of μ and ρ. Along these lines, we gain that

$$\Psi_1 = \kappa(\mu - \bar{\mu}) \tag{k}$$

Table 7.3: $\mu \neq 0 = \rho$

$\mu - \bar{\mu} = 0$ Type III	$\mu + \bar{\mu} = 0$ Type III
$\kappa = 0 \neq \nu$ $\Psi_3 \neq 0 \neq \Psi_4$	$\kappa = 0 \neq \nu$ $\Psi_3 \neq 0 \neq \Psi_4$
$d\lambda_2 \neq 0$	$d\lambda_2 = 0$

The Class III is presented in the Table 7.3. All these cases are characterized by $\Psi_2 = \Lambda = 0$ which arises from NPE (q) for $\rho = \tau = 0$, while in these Type III cases we know that $\Psi_1 = 0$. Finally, after all the previous discussion, we postulate the following theorem.

Theorem: *Petrov Types III, D and N admit both $K^2_{\mu\nu}, K^3_{\mu\nu}$ Canonical Forms of Killing tensor with $\lambda_7 = 0$.*

7.2 Considerations and Type D Solution

In this work we are focused on Type D solutions of Class I only for the case $q = +1$. The Killing tensor in this case admits 4 distinct eigenvalues, although, the annihilation of λ_7 yields a double eigenvalue, which is λ_2.

Our solution is of Type D and the components of Weyl tensor are connected by the relation $\Psi_0\Psi_4 = 9{\Psi_2}^2$ with $\Psi_2 = \Lambda$. Generally speaking, Type D spacetimes have only one non-zero Weyl component, the Ψ_2. However, there are two other versions as well. The first version is characterized by the relation $3\Psi_2\Psi_4 = 2\Psi_3^2$, where $\Psi_0 = \Psi_1 = 0$, and the second version is the same as ours where $\Psi_0\Psi_4 = 9{\Psi_2}^2$ with $\Psi_1 = \Psi_3 = 0$ [45]. At last, both versions are equivalent and could be obtained with two classes of rotations. Chandrasekhar and Xanthopoulos [90] proved that in a chosen null-tetrad frame, the type D character of our case could be obtained with two classes of rotations around l^μ, n^μ. We operated the same rotation in Appendix F.

In this section we demonstrate these useful relations that determine our solution. Also, taking advantage by the rotation with l^μ fixed, we obtain the key relations. **This is the maximal utilization of symmetry that one could**

gain from a rotation around the null tetrad frame with the 2nd Canonical form of the Killing Tensor with λ_7,

$$\sigma = 0 = \lambda$$
$$\mu = 0 = \rho$$
$$\Psi_0\Psi_4 = 9\Psi_2^2$$
$$\Psi_2 = \Lambda = Constant$$
$$\Psi_1 = 0 = \Psi_3$$
$$\kappa\nu = \tau\pi$$
$$\bar{\alpha} + \beta = 0$$
$$\epsilon + \bar{\epsilon} = 0$$
$$\gamma + \bar{\gamma} = 0$$

The NPEs, BI, IC, with the substitution of the relations above, are given by

$$\bar{\delta}\kappa = \bar{\kappa}\tau + \kappa((\alpha - \bar{\beta}) - \pi) \quad \text{(a)}$$
$$\delta\kappa = \kappa(\tau - \bar{\pi} - (\bar{\alpha} - \beta)) - \Psi_o \quad \text{(b)}$$
$$D\tau = \Delta\kappa + \tau(\epsilon - \bar{\epsilon}) - 2\kappa\gamma \quad \text{(c)}$$
$$D\nu - \Delta\pi = \pi(\gamma - \bar{\gamma}) - 2\nu\epsilon \quad \text{(i)}$$
$$\bar{\delta}\pi = -\pi(\pi + \alpha - \bar{\beta}) + \nu\bar{\kappa} \quad \text{(g)}$$
$$\delta\tau = \tau(\tau - \bar{\alpha} + \beta) - \bar{\nu}\kappa \quad \text{(p)}$$
$$-\delta\pi = \pi(\bar{\pi} - \bar{\alpha} + \beta) - \kappa\nu + \Psi_2 + 2\Lambda \quad \text{(h)}$$
$$\delta\nu = -\bar{\nu}\pi + \nu(\tau + (\bar{\alpha} - \beta)) \quad \text{(n)}$$
$$-\bar{\delta}\tau = -\tau(\bar{\tau} + \alpha - \bar{\beta}) + \nu\kappa - \Psi_2 - 2\Lambda \quad \text{(q)}$$
$$\Psi_1 = 0 \quad \text{(k)}$$
$$\Psi_3 = 0 \quad \text{(m)}$$
$$D\alpha - \bar{\delta}\epsilon = \alpha(\bar{\epsilon} - 2\epsilon) - \bar{\beta}\epsilon - \bar{\kappa}\gamma + \pi\epsilon \quad \text{(d)}$$
$$D\beta - \delta\epsilon = -\beta\bar{\epsilon} - \kappa\gamma - \epsilon(\bar{\alpha} - \bar{\pi}) \quad \text{(e)}$$
$$\Delta\alpha - \bar{\delta}\gamma = \nu\epsilon + \alpha\bar{\gamma} + \gamma(\bar{\beta} - \bar{\tau}) \quad \text{(r)}$$
$$-\Delta\beta + \delta\gamma = \gamma\tau - \epsilon\bar{\nu} - \beta(\gamma - \bar{\gamma}) \quad \text{(o)}$$
$$\delta\alpha - \bar{\delta}\beta = \alpha(\bar{\alpha} - \beta) - \beta(\alpha - \bar{\beta}) \quad \text{(l)}$$
$$D\gamma - \Delta\epsilon = \alpha(\tau + \bar{\pi}) + \beta(\bar{\tau} + \pi) \quad \text{(f)}$$
$$\bar{\delta}\nu = -\nu(\alpha - \bar{\beta} + \pi - \bar{\tau}) + \Psi_4 \quad \text{(j)}$$

The Bianchi Identities become

$$\bar{\delta}\Psi_0 = (4\alpha - \pi)\Psi_0 + 3\kappa\Psi_2 \tag{I}$$
$$0 = 0 \tag{II}$$
$$3\pi\Psi_2 = \kappa\Psi_4 \tag{III}$$
$$D\Psi_4 = -4\epsilon\Psi_4 \tag{IV}$$
$$\Delta\Psi_0 = 4\gamma\Psi_0 \tag{V}$$
$$\nu\Psi_0 = 3\tau\Psi_2 \tag{VI}$$
$$0 = 0 \tag{VII}$$
$$-\delta\Psi_4 = 3\nu\Psi_2 + (4\beta - \tau)\Psi_4 \tag{VIII}$$

The Integrability conditions of λ_0 take the form

$$2Q[\pi\bar{\pi} - \tau\bar{\tau}] = (\pi + \bar{\tau})(\kappa - q\bar{\nu}) + (\bar{\pi} + \tau)(\bar{\kappa} - q\nu) \qquad (CR1 : \lambda_0)$$
$$Q[D(\bar{\pi} - \tau) - (\bar{\pi} - \tau)(\epsilon - \bar{\epsilon})] = D(\kappa - q\bar{\nu}) - (\epsilon - \bar{\epsilon})(\kappa - q\bar{\nu}) \qquad (CR2 : \lambda_0)$$
$$Q[\Delta(\bar{\pi} - \tau) - (\gamma - \bar{\gamma})(\bar{\pi} - \tau)] = \Delta(\kappa - q\bar{\nu}) - (\gamma - \bar{\gamma})(\kappa - q\bar{\nu}) \qquad (CR3 : \lambda_0)$$

$$Q[\bar{\delta}(\bar{\pi} - \tau) - \delta(\pi - \bar{\tau}) - (\bar{\pi} - \tau)(\alpha - \bar{\beta}) + (\pi - \bar{\tau})(\bar{\alpha} - \beta)]$$
$$= \delta(q\nu - \bar{\kappa}) - \bar{\delta}(q\bar{\nu} - \kappa) + (q\bar{\nu} - \kappa)(\alpha - \bar{\beta}) - (q\nu - \bar{\kappa})(\bar{\alpha} - \beta) \qquad (CR4 : \lambda_0)$$

The Integrability conditions of λ_1 take the form

$$Q\frac{q}{2}[(\pi + \bar{\tau})(\kappa - q\bar{\nu}) + (\bar{\pi} + \tau)(\bar{\kappa} - q\nu)] = (\pi\bar{\pi} - \tau\bar{\tau}) \qquad (CR1 : \lambda_1)$$
$$qQ[D(\kappa - q\bar{\nu}) - (\epsilon - \bar{\epsilon})(\kappa - q\bar{\nu})] = D(\bar{\pi} - \tau) - (\bar{\pi} - \tau)(\epsilon - \bar{\epsilon}) \qquad (CR2 : \lambda_1)$$
$$qQ[\Delta(\kappa - q\bar{\nu}) - (\gamma - \bar{\gamma})(\kappa - q\bar{\nu})] = \Delta(\bar{\pi} - \tau) - (\gamma - \bar{\gamma})(\bar{\pi} - \tau) \qquad (CR3 : \lambda_1)$$

$$qQ[\delta(q\nu - \bar{\kappa}) - \bar{\delta}(q\bar{\nu} - \kappa) + (q\bar{\nu} - \kappa)(\alpha - \bar{\beta}) - (q\nu - \bar{\kappa})(\bar{\alpha} - \beta)]$$
$$= \bar{\delta}(\bar{\pi} - \tau) - \delta(\pi - \bar{\tau}) - (\bar{\pi} - \tau)(\alpha - \bar{\beta}) + (\pi - \bar{\tau})(\bar{\alpha} - \beta) \qquad (CR4 : \lambda_1)$$

At this point it should be noted that due to the annihilation of μ and ρ coefficients, the eigenvalue λ_2 is constant. So, we don't take any information from its commutation relations. Also, it is obvious that the corresponding commutation relations yield the same information since the condition $qQ^2 - 1 = 0$ is not valid for both $K^2_{\mu\nu}$ and $K^3_{\mu\nu}$. Hence, the CR are summarized below

$$\bar{\pi}\pi - \tau\bar{\tau} = 0 \tag{7.19}$$
$$\bar{\kappa}\kappa - \bar{\nu}\nu = 0 \tag{7.20}$$
$$D(\kappa - q\bar{\nu}) - (\epsilon - \bar{\epsilon})(\kappa - q\bar{\nu}) = 0 \tag{7.21}$$
$$D(\bar{\pi} - \tau) - (\epsilon - \bar{\epsilon})(\bar{\pi} - \tau) = 0 \tag{7.22}$$
$$\Delta(\kappa - q\bar{\nu}) - (\gamma - \bar{\gamma})(\kappa - q\bar{\nu}) = 0 \tag{7.23}$$
$$\Delta(\bar{\pi} - \tau) - (\gamma - \bar{\gamma})(\bar{\pi} - \tau) = 0 \tag{7.24}$$
$$\delta(q\nu - \bar{\kappa}) - \bar{\delta}(q\bar{\nu} - \kappa) + (q\bar{\nu} - \kappa)(\alpha - \bar{\beta}) - (q\nu - \bar{\kappa})(\bar{\alpha} - \beta) = 0 \tag{7.25}$$
$$\bar{\delta}(\bar{\pi} - \tau) - \delta(\pi - \bar{\tau}) - (\bar{\pi} - \tau)(\alpha - \bar{\beta}) + (\pi - \bar{\tau})(\bar{\alpha} - \beta) = 0 \tag{7.26}$$

7.2.1 Frobenius Theorem

Considering the relations (7.6), (7.7), (7.8) we can make a suitable choice for our spin coefficients. One possible solution is the following,

$$\bar{\pi} + \tau = 0 \tag{7.27}$$

$$\kappa + q\bar{\nu} = 0 \tag{7.28}$$

This solution regards only the form $K^2_{\mu\nu}$ and we already have (7.29) at our disposal due to the rotation.

$$\bar{\alpha} + \beta = 0 \tag{7.29}$$

The substitution of (7.27), (7.28) in the equation $\kappa\nu = \pi\tau$ dictates $q = +1$. We shall now proceed to the implication of the Frobenius Integrability theorem.

The Cartan's structure equations are

$$d\theta^1 = -\bar{\pi}\theta^1 \wedge \theta^3 - \pi\theta^1 \wedge \theta^4 - \bar{\nu}\theta^2 \wedge \theta^3 - \nu\theta^2 \wedge \theta^4 \tag{7.30}$$

$$d\theta^2 = \kappa\theta^1 \wedge \theta^3 + \bar{\kappa}\theta^1 \wedge \theta^4 + \tau\theta^2 \wedge \theta^3 + \bar{\tau}\theta^2 \wedge \theta^4 \tag{7.31}$$

$$d\theta^3 = -(\epsilon - \bar{\epsilon})\theta^1 \wedge \theta^3 - (\gamma - \bar{\gamma})\theta^2 \wedge \theta^3 + (\alpha - \bar{\beta})\theta^3 \wedge \theta^4 \tag{7.32}$$

$$d\theta^4 = -(\epsilon - \bar{\epsilon})\theta^1 \wedge \theta^4 - (\gamma - \bar{\gamma})\theta^2 \wedge \theta^4 - (\bar{\alpha} - \beta)\theta^3 \wedge \theta^4 \tag{7.33}$$

It follows that

$$d\theta^1 \wedge \theta^1 \wedge \theta^2 = 0 \tag{7.34}$$

$$d\theta^2 \wedge \theta^1 \wedge \theta^2 = 0 \tag{7.35}$$

$$d(\theta^3 - \theta^4) \wedge (\theta^3 - \theta^4) \wedge (\theta^3 + \theta^4) = 0 \tag{7.36}$$

$$d(\theta^3 + \theta^4) \wedge (\theta^3 - \theta^4) \wedge (\theta^3 + \theta^4) = 0, \tag{7.37}$$

which, on account of Frobenius Integrability theorem, implies the existence of a local coordinate system (t, z, x, y) such that

$$\theta^1 = (L - N)dt + (M - P)dz \tag{7.38}$$

$$\theta^2 = (L + N)dt + (M + P)dz \tag{7.39}$$

$$\theta^3 = Sdx + iRdy \tag{7.40}$$

$$\theta^4 = Sdx - iRdy, \tag{7.41}$$

where L, N, M, P, S, R are real valued functions of (t, z, x, y) [1]. Next, if one replaces the differential forms in (7.30)-(7.33) by their values (7.38)-(7.41) and equates the corresponding coefficients of the differentials it follows that

$$R_t = R_z = S_t = S_z = 0 \Rightarrow \gamma - \bar{\gamma} = \epsilon - \bar{\epsilon} = 0 \tag{7.42}$$

$$M_t = L_z \tag{7.43}$$

[1] At this point it should be noted that the lower-case indices denote the derivation with respect to coordinates.

$$P_t = N_z \tag{7.44}$$

$$M_x L - L_x M = 0 = M_y L - L_y M = 0 \tag{7.45}$$

$$P_x N - N_x P = 0 = P_y N - N_y P = 0 \tag{7.46}$$

$$\bar{\pi} = -\tau = \frac{\delta Z}{2Z} \tag{7.47}$$

$$\kappa = -\bar{\nu} = \frac{(M_x N - N_x M) + (P_x L - L_x P)}{4ZS} - i\frac{(M_y N - N_y M) + (P_y L - L_y P)}{4ZR} \tag{7.48}$$

$$2\alpha = \alpha - \bar{\beta} = \frac{-1}{2}(\frac{(\delta + \bar{\delta})R}{R} - \frac{(\delta - \bar{\delta})S}{S}) \tag{7.49}$$

$$Z \equiv PL - MN \tag{7.50}$$

Taking advantage of relations (7.45), (7.46), we get

$$L = A(t, z)M \tag{7.51}$$

$$N = B(t, z)P \tag{7.52}$$

and substituting them into (7.43)-(7.44), we get the following relations for spin coefficients and the corresponding simplifications as well,

$$M_t = (AM)_z \tag{7.53}$$

$$P_t = (BP)_z \tag{7.54}$$

$$\bar{\pi} = -\tau = \frac{\delta(PM)}{2PM} \tag{7.55}$$

$$\kappa = -\bar{\nu} = \frac{1}{2}(\frac{\delta P}{P} - \frac{\delta M}{M}) \tag{7.56}$$

$$2\alpha = \alpha - \bar{\beta} = -\frac{1}{2}(\frac{(\delta + \bar{\delta})R}{R} - \frac{(\delta - \bar{\delta})S}{S}) \tag{7.57}$$

$$Z \equiv PL - MN = (A - B)PM \tag{7.58}$$

The results of the implication of the Frobenius theorem have a great impact in the NPEs, BI, IC. The Newman-Penrose equations become

$$\bar{\delta}\kappa = \bar{\kappa}\tau + \kappa((\alpha - \bar{\beta}) - \pi) \tag{a}$$

$$\delta\kappa = \kappa(\tau - \bar{\pi} - (\bar{\alpha} - \beta)) - \Psi_o \tag{b}$$

$$\bar{\delta}\pi = -\pi(\pi + \alpha - \bar{\beta}) + \nu\bar{\kappa} \tag{g}$$

$$\delta\tau = \tau(\tau - \bar{\alpha} + \beta) - \bar{\nu}\kappa \tag{p}$$

$$-\delta\pi = \pi(\bar{\pi} - \bar{\alpha} + \beta) - \kappa\nu + \Psi_2 + 2\Lambda \tag{h}$$

$$\delta\nu = -\bar{\nu}\pi + \nu(\tau + (\bar{\alpha} - \beta)) \tag{n}$$

$$-\bar{\delta}\tau = -\tau(\bar{\tau} + \alpha - \bar{\beta}) + \nu\kappa - \Psi_2 - 2\Lambda \tag{q}$$

$$D\alpha = D\beta = 0 \tag{d}$$

$$\Delta\alpha = \Delta\beta = 0 \tag{r}$$

$$\delta\alpha - \bar{\delta}\beta = \alpha(\bar{\alpha} - \beta) - \beta(\alpha - \bar{\beta}) \tag{l}$$

$$\bar{\delta}\nu = -\nu(\alpha - \bar{\beta} + \pi - \bar{\tau}) + \Psi_4 \tag{j}$$

Bianchi Identities require a reformation in order to be functionable. Regarding this, it is easy to correlate Ψ_0 with Ψ_4 combining BI (III) with BI (VI). Next, we aim to abolish Ψ_4 by our relations. Hence, we multiply BI (IV) with π and with the usage of $\kappa\nu = \pi\tau$ we get

$$3\kappa\Psi_2 = \pi\Psi_0, \tag{VI}$$

The latter, combined with BI (I), gives

$$\bar{\delta}\Psi_0 = 4\alpha\Psi_0 \tag{I}$$

$$D\Psi_0 = 0 \tag{IV}$$

$$\Delta\Psi_0 = 0, \tag{V}$$

where the relations between the Weyl components are given by

$$\Psi_0 = \Psi_4^* \tag{7.59}$$

$$\Psi_4\Psi_4^* = \Psi_0\Psi_0^* = 9\Lambda^2 \tag{7.60}$$

At last, the Integrability conditions resulted to be the following,

$$D\kappa = \Delta\kappa = D\nu = \Delta\nu = 0 \tag{7.61}$$

$$D\pi = \Delta\pi = D\tau = \Delta\tau = 0 \tag{7.62}$$

$$\delta\bar{\kappa} - \bar{\delta}\kappa = \kappa(\alpha - \bar{\beta}) - \bar{\kappa}(\bar{\alpha} - \beta) \tag{7.63}$$

$$\bar{\delta}\bar{\pi} - \delta\pi = \bar{\pi}(\alpha - \bar{\beta}) - \pi(\bar{\alpha} - \beta) \tag{7.64}$$

The above relation (7.63) can be obtained by the NPEs (a) and (b). The relations (d), (r), (7.61)-(7.62), clarify that our metric doesn't depend from t, z since every spin coefficient is annihilated both by D, Δ. As we know, the type D solutions admit a Riemannian-Maxwellian invertible structure, hence, there is an invertible Abelian two-parameter isometry group. This has been proved by [17], [91]. Considering that the vectors ∂_t, ∂_z, or a combination of these two, result to be commutative Killing vectors, then our equations can be expressed as follows,

Newman Penrose Equations

$$(\delta + \bar{\delta})(\pi + \bar{\pi}) = -(\pi + \bar{\pi})^2 - (\kappa + \bar{\kappa})^2 - (\pi - \bar{\pi})[(\alpha - \bar{\beta}) - (\bar{\alpha} - \beta)] - 6\Psi_2 \tag{7.65}$$

$$(\delta - \bar{\delta})(\pi - \bar{\pi}) = (\pi - \bar{\pi})^2 + (\kappa - \bar{\kappa})^2 + (\pi + \bar{\pi})[(\alpha - \bar{\beta}) + (\bar{\alpha} - \beta)] - 6\Psi_2 \tag{7.66}$$

$$(\delta + \bar{\delta})(\pi - \bar{\pi}) = -(\pi - \bar{\pi})(\pi + \bar{\pi}) + (\kappa + \bar{\kappa})(\kappa - \bar{\kappa}) - (\pi + \bar{\pi})[(\alpha - \bar{\beta}) - (\bar{\alpha} - \beta)] \tag{7.67}$$

$$(\delta - \bar{\delta})(\pi + \bar{\pi}) = (\pi - \bar{\pi})(\pi + \bar{\pi}) - (\kappa + \bar{\kappa})(\kappa - \bar{\kappa}) + (\pi - \bar{\pi})[(\alpha - \bar{\beta}) + (\bar{\alpha} - \beta)] \tag{7.68}$$

$$(\delta+\bar{\delta})(\kappa+\bar{\kappa}) = -2(\pi+\bar{\pi})(\kappa+\bar{\kappa})+(\kappa-\bar{\kappa})[(\alpha-\bar{\beta})-(\bar{\alpha}-\beta)]-(\Psi_0+\Psi_0^*) \quad (7.69)$$

$$(\delta-\bar{\delta})(\kappa-\bar{\kappa}) = 2(\pi-\bar{\pi})(\kappa-\bar{\kappa})+(\kappa+\bar{\kappa})[(\alpha-\bar{\beta})+(\bar{\alpha}-\beta)]-(\Psi_0+\Psi_0^*) \quad (7.70)$$

$$(\delta-\bar{\delta})(\kappa+\bar{\kappa}) = -(\pi+\bar{\pi})(\kappa-\bar{\kappa})+(\pi-\bar{\pi})(\kappa+\bar{\kappa})-(\kappa-\bar{\kappa})[(\alpha-\bar{\beta})+(\bar{\alpha}-\beta)]-(\Psi_0-\Psi_0^*) \quad (7.71)$$

$$(\delta+\bar{\delta})(\kappa-\bar{\kappa}) = -(\pi+\bar{\pi})(\kappa-\bar{\kappa})+(\pi-\bar{\pi})(\kappa+\bar{\kappa})+(\kappa+\bar{\kappa})[(\alpha-\bar{\beta})-(\bar{\alpha}-\beta)]-(\Psi_0-\Psi_0^*) \quad (7.72)$$

$$\delta(\alpha-\bar{\beta})+\bar{\delta}(\bar{\alpha}-\beta) = 2(\alpha-\bar{\beta})(\bar{\alpha}-\beta) \quad (7.73)$$

Bianchi Identities

$$\bar{\delta}\Psi_0 = 4\alpha\Psi_0 \quad (I)$$

$$3\kappa\Psi_2 = \pi\Psi_0 \quad (VI)$$

Using the relations for spin coefficients

$$\bar{\pi} = -\tau = \frac{\delta(PM)}{2PM} \quad (7.74)$$

$$\kappa = -\bar{\nu} = \frac{1}{2}\left(\frac{\delta P}{P} - \frac{\delta M}{M}\right) \quad (7.75)$$

$$2\alpha = \alpha - \bar{\beta} = -\frac{1}{2}\left(\frac{(\delta+\bar{\delta})R}{R} - \frac{(\delta-\bar{\delta})S}{S}\right) \quad (7.76)$$

$$Z \equiv PL - MN = (A-B)PM \quad (7.77)$$

$$(\delta+\bar{\delta}) = \frac{\partial_x}{S} \quad (7.78)$$

$$(\delta-\bar{\delta}) = (-i)\frac{\partial_y}{R} \quad (7.79)$$

our NPEs are listed below,

$$12\Psi_2 = -\frac{1}{PR}\left[\left[\frac{P_y}{R}\right]_y + \frac{R_x}{S}\frac{P_x}{S}\right] - \frac{1}{MR}\left[\left[\frac{M_y}{R}\right]_y + \frac{R_x}{S}\frac{M_x}{S}\right] \quad (7.80)$$

$$12\Psi_2 = -\frac{1}{PS}\left[\left[\frac{P_x}{S}\right]_x + \frac{S_y}{R}\frac{P_y}{R}\right] - \frac{1}{MS}\left[\left[\frac{M_x}{S}\right]_x + \frac{S_y}{R}\frac{M_y}{R}\right] \quad (7.81)$$

$$2(\Psi_0+\Psi_0^*) = \frac{1}{PR}\left[\left[\frac{P_y}{R}\right]_y + \frac{R_x}{S}\frac{P_x}{S}\right] - \frac{1}{MR}\left[\left[\frac{M_y}{R}\right]_y + \frac{R_x}{S}\frac{M_x}{S}\right] \quad (7.82)$$

$$2(\Psi_0+\Psi_0^*) = -\frac{1}{PS}\left[\left[\frac{P_x}{S}\right]_x + \frac{S_y}{R}\frac{P_y}{R}\right] + \frac{1}{MS}\left[\left[\frac{M_x}{S}\right]_x + \frac{S_y}{R}\frac{M_y}{R}\right] \quad (7.83)$$

$$2(-i)(\Psi_0 - \Psi_0^*) = \frac{1}{PR}\left[\left[\frac{P_x}{S}\right]_y - \frac{R_x}{S}\frac{P_y}{R}\right] - \frac{1}{MR}\left[\left[\frac{M_x}{S}\right]_y - \frac{R_x}{S}\frac{M_y}{R}\right] \quad (7.84)$$

$$2(-i)(\Psi_0 - \Psi_0^*) = \frac{1}{PS}\left[\left[\frac{P_y}{R}\right]_x - \frac{S_y}{R}\frac{P_x}{S}\right] - \frac{1}{MS}\left[\left[\frac{M_y}{R}\right]_x - \frac{S_y}{R}\frac{M_x}{S}\right] \quad (7.85)$$

$$0 = \frac{1}{PR}\left[\left[\frac{P_x}{S}\right]_y - \frac{R_x}{S}\frac{P_y}{R}\right] + \frac{1}{MR}\left[\left[\frac{M_x}{S}\right]_y - \frac{R_x}{S}\frac{M_y}{R}\right] \quad (7.86)$$

$$0 = \frac{1}{PS}\left[\left[\frac{P_y}{R}\right]_x - \frac{S_y}{R}\frac{P_x}{S}\right] + \frac{1}{MS}\left[\left[\frac{M_y}{R}\right]_x - \frac{S_y}{R}\frac{M_x}{S}\right] \quad (7.87)$$

$$\left[\frac{R_x}{S}\right]_x + \left[\frac{S_y}{R}\right]_y = 0 \quad (7.88)$$

7.2.2 Separation of Hamilton-Jacobi Equation

It's time to imply the separation of Hamilton-Jacobi equation. Since our metric functions have no dependency on t, z, the Hamilton-Jacobi action is soluble with the most simple possible way [55].

However, a more generic separation of variables in Hamilton-Jacobi equation was already achieved by Shapovalov [92] and Bagrov [93] who made known a family of spacetimes with N-parametric Abelian group of motions, where N=0,1,2,3. Also, a complete separation of Hamilton-Jacobi equation in four dimensions was achieved by Katanaev [94].

The HJ action and the corresponding HJ equation are presented,

$$\mathcal{S} = at - bz + S_1(x) + S_2(y) \quad (7.89)$$

$$\bar{m}^2 = g^{\mu\nu}\frac{\partial \mathcal{S}}{\partial x^\mu}\frac{\partial \mathcal{S}}{\partial x^\nu}, \quad (7.90)$$

The inverse metric is

$$g^{\mu\nu} = \begin{pmatrix} \frac{P^2-M^2}{2Z^2} & \frac{AM^2-BP^2}{2Z^2} & 0 & 0 \\ \frac{AM^2-BP^2}{2Z^2} & \frac{B^2P^2-A^2M^2}{2Z^2} & 0 & 0 \\ 0 & 0 & -\frac{1}{2S^2} & \\ 0 & 0 & 0 & -\frac{1}{2R^2} \end{pmatrix}. \quad (7.91)$$

Using these previous relations we finally take:

$$2\bar{m}^2 = -\frac{\mathcal{S}_y^2}{R^2} - \frac{\mathcal{S}_x^2}{S^2} + \frac{\tilde{B}^2}{M^2} - \frac{\tilde{A}^2}{P^2} \quad (7.92)$$

The new tilded quantities are constants since they are related with constants A, B and the constants of motion due to the action of the commutative Killing vectors ∂_t, ∂_z.

$$\tilde{A} \equiv \frac{a + Ab}{A - B} \quad (7.93)$$

$$\tilde{B} \equiv \frac{a + Bb}{A - B} \tag{7.94}$$

The separation allows us to introduce the function $\Omega^2 \equiv \Phi(x) + \Psi(y)$,

$$2\Omega^2 \bar{m}^2 = -\frac{\Omega^2}{R^2}\mathcal{S}_y^2 - \frac{\Omega^2}{S^2}\mathcal{S}_x^2 + \frac{\Omega^2}{M^2}\tilde{B}^2 - \frac{\Omega^2}{P^2}\tilde{A}^2 \tag{7.95}$$

Moving forward without loss of generality, the separation of HJ equation takes place as

$$\frac{\Omega}{S} = D_S(x) \tag{7.96}$$

$$\frac{\Omega}{R} = D_R(y) \tag{7.97}$$

$$\frac{\Omega}{M} = C_M(x) \tag{7.98}$$

$$\frac{\Omega}{P} = C_P(y) \tag{7.99}$$

We shall continue with the solution of the NPEs (7.84)-(7.87) considering the relations (7.96)-(7.99), then we take

$$\Psi_0 - \Psi_0^* = 0 = \left[\frac{\Omega_x}{\Omega}\right]_y - \frac{\Omega_x}{\Omega}\frac{\Omega_y}{\Omega} \tag{7.100}$$

Equivalently, we have

$$\Psi_0 - \Psi_0^* = 0 = \Phi_x \Psi_y \tag{7.101}$$

At this point, we should indicate that there is no essential difference between the two choices that the last relation yielded. We choose $\Phi_x = 0$. The separation process provides us with the relations (7.96)-(7.99). Based on the latter, and on our previous choice, we get

$$R(x, y) \to R(y)$$

$$P(x, y) \to P(y)$$

Thus, the real and imaginary parts of Bianchi Identity (VI) could be rewritten as follows if we take advantage of the annihilation of the imaginary part of Ψ_0,

$$2\Psi_0 \frac{\Omega_x}{\Omega} - \frac{C_{M\,x}}{C_M}[3\Psi_2 + \Psi_0] = 0 \tag{7.102}$$

$$2\Psi_0 \frac{\Omega_y}{\Omega} - \frac{C_{P\,y}}{C_P}[3\Psi_2 + \Psi_0] = 0 \tag{7.103}$$

The relation $\Phi(x) = 0 = \Omega_x$ will reform the real part of Bianchi Identity (VI) yielding two possible choices

$$\frac{C_{M\,x}}{C_M}[3\Psi_2 + \Psi_0] = 0 \tag{7.104}$$

Also, we must denote that the annihilation of the bracket is the only acceptable choice[2]. However, our choice and the equation (7.103) implies that Ω is constant. As an immediate impact,

$$\alpha - \bar{\beta} = 0, \tag{7.105}$$

since $R = R(y)$ and $S = S(x)$ due to the choice that was made during the separation of variables. Also, the Weyl components are equal to the cosmological constant, $\Psi_0 = \Psi_4^* = -3\Psi_2 = -3\Lambda$. At last, the only equations that we have to confront are the following,

$$12\Psi_2 = -4\Psi_0 = -\frac{1}{MS}\left[\frac{M_x}{S}\right]_x \tag{7.106}$$

$$12\Psi_2 = -4\Psi_0 = -\frac{1}{PR}\left[\frac{P_y}{R}\right]_y \tag{7.107}$$

One could observe that the two equations have the same form if we substitute $M \to P$ and $S \to R$. Hence, we may continue with the treatment only of (7.107). Let's present more properly the non-linear differential equation of second order

$$\frac{P_{yy}}{P} - \frac{P_y}{P}\frac{R_y}{R} + 12\Lambda R^2 = 0 \tag{7.108}$$

7.2.3 General solution

In this section different solutions emerged by solving the last differential equation. The functions of the metric are determined by the two non-linear differential equations of second order resulting to different spacetimes. Since the two differential equations have the same form, the solutions for P, R are the same with the solutions for M, S respectively.

One could observe that the differetial equation is a 2nd order non-linear *autonomous equation* since it does not contain the depended coordinate y (or x in case of (7.106)) implicitly [95]. Such equations encompass symmetry in spatial translations since they remain unchanged under a translation such that $y \to y + const$.

In this subsection we will give the general solutions for equation (7.107) **assuming that the cosmological constant is positive although different solutions could also be obtained with a negative sign of the cosmological constant**. We are going to achieve this by correlating the function $P(y)$ with function $R(y)$. One of the most generic way to correlate these two functions is through the following separation

$$P(y) = g(y)T(y) \tag{7.109}$$

[2]Appendix A.

Equivalently, the corresponding relation for (7.106) is $M(x) = y(x)\tilde{T}(x)$. Considering (7.109) the differential equation (7.108) could be rewritten as follows,

$$\frac{T_{yy}}{T} + \frac{T_y}{T}\left[2\frac{g_y}{g} - \frac{R_y}{R}\right] + \left[\frac{g_{yy}}{g} - \frac{g_y}{g}\frac{R_y}{R} + 12\Lambda R^2\right] = 0 \tag{7.110}$$

Regarding this, we could make two choices in order to determine a general solution. Both choices scope to correlate $g(y)$ with the function $R(y)$. **Choice 1** annihilates the first square bracket and **Choice 2** the second bracket. The first choice gives a specific expression for $g(y)$ in terms of $R(y)$, while in the second one there are different ways to correlate these functions resolving the differential equation of the second bracket. Let's proceed with **Choice 1**.

Choice 1

With this choice we obtain the following relation annihilating the first bracket, where G is a constant of integration,

$$g(y) = G\sqrt{R(y)} \tag{7.111}$$

$$\frac{T_{yy}}{T} + \frac{1}{2}\left[\frac{R_{yy}}{R} - \frac{3}{2}\left(\frac{R_y}{R}\right)^2 + 24\Lambda R^2\right] = 0 \tag{7.112}$$

We will solve the latter with separation of the variables inserting a non-zero constant F. Then, we take the following,

$$T_{yy} - FT = 0 \tag{7.113}$$

$$\frac{R_{yy}}{R} - \frac{3}{2}\left(\frac{R_y}{R}\right)^2 + 24\Lambda R^2 + 2F = 0 \tag{7.114}$$

One could observe that the solution of (7.113) is a second order differential equation which gives the following results.

$$\boldsymbol{F \neq 0}, \qquad T(y) = \tau_1 e^{\sqrt{F}y} + \tau_2 e^{-\sqrt{F}y} \tag{7.115}$$

$$\boldsymbol{F = 0}, \qquad T(y) = \tau_1 y + \tau_2 \tag{7.116}$$

The solution of (7.114) was obtained with the method that we describe in Appendix B and results to an integral whose explicit form is not obvious. We used the integrals at p. 97 from [96], [97].

$$R_y = \sqrt{48\Lambda}R\sqrt{4\tilde{F} + \tilde{K}R - R^2} \quad \rightarrow \quad \frac{dR}{R\sqrt{4\tilde{F} + \tilde{K}R - R^2}} = \sqrt{48\Lambda}dy \tag{7.117}$$

The tilded constants are defined as follows,

$$\tilde{F} = \frac{F}{48\Lambda} \qquad \tilde{K} = \frac{K}{48\Lambda}$$

It is important to note that the integral of (7.117) has to be handled carefully. We shall separate the cosmological constant term and we incorporate it as a component of the variable y. **This is a significant step because the cosmological constant is linked to the Weyl components. Therefore, eliminating the cosmological constant, we ought to obtain conformally flat spacetimes with appropriate choice of constants.**

Additionally, we have the flexibility to determine the constants of integration F and K, without encountering singularities since they contain constants of integration.

Furthermore, the integral (7.117) gives different results depending on the value of constant F and on the value of the negative discriminant,

$$\Delta = -(16\tilde{F} + \tilde{K}^2) \tag{7.118}$$

Thus, we can take four different solutions which are presented in Appendix C. The general metric is resulted from **Choice 1** has the following form,

$$\begin{aligned} \boldsymbol{ds^2 = 2\left[Y^2 S(x)\tilde{T}^2(x)(Adt + dz)^2 - S^2(x)dx^2\right]} \\ \boldsymbol{- 2\left[G^2 R(y)T^2(y)(Bdt + dz)^2 + R^2(y)dy^2\right]} \end{aligned} \tag{7.119}$$

We should denote that the final form of $R(y)$ is determined by the integral of (7.117) depending on the discriminant Δ and the values of $\tilde{F}$. On the other hand, the function $T(y)$ is determined by relations (7.115)-(7.116) and it is depended on the value of constant F. In this manner we can construct the y part of the metric in full detail. We obtain the exact same form for x part where the metric functions $S(x)$ and $M(x)$ satisfy the corresponding relations. Along these lines, the integration constants are different, the constants F and G are replaced by H and Y accordingly.

Choice 2

The second choice grants us the freedom to select multiples methods of solution. The usual method of solution is to correlate functions $g(y)$ and $R(y)$ annihilating the second bracket of (7.110).

It is worth noting that the solution of the differential equation of the second bracket happens to be the same solution of the main differential equation (7.108) since it has the exact same form. **However, we have to be cautious because if we solve the differential equation (7.108) correlating the function $P(y)$ with $R(y)$ directly as in Choice 3 we lack the dependence of $T(y)$.** Let us proceed now with (7.120)

$$\left[\frac{g_{yy}}{g} - \frac{g_y}{g}\frac{R_y}{R} + 12\Lambda R^2\right] = 0 \tag{7.120}$$

In order to solve the most generic case for this one, we choose the relation between g and R

$$\frac{g_y}{g} = \zeta\frac{R_y}{R} \tag{7.121}$$

Then we get

$$\zeta\frac{R_{yy}}{R} + \zeta(\zeta-2)\left(\frac{R_y}{R}\right)^2 + 12\Lambda R^2 = 0, \tag{7.122}$$

and using the method of Appendix B, we take

$$R_y = R^2\sqrt{KR^{-2\zeta} - \frac{12\Lambda}{\zeta^2}} \qquad for\ \ \zeta \neq 0, \tag{7.123}$$

where K is a constant of integration. This integral could be solved only case by case. The remaining terms in equation (7.110) determine the differential equation for $T(y)$,

$$\frac{T_{yy}}{T_y} = (1-2\zeta)\frac{R_y}{R} \tag{7.124}$$

Once we solve the integral for $R(y)$, we can insert it to the latter equation and finally we can obtain the expression for $P(y)$. Then, the same could follow for the differential equation (7.106).

The cases that could provide us with useful results are four apparently. Actually the choice of ζ determines the form of the integral (7.123). There are three manageable cases for $\zeta = +\frac{1}{2}, \pm 1$. All cases are presented in Appendix D.

Choice 3

This choice emerged as a special case of the **Choice 2**. In **Choice 2** we choose to solve the relation (7.120) assuming that $g(y) = GR^{\zeta}(y)$. Furthermore the relation (7.120) is the same 2nd order nonlinear differential equation with (7.107), hence, we assume the following solution.

$$P(y) = R(y)^{\zeta} \tag{7.125}$$

In this case, (7.107) turned out to be the same with (7.122). We follow the same methodology to deal with this differential equation as in the previous choice. Thus, the solution is again the relation (7.123). The only difference for our metric in **Choice 3** is that there is not a function such that $T(y) = \int R^{1-2\zeta}(y)dy$ or we can consider it as $T(y) = 1$

Concluding, this choice yields three different solutions as a special case of **Choice 2**. We will present the metrics of this choice in full detail a few pages below.

7.3 New Type D Solution in Vacuum with $\Lambda > 0$

In this subsection, we will present our metrics for all choices in full detail. The exact solution we present belong to type D in vacuum with a cosmological constant, where $\kappa \neq 0 = \sigma$. In this context, we claim that our solution is unique and does not belong to the already most general families:

- Our solutions do not belong to Kinnersley's solutions since he investigated all type D solutions in vacuum without a cosmological constant [49].
- Our solutions are not part of the Debever-Plebański-Demiański family of metrics since the Goldberg-Sachs theorem does not apply in our case due to the non-zero value of spin coefficient κ [52], [98], [99].
- In our solution the Principal Null Directions of Weyl tensor ($\hat{n}^\mu$, $\hat{l}^\mu$) are non geodesic ($\kappa, \nu \neq 0$), but they are shearfree ($\sigma, \lambda = 0$).

After conducting an exhaustive investigation, we can conclude that we have discovered a new type D solution of Einstein's Field Equations in vacuum with cosmological constant $\Lambda > 0$, where the Goldberg-Sachs theorem does not hold due to the combination $\kappa \neq 0 = \sigma$. The characteristics that make our solution unique is that the Principal Null Directions of Weyl tensor ($\hat{n}^\mu$, $\hat{l}^\mu$) are not geodesic ($\kappa, \nu \neq 0$) but they are shearfree ($\sigma, \lambda = 0$) [18]. **Other solutions, where the Goldberg-Sachs theorem does not apply, were found by Plebański-Hacyan [42] and Garcia-Plebański [100] in electrovacuum with $\Lambda < 0$.**

At last, as Stephani et al. noted about the aligned[3] case in [19], at Chapter 26 on p. 409:

"The case $\kappa = 0$, $\sigma = 0$, has been excluded by Kozarzewski(1965) [101], so only $\kappa \neq 0$, $\sigma = 0$, remains to be studied. If the two null eigenvectors of a type D solution are aligned with the eigenvectors of the Maxwell tensor, then they must both be geodesic and shearfree; this is not true if a cosmological constant Λ is admitted (see Garcia D. and Plebański (1982a) [100] and Plebański and Hacyan (1979) [42], where also some solutions are given)..."

Using the two classes of rotation around the null tetrad frame that were operated by Chandrasekhar and Xanthopoulos in [90], we manage to obtain the only non-zero Weyl component to be $\Psi_2 \neq 0$. Hence we checked if our new null tetrads n^μ, l^μ are geodesic.

[3] Generally speaking, the term *aligned* is referred to the alignment of the null tetrads n^μ and l^μ with the two repeated principal null directions of a type D spacetime, which are the real null tetrads of the only non-zero component of the Weyl tensor (Ψ_2). Although, Stephani et.al referred to the alignment of (at least) one of the null eigenvectors of the Maxwell field with the repeated Weyl null eigenvector.

The proof, given in Appendix F, shows that the new null tetrads are not geodesic unless if $\kappa = -\bar{\nu} = -\tau = \bar{\pi}$. This is the case if the function $M(x)$ is constant as we can see from relations (7.74) and (7.75). Namely, it is

$$\hat{n}^{\nu}\hat{n}_{\mu;\nu} = -(\kappa + \bar{\kappa} + \tau + \bar{\tau})(n_{\mu} + l_{\mu}) - 2(\bar{\kappa} + \bar{\tau})m_{\mu} - 2(\kappa + \tau)\bar{m}_{\mu} \qquad (7.126)$$

$$\hat{l}^{\nu}\hat{l}_{\mu;\nu} = -(\kappa + \bar{\kappa} + \tau + \bar{\tau})(n_{\mu} + l_{\mu}) - 2(\bar{\kappa} + \bar{\tau})m_{\mu} - 2(\kappa + \tau)\bar{m}_{\mu} \qquad (7.127)$$

In the next subsection, we will list all the exact solutions that we obtained. We have the opinion that the characteristics of the solutions, noted above, concern more general spacetimes that one could obtain solving the cumbersome system of equations (7.80)-(7.88).

The separation of variables in Hamilton-Jacobi equation gave us some of these spacetimes which are presented below. **All these spacetimes are 2-product spaces with constant curvature, consequently, they admit a 6-dimensional simple transitive group of motion**. Furthermore, there are coordinate systems where all these metrics can be reduced to the following general metric (Schmidt's method) [43], [19],

General metric

$$ds^2 = \Omega_1 \left[\Sigma^2(x, g_1)dt^2 - dx^2\right] - \Omega_2 \left[\Sigma^2(y, g_2)dz^2 + dy^2\right] \qquad (7.128)$$

where $\Omega_1, \Omega_2, g_1, g_2$ are constants of integration and Σ^2 is a arbitrary function. Thus, all of our metrics must be reduced in this form.

Let's return now to the presentation of our resulted metrics. The method of solution was presented only for the differential equation (7.107) since its form is exactly the same with (7.106). It is obvious that these equations have the same form. If one substitutes $M(x)$ for $P(y)$ and $S(x)$ for $R(y)$, then we get the same equation. Following this, there is a need to clarify the correspondences between constants of integration,

$$F \to H \quad K \to V \quad G \to Y \quad C_y \to C_x \quad \tau_1 \to \tau_3 \quad \tau_2 \to \tau_4 \qquad (7.129)$$

Finally, we have to dictate a coordinate transformation that simplifies our metrics. It is true that the quantities $Adt + dz$ and $Bdt + dz$ do not provide any further information due to their form. So, a coordinate transformation such the following does not change the metric, but simplifies it,

$$\tilde{t} = At + z \qquad (7.130)$$

$$\tilde{z} = Bt + z \qquad (7.131)$$

The following subsections contain the obtained metrics. We categorize them based on the choice that we made in order to solve the differential equation (7.107). In appendices C, D, we give more details about these metrics.

7.3.1 Choice 1 solution: $\tilde{F} = 0$

This case concerns the first choice. The case where $\tilde{F} = 0$ gives the following metric which is a quite special solution,

$$ds^2 = \frac{8\tilde{V}Y^2(\tau_3 x + \tau_4)^2}{\tilde{V}^2(\sqrt{48\Lambda}x - C_x)^2 + 4}d\tilde{t}^2 - \frac{32\tilde{V}^2}{\left[\tilde{V}^2(\sqrt{48\Lambda}x - C_x)^2 + 4\right]^2}dx^2 \tag{7.132}$$
$$-\frac{8\tilde{K}G^2(\tau_1 y + \tau_2)^2}{\tilde{K}^2(\sqrt{48\Lambda}y - C_y)^2 + 4}d\tilde{z}^2 - \frac{32\tilde{K}^2}{\left[\tilde{K}^2(\sqrt{48\Lambda}y - C_y)^2 + 4\right]^2}dy^2$$

Where the constants $G, Y, F, K, H, V, \tau_1, \tau_2, \tau_3, \tau_4, C_x, C_y$ are constants of integration and $\tilde{F}$, $\tilde{K}$, $\tilde{H}$, $\tilde{V}$ are defined by $\tilde{F} \equiv \frac{F}{48\Lambda}$, $\tilde{K} \equiv \frac{K}{48\Lambda}$, $\tilde{H} \equiv \frac{H}{48\Lambda}$ and $\tilde{V} \equiv \frac{V}{48\Lambda}$.

Conformally flat Spacetime ($\Lambda = 0$)

Now, we are going to add a few lines of analysis about this metric. At first glance, the above metric is not conformally flat when $\Lambda \to 0$,

$$ds^2 = \frac{8\tilde{V}Y^2(\tau_3 x + \tau_4)^2}{\tilde{V}^2 C_x^2 + 4}d\tilde{t}^2 - \frac{32\tilde{V}^2}{\left(\tilde{V}^2 C_x^2 + 4\right)^2}dx^2 - \frac{8\tilde{K}G^2(\tau_1 y + \tau_2)^2}{\tilde{K}^2 C_y^2 + 4}d\tilde{z}^2 - \frac{32\tilde{K}^2}{\left(\tilde{K}^2 C_y^2 + 4\right)^2}dy^2 \tag{7.133}$$

Although, we can make appropriate choices for the constants in order to simplify the form of the metric. For this reason we choose the component of dx^2, dy^2 to be equal to one, and also, we take $\tau_1 = \tau_2$ and $\tau_3 = \tau_4$. Hence, we obtain

$$\sqrt{32}\tilde{V} = \tilde{V}^2 C_x^2 + 4$$
$$\sqrt{32}\tilde{K} = \tilde{K}^2 C_y^2 + 4$$

Using now the latter relations along with $\sqrt{2}Y^2\tau_3^2 = \sqrt{2}G^2\tau_1^2 = 1$ we take

$$ds^2 = (x+1)^2 d\tilde{t}^2 - dx^2 - (y+1)^2 d\tilde{z}^2 - dy^2 \tag{7.134}$$

This metric is a conformally flat spacetime which describes a hyperbola in $\hat{t}, \hat{x}$ plane with $\hat{x}^2 - \hat{t}^2 = (x+1)^2$. Using now the transformations

$$\hat{t} = \pm(x+1)\sinh\tilde{t}$$
$$\hat{x} = \pm(x+1)\cosh\tilde{t}$$
$$\hat{z} = \pm(y+1)\sin\tilde{z}$$
$$\hat{y} = \pm(y+1)\cos\tilde{z},$$

the metric transforms to Minkowski spacetime in "hat" coordinates for both the plus (+) or minus (-) branch [45],

$$ds^2 = d\hat{t}^2 - d\hat{x}^2 - d\hat{y}^2 - d\hat{z}^2 \tag{7.135}$$

Regarding this, the transformations for the (+) branch concern the region $\hat{x} > |\hat{t}|$, where $x \in (0, \infty)$ and $\tilde{t} \in (-\infty, \infty)$. Hence, we need another patch for the negative region of $\hat{x}$. The latter is satisfied for the (-) sign in the transformations. Griffiths and Podolsky [45] also present the inverse transformation where both patches are satisfied.

Moreover, the curves with $\tilde{t} = const$ are straight lines through the origin in $\hat{t}, \hat{x}$ plane. The curves of $x = \frac{1}{\alpha}$ describe hyperbolas which are wordlines of points with constant uniform acceleration α. The points in these "wordlines" have constant acceleration and this metric is called *uniformly accelerated metric* [102]. The boundaries of the null cone are the lines $\hat{t} = \pm\hat{x}$.

Asymptotically flat Spacetime

Returning to the initial general metric (7.132) in the equivalent form

$$\begin{aligned} ds^2 = & \frac{2\tilde{V}Y^2(\tau_3 x + \tau_4)^2}{1 + \left[\tilde{V}\sqrt{3\Lambda}x - \frac{\tilde{V}C_x}{16}\right]^2} d\tilde{t}^2 - \frac{2\tilde{V}^2 dx^2}{\left[1 + \left[\tilde{V}\sqrt{3\Lambda}x - \frac{\tilde{V}C_x}{16}\right]^2\right]^2} \\ & - \frac{2\tilde{K}G^2(\tau_1 x + \tau_2)^2}{1 + \left[\tilde{K}\sqrt{3\Lambda}y - \frac{\tilde{K}C_y}{16}\right]^2} d\tilde{z}^2 - \frac{2\tilde{K}^2 dy^2}{\left[1 + \left[\tilde{K}\sqrt{3\Lambda}y - \frac{\tilde{K}C_y}{16}\right]^2\right]^2} \end{aligned} \tag{7.132}$$

we can apply the following tranformations,

$$\begin{aligned} \sqrt{\tilde{V}}Y\tilde{t} &= \tau \\ \tilde{V}\sqrt{3\Lambda}x - \frac{\tilde{V}C_x}{16} &= \sinh v \\ \sqrt{\tilde{K}}G\tilde{z} &= \zeta \\ \tilde{K}\sqrt{3\Lambda}y - \frac{\tilde{K}C_y}{16} &= \sinh w \end{aligned}$$

in order to write it in the form

$$ds^2 = \frac{2}{3\Lambda}\left[\frac{(\tilde{\tau}_3 \sinh v + \tilde{\tau}_4)^2}{\cosh^2 v} d\tau^2 - dv^2 - \frac{(\tilde{\tau}_1 \sinh w + \tilde{\tau}_2)^2}{\cosh^2 w} d\zeta^2 - dw^2\right] \tag{7.136}$$

At last, we have obtained the desirable form of the metric, where the components of the differential coordinates $d\tau^2, d\zeta^2$ are depended by $\tanh v$ and $\tanh w$ accordingly. When these two coordinates v, w tend to ∞ their corresponding functions $\tanh v, \tanh w \to 1$ providing us with an asymptotically flat spacetime. **The relation (7.136) is an example of how a 2-product space can be reduced into the form of the general metric of the 2-product spaces with constant curvature eq. (7.128).**

7.3.2 Choice 1 solution: $\tilde{F} > 0$, $\Delta < 0$

This solution concerns also the same choice. When the constant $\tilde{F} > 0$, the discriminant can only be negative and the solution becomes

$$
\begin{aligned}
ds^2 = & \frac{16\tilde{H}Y^2\left[\tau_3 e^{\sqrt{H}x} + \tau_4 e^{-\sqrt{H}x}\right]^2}{\left(16\tilde{H}+\tilde{V}^2\right)\cosh\left(\sqrt{4\tilde{H}}(\sqrt{48\Lambda}x - C_x)\right) - \tilde{V}} d\tilde{t}^2 \\
& - \frac{2(8\tilde{H})^2}{\left[\left(16\tilde{H}+\tilde{V}^2\right)\cosh\left(\sqrt{4\tilde{H}}(\sqrt{48\Lambda}x - C_x)\right) - \tilde{V}\right]^2} dx^2 \\
& - \frac{16\tilde{F}G^2\left[\tau_1 e^{\sqrt{F}y} + \tau_2 e^{-\sqrt{F}y}\right]^2}{\left(16\tilde{F}+\tilde{K}^2\right)\cosh\left(\sqrt{4\tilde{F}}(\sqrt{48\Lambda}y - C_y)\right) - \tilde{K}} d\tilde{z}^2 \\
& - \frac{2(8\tilde{F})^2}{\left[\left(16\tilde{F}+\tilde{K}^2\right)\cosh\left(\sqrt{4\tilde{F}}(\sqrt{48\Lambda}y - C_y)\right) - \tilde{K}\right]^2} dy^2
\end{aligned}
\tag{7.137}
$$

Where the constants $G, Y, F, K, H, V, \tau_1, \tau_2, \tau_3, \tau_4, C_x, C_y$ are constants of integration and $\tilde{H}, \tilde{V}$ are defined by $\tilde{H} \equiv \frac{H}{48\Lambda}$, $\tilde{V} \equiv \frac{V}{48\Lambda}$.

7.3.3 Choice 1 solution: $\tilde{F} < 0$, $\Delta < 0$

For negative constant $\tilde{F}$, there are two choices for the discriminant but only the first one is manageable. The metric gets the form

$$
\begin{aligned}
ds^2 = & \frac{2(8|\tilde{H}|)Y^2\left[\tau_3 e^{i\sqrt{|H|}x} + \tau_4 e^{-i\sqrt{|H|}x}\right]}{\tilde{V} + \sqrt{\tilde{V}^2 - 16|\tilde{H}|}\sin\left(\sqrt{4|\tilde{H}|}(\sqrt{48\Lambda}x - C_x)\right)} d\tilde{t}^2 \\
& - \frac{2(8|\tilde{H}|)^2}{\left[\tilde{V} + \sqrt{\tilde{V}^2 - 16|\tilde{H}|}\sin\left(\sqrt{4|\tilde{H}|}(\sqrt{48\Lambda}x - C_x)\right)\right]^2} dx^2 \\
& - \frac{2(8|\tilde{F}|)G^2\left[\tau_1 e^{i\sqrt{|F|}y} + \tau_2 e^{-i\sqrt{|F|}y}\right]}{\tilde{K} + \sqrt{\tilde{K}^2 - 16|\tilde{F}|}\sin\left(\sqrt{4|\tilde{F}|}(\sqrt{48\Lambda}y - C_y)\right)} d\tilde{z}^2 \\
& - \frac{2(8|\tilde{F}|)^2}{\left[\tilde{K} + \sqrt{\tilde{K}^2 - 16|\tilde{F}|}\sin\left(\sqrt{4|\tilde{F}|}(\sqrt{48\Lambda}y - C_y)\right)\right]^2} dy^2
\end{aligned}
\tag{7.138}
$$

Where the constants $G, Y, F, K, H, V, \tau_1, \tau_2, \tau_3, \tau_4, C_x, C_y$ are constants of integration and $\tilde{F}$, $\tilde{K}$, $\tilde{H}$, $\tilde{V}$ are defined by $\tilde{F} \equiv \frac{F}{48\Lambda}$, $\tilde{K} \equiv \frac{K}{48\Lambda}$, $\tilde{H} \equiv \frac{H}{48\Lambda}$ and $\tilde{V} \equiv \frac{V}{48\Lambda}$.

7.3.4 Choice 2 solution: $\zeta = \frac{1}{2}$

At this point, we present the obtained solutions which concern the second choice. In general the solutions of these two choices should be different because they satisfy independent differential equations. Although, this solution is the same with the first solution of Choice 1, where $\tilde{F} = 0$. This is possible since the solution of the first choice described by (7.111) happens to solve the differential equation of the second choice.

$$ds^2 = \frac{2(4\tilde{V})Y^2(\tau_3 x + \tau_4)^2}{\tilde{V}^2(\sqrt{48\Lambda}x - C_x)^2 + 4} d\tilde{t}^2 - \frac{2(4\tilde{V})^2}{\left[\tilde{V}^2(\sqrt{48\Lambda}x - C_x)^2 + 4\right]^2} dx^2 \tag{7.139}$$

$$- \frac{2(4\tilde{K})G^2(\tau_1 y + \tau_2)^2}{\tilde{K}^2(\sqrt{48\Lambda}y - C_y)^2 + 4} d\tilde{z}^2 - \frac{2(4\tilde{K})^2}{\left[\tilde{K}^2(\sqrt{48\Lambda}y - C_y)^2 + 4\right]^2} dy^2$$

Where the constants $G, Y, F, K, H, V, \tau_1, \tau_2, \tau_3, \tau_4, C_x, C_y$ are constants of integration and $\tilde{F}$, $\tilde{K}$, $\tilde{H}$, $\tilde{V}$ are defined by $\tilde{F} \equiv \frac{F}{48\Lambda}$, $\tilde{K} \equiv \frac{K}{48\Lambda}$, $\tilde{H} \equiv \frac{H}{48\Lambda}$ and $\tilde{V} \equiv \frac{V}{48\Lambda}$.

7.3.5 Choice 2 solution: $\zeta = +1$

If we adjust the value of ζ in equation (7.123) we take a new function for $R(y)$ which yields a new metric,

$$ds^2 = \frac{2Y^2\tilde{V}}{\cosh^2\tilde{x}}\left[\tilde{C}_x + \frac{\tilde{x}}{2} + \frac{\sinh(2\tilde{x})}{4}\right]^2 d\tilde{t}^2 - \frac{2\tilde{V}}{\cosh^2\tilde{x}} dx^2$$
$$- \frac{2G^2\tilde{K}}{\cosh^2\tilde{y}}\left[\tilde{C}_y + \frac{\tilde{y}}{2} + \frac{\sinh(2\tilde{y})}{4}\right]^2 d\tilde{z}^2 - \frac{2\tilde{K}}{\cosh^2\tilde{y}} dy^2 \tag{7.140}$$

where the quantities $\tilde{x}, \tilde{y}$ are defined as follows for reasons of convenience

$$\tilde{x} = \sqrt{\tilde{K}}(\sqrt{12\Lambda}x - C_x) \tag{7.141}$$

$$\tilde{y} = \sqrt{\tilde{V}}(\sqrt{12\Lambda}y - C_y) \tag{7.142}$$

Where the constants $G, Y, \tilde{K}, \tilde{V}, C_x, C_y$ are constants of integration and $\tilde{K}$, $\tilde{V}$ are defined by $\tilde{K} \equiv \frac{K}{48\Lambda}$ and $\tilde{V} \equiv \frac{V}{48\Lambda}$.

7.3.6 Choice 2 solution: $\zeta = -1$

This is the final solution of those which concern the second choice. The metric for this case is the following

$$ds^2 = 2Y^2\left[\frac{12\Lambda x - C_x}{\tilde{V}} + C_1\sqrt{\tilde{V}}\sqrt{1-\left(\frac{\sqrt{12\Lambda}x - C_x}{\tilde{V}}\right)^2}\right]^2 d\tilde{t}^2 - \frac{2dx^2}{\tilde{V} - (\sqrt{12\Lambda}x - C_x)^2} - 2G^2\left[\frac{12\Lambda y - C_y}{\tilde{K}} + C_2\sqrt{\tilde{K}}\sqrt{1-\left(\frac{\sqrt{12\Lambda}y - C_y}{\tilde{K}}\right)^2}\right]^2 d\tilde{z}^2 - \frac{2dy^2}{\tilde{K} - (\sqrt{12\Lambda}y - C_y)^2} \quad (7.143)$$

Where the constants $G, Y, \tilde{K}, \tilde{V}, C_x, C_y, C_1, C_2$ are constants of integration and $\tilde{K}$, $\tilde{V}$ are defined by $\tilde{K} \equiv \frac{K}{48\Lambda}$ and $\tilde{V} \equiv \frac{V}{48\Lambda}$.

7.3.7 Choice 3 solution: $\zeta = \frac{1}{2}$

The metric functions for the choice $\zeta = \frac{1}{2}$ take the following forms

$$P^2(y) = R(y) = \frac{4\tilde{K}}{\tilde{K}(\sqrt{48\Lambda}y - C_y)^2 + 4} \quad (7.144)$$

$$M^2(x) = S(x) = \frac{4\tilde{V}}{\tilde{V}(\sqrt{48\Lambda}x - C_x)^2 + 4}, \quad (7.145)$$

and the metric results to

$$ds^2 = \frac{2(4\tilde{V})}{\tilde{V}(\sqrt{48\Lambda}x - C_x)^2 + 4}\left[d\tilde{t}^2 - \frac{4\tilde{V}dx^2}{\tilde{V}(\sqrt{48\Lambda}x - C_x)^2 + 4}\right] - \frac{2(4\tilde{K})}{\tilde{K}(\sqrt{48\Lambda}y - C_y)^2 + 4}\left[d\tilde{z}^2 + \frac{4\tilde{K}dy^2}{\tilde{K}(\sqrt{48\Lambda}y - C_y)^2 + 4}\right] \quad (7.146)$$

Where the constants $\tilde{K}, \tilde{V}, C_x, C_y$ are constants of integration and $\tilde{K}$, $\tilde{V}$ are defined by $\tilde{K} \equiv \frac{K}{48\Lambda}$ and $\tilde{V} \equiv \frac{V}{48\Lambda}$.

7.3.8 Choice 3 solution: $\zeta = +1$

In this case the metric functions are turned out to be as follows,

$$P^2(y) = R^2(y) = \tilde{K}\left(1 - \tanh^2 \tilde{y}\right) = \frac{\tilde{K}}{\cosh^2 \tilde{y}} \quad (7.147)$$

$$M^2(x) = S^2(x) = \tilde{V}\left(1 - \tanh^2 \tilde{x}\right) = \frac{\tilde{V}}{\cosh^2 \tilde{x}} \quad (7.148)$$

where the quantities $\tilde{x}, \tilde{y}$ are defined as follows for reasons of convenience

$$\tilde{x} = \sqrt{\tilde{K}}(\sqrt{12\Lambda}x - C_x) \tag{7.149}$$

$$\tilde{y} = \sqrt{\tilde{V}}(\sqrt{12\Lambda}y - C_y) \tag{7.150}$$

Therefore,

$$ds^2 = \frac{2\tilde{V}}{\cosh^2 \tilde{y}}\left(d\tilde{t}^2 - dx^2\right) - \frac{2\tilde{K}}{\cosh^2 \tilde{y}}\left(d\tilde{z}^2 + dy^2\right) \tag{7.151}$$

Where the constants $\tilde{K}, \tilde{V}, C_x, C_y$ are constants of integration and $\tilde{K}$, $\tilde{V}$ are defined by $\tilde{K} \equiv \frac{K}{48\Lambda}$ and $\tilde{V} \equiv \frac{V}{48\Lambda}$.

7.3.9 Choice 3 Solution: $\zeta = -1$ (Carter's Case [D])

Carter's Case [D] is a widely known solution and it is a special case of Carter's Family $[\tilde{\mathcal{A}}]$ (p. 27). In this case for $\zeta = -1$ we have

$$P^2(y) = \frac{1}{R^2(y)} = \tilde{K} - (\sqrt{12\Lambda}y - D_y)^2 \tag{7.152}$$

In the same fashion we can obtain the relation for $M(x)$ and $S(x)$,

$$M^2(x) = \frac{1}{S^2(x)} = \tilde{V} - (\sqrt{12\Lambda}x - D_x)^2, \tag{7.153}$$

where the quantities K, V, D_y, D_x are constants of integration. In order to study this metric we make the choice,

$$D_x = \sqrt{12\Lambda}C_x \qquad D_y = \sqrt{12\Lambda}C_y$$

The constants of integration C_x, C_y have been chosen with a specific manner multiplied by $\sqrt{12\Lambda}$, since the annihilation of Λ will give us a conformally flat spacetime. Applying the latter choice for the constants of integration and substituting in the metric components we get the final relation

$$\begin{aligned} ds^2 = 2\left[\tilde{V} - 12\Lambda(x - C_x)^2\right] d\tilde{t}^2 - \frac{2dx^2}{\tilde{V} - 12\Lambda(x - C_x)^2} \\ - 2\left[\tilde{K} - 12\Lambda(y - C_y)^2\right] d\tilde{z}^2 - \frac{2dy^2}{\tilde{K} - 12\Lambda(y - C_y)^2} \end{aligned} \tag{7.154}$$

7.4 Geodesics and Constants of Motion

In this section we will present the equations of geodesic and the constants of motion. Our line of attack contains the Hamilton-Jacobi equation for the general form of the 2-product space. With this manner we can correlate our metric functions with the constants of motion. We give the geodesics in a general form assuming that our metric is described by

$$ds^2 = 2\left[M^2(x)d\tilde{t}^2 - S^2(x)dx^2\right] - 2\left[P^2(y)d\tilde{z}^2 + R^2(y)dy^2\right] \tag{7.155}$$

This consideration is valid since all metrics of the previous analysis are direct products of 2-dimensional spaces. Hence, the final formulas of paragraphs (7.4) and (7.5) would be applied in any of our metrics.

The equation of geodesics fundamentally describes the phenomenon of absence of the acceleration that an observer feels along a geodesic line. Namely, a geodesic line of a gravitational field describes a "free fall" in the gravitational field and can be expressed by the equation of geodesics. In this chapter our focus resides to take advantage of the symmetries in order to obtain the Integration Constants of Motion and the geodesic lines with respect to an affine parameter λ,

$$u^\mu u_{\nu;\mu} = 0 \tag{7.156}$$

We define the 4-velocity vector of the observer of mass m as

$$u^\mu \equiv \dot{x}^\mu = k_1 n^\mu + k_2 l^\mu + k_3 m^\mu + k_4 \bar{m}^\mu. \tag{7.157}$$

The derivation of the displacement vector is performed with respect to the affine parameter λ. The affine parameter is related to the proper time by

$$\tau = \bar{m}\lambda \tag{7.158}$$

Our Killing tensor is not a conformal one, hence, the only two possible cases, which are allowed for the geodesic lines, are to be either spacelike or timelike. Additionally, the norm of the vector is expressed below,

$$k_1 k_2 - k_3 \bar{k}_3 = \pm\frac{1}{2}, \tag{7.159}$$

where the sign (+) is for timelike orbits and the (-) for spacelike orbits. Unravelling this, we take

$$4k_1 k_2 - (k_3 + \bar{k_3})^2 + (k_3 - \bar{k}_3)^2 = \pm 4. \tag{7.160}$$

The geodesic equation could be easily obtained by solving the Euler-Lagrange equations. The most suitable Lagrangian for the study of geodesics is

$$\mathcal{L} = \frac{1}{2} g_{\mu\nu}\dot{x}^\mu \dot{x}^\nu \tag{7.161}$$

7.4.1 Hamilton-Jacobi Action

The symmetries of the problem allow us to gain expressions for the 4-velocity vector of the observer, as a result of the separation of variables of the Hamilton-Jacobi equation. Given that the coordinates are functions of the affine parameter, the action and the inverse metric could be expressed as

$$\mathcal{S} = \frac{\bar{m}^2}{2}\lambda + E\tilde{t} - L\tilde{z} + S_1(x) + S_2(y) \tag{7.162}$$

$$g^{\mu\nu} = \begin{pmatrix} \frac{1}{2M^2(x)} & 0 & 0 & 0 \\ 0 & -\frac{1}{2P^2(y)} & 0 & 0 \\ 0 & 0 & -\frac{1}{2S^2(x)} & 0 \\ 0 & 0 & 0 & -\frac{1}{2R^2(y)} \end{pmatrix}$$

The Hamilton-Jacobi equation is given by

$$\frac{\partial \mathcal{S}}{\partial \lambda} = \frac{1}{2} g^{\mu\nu} \frac{\partial \mathcal{S}}{\partial x^\mu} \frac{\partial \mathcal{S}}{\partial x^\nu}. \tag{7.163}$$

If we elaborate the derivations of the action, we take the relations below

$$2\bar{m}^2 = \frac{E^2}{M^2(x)} - \frac{L^2}{P^2(y)} - \frac{\mathcal{S}_y^2}{R^2(y)} - \frac{\mathcal{S}_x^2}{S^2(x)} \tag{7.164}$$

7.4.2 4th constant of motion or Carter's constant

One way to define the fourth constant of motion, denoted as $\mathcal{K}$, is through the separation of variables in the Hamilton-Jacobi equation. This approach yields both the definition of the fourth constant of motion and it allows us to obtain integrated geodesics.

This constant is also referred to as Carter's Constant, it is named after the first discovery of the separation of Hamilton-Jacobi equation using Boyer-Lindquist coordinates for the Kerr metric by Carter. In the next section, we will explore an alternative definition of this constant using the Killing tensor [20], [103],

$$\mathcal{K} \equiv \frac{\mathcal{S}_y^2}{R^2(y)} + \frac{L^2}{P^2(y)} = -\frac{\mathcal{S}_x^2}{S^2(x)} + \frac{E^2}{M^2(x)} - 2\bar{m}^2 \tag{7.165}$$

In our coordinate system though, the HJ equation is not uniquely separated, unlike Kerr geometry, since the mass $\bar{m}$ could be located in either the 'x part,' the 'y part,' or in both sides. At Kerr geometry the transformation in Boyer-Lindquist coordinates guides us uniquely to the separation of HJ equation in "r part" and in "θ part".

Concerning our case, the first we thought would be that the mass should be distributed on both sides equivalently. **However, after investigating the separation of the HJ equation in metrics with spherical or polar symmetry,**

we observe that in the equatorial plane ($\theta = \frac{\pi}{2}$) Carter's constant is depends solely by the angular momentum L without any additional mass term [104]. This observation is also applicable to Schwarzschild metric.

In the next chapter, we will encounter a similar phenomenon in the reduction of **Carter's Case [D]** to Nariai spacetime, where the second part constitutes a spherical surface.

7.4.3 Geodesics

The canonical momentum is correlated with the 4-velocity of the observer as follows.

$$p_\mu = g_{\mu\nu}u^\nu = g_{\mu\nu}\dot{x}^\nu \tag{7.166}$$

The latter yields the following relations

$$p_{\tilde{t}} = 2M^2(x)\dot{\tilde{t}} \tag{7.167}$$

$$p_{\tilde{z}} = 2P^2(y)\dot{\tilde{z}} \tag{7.168}$$

$$p_x = 2S^2(x)\dot{x} \tag{7.169}$$

$$p_y = 2R^2(y)\dot{y} \tag{7.170}$$

The normalizing condition of the system is equivalent with the conservation of the rest mass.

$$\bar{m}^2 = g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu \tag{7.171}$$

Along these lines, the Hamiltonian is defined by

$$\mathcal{H} \equiv p_\mu\dot{x}^\mu - \mathcal{L} = \frac{1}{2}g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu = \mathcal{L}. \tag{7.172}$$

The Hamiltonian is a conserved quantity of the problem since it is correlated with the conserved rest mass. Furthermore, the momentum is the derivative of the action. Hence, using the relations (7.169) and (7.170), we take expressions for p_x, p_y [4]. Considering that the components of the 4-vector momentum is the partial derivative of the action, it could be expressed as

$$p_\mu = \left(E, -L, S(x)\left[\frac{E^2}{M^2(x)} - \mathcal{K} - 2\bar{m}^2\right]^{1/2}, R(y)\left[\mathcal{K} - \frac{L^2}{P^2(y)}\right]^{1/2}\right) \tag{7.173}$$

The comparison between the latter and the relations (7.156)-(7.159) results to the geodesic equations

$$\dot{\tilde{t}} = \frac{E}{2M^2(x)} \tag{7.174}$$

$$\dot{\tilde{z}} = \frac{L}{2P^2(y)} \tag{7.175}$$

[4] The sign of the square roots could be chosen independently, although for reasons of convenience we take the positive sign for both cases.

$$\dot{x} = \frac{1}{2S(x)} \left[\frac{E^2}{M^2(x)} - \mathcal{K}_+ \right]^{1/2} \tag{7.176}$$

$$\dot{y} = \frac{1}{2R(y)} \left[\mathcal{K} - \frac{L^2}{P^2(y)} \right]^{1/2} \tag{7.177}$$

The above relations describe all possible geodesic lines with respect to an affine parameter, which is denoted as λ. The new constant is defined as $\mathcal{K}_+ \equiv \mathcal{K} + 2\bar{m}^2$ which combines the 4th constant of motion (Carter's constant) with the conserved mass. We finally express the time derivative of our coordinates with respect to the affine parameter λ in terms of constants of motion and the functions. In this general form of geodesics, one could easily substitute the functions of metric in order to obtain the geodesic equations of each new solution.

7.4.4 Unique points x_+ and y_- for geodesics

The following equations are obtained when we focus on studying the system dynamically at specific points. For example, there exists a point x_+ where the derivative of $x(\lambda)$ vanishes, i.e,

$$\mathcal{K}_+ = \frac{E^2}{M(x_+)} \qquad \rightarrow \qquad \dot{x} = 0 \tag{7.178}$$

This same operation could also be applied for the unique point y_- where the derivative of $y(\lambda)$ vanishes as well,

$$\mathcal{K} = \frac{L^2}{P^2(y_-)} \qquad \rightarrow \qquad \dot{y} = 0 \tag{7.179}$$

On the other hand, the fourth constant of motion is also associated with the metric functions, the energy, or the angular momentum per unit mass when focusing on a particular point. **Furthermore, this is a more straightforward way to define the Carter's constant.** Therefore, this way we present these relationships in a more general form.

7.5 Killing Tensor and Constants of Motion

In this section we will reveal the role of the Killing tensor in the dynamics of a Hamiltonian system. The eigenvalues of our canonical forms are correlated with the constants of motions.

At first we are going to acquire the relations of the eigenvalues $\lambda_0 \pm \lambda_1$ in terms of the metric functions $M^2(x), P^2(y)$. The real parts of the reformed relations (7.6) and (7.11) have the forms

$$(\delta + \bar{\delta})\lambda_0 = 2\left[\lambda_0(\pi + \bar{\pi}) - (\kappa + \bar{\kappa})(\lambda_1 + \lambda_2)\right] \tag{7.180}$$

$$(\delta + \bar{\delta})\lambda_0 = 2\left[\lambda_0(\bar{\pi} - \pi) - (\kappa - \bar{\kappa})(\lambda_1 + \lambda_2)\right] \tag{7.181}$$

$$(\delta + \bar{\delta})\lambda_1 = -2\left[\lambda_0(\kappa + \bar{\kappa}) - (\pi + \bar{\pi})(\lambda_1 + \lambda_2)\right] \tag{7.182}$$

$$(\delta + \bar{\delta})\lambda_1 = -2\left[\lambda_0(\kappa - \bar{\kappa}) - (\bar{\pi} - \pi)(\lambda_1 + \lambda_2)\right] \tag{7.183}$$

After the integration, we obtain the relations below with $\lambda_\pm$ to be constants of integration. The non-constant eigenvalues of the Killing tensor[5] are described by the following relations.

$$\lambda_0 + \lambda_1 = \lambda_+ M^2(x) \tag{7.184}$$

$$\lambda_0 - \lambda_1 = \lambda_- P^2(y) \tag{7.185}$$

It is clear now that our eigenvalues are depended on the non-ignorable coordinates. **Besides, Woodhouse has shown that the separation takes place in the direction of the eigenvectors of the Killing tensor [105].** Next, we shall determine the 4th constant of motion using the relation

$$K^{\mu\nu} p_\mu p_\nu = \mathcal{K} \tag{7.186}$$

The inverse Killing tensor is

$$K^{\mu\nu} = \begin{pmatrix} \frac{\lambda_0}{\lambda_0^2-\lambda_1^2} & -\frac{\lambda_1}{\lambda_0^2-\lambda_1^2} & 0 & 0 \\ -\frac{\lambda_1}{\lambda_0^2-\lambda_1^2} & \frac{\lambda_0}{\lambda_0^2-\lambda_1^2} & 0 & 0 \\ 0 & 0 & 0 & -\frac{1}{\lambda_2} \\ 0 & 0 & -\frac{1}{\lambda_2} & 0 \end{pmatrix}, \tag{7.187}$$

while the vector of the observer is given by the relation (7.173),

$$p_\mu = \left(E, -L, S(x) \left[\frac{E^2}{M^2(x)} - \mathcal{K}_+ \right]^{1/2}, R(y) \left[\mathcal{K} - \frac{L^2}{P^2(y)} \right]^{1/2} \right) \tag{7.188}$$

The last three equations yield the final outcome,

$$\frac{1}{2} \left[\frac{(E+L)^2}{\lambda_0 - \lambda_1} + \frac{(E-L)^2}{\lambda_0 + \lambda_1} \right] - \frac{2R(y)S(x)}{\lambda_2} \sqrt{\left(\frac{E^2}{M^2(x)} - \mathcal{K}_+ \right) \left(\mathcal{K} - \frac{L^2}{P^2(y)} \right)} = \mathcal{K} \tag{7.189}$$

The last equations shine a spotlight on the significance of the entanglement of a Killing tensor in a Hamiltonian system. The employing of a Killing tensor guarantees the existence of hidden symmetries, like the Carter's constant $\mathcal{K}$ which represents the fourth constant of motion. Apparently, there are two ways to acquire expressions for the Carter's constant.

In cases where the Hamilton-Jacobi (HJ) equation is separable Carter's constant allows for the separation of the equation into two parts, each containing terms related to the non-ignorable coordinates. One part equals to $\mathcal{K}$ and the other is equal to its negative value. This approach helps us to understand the significance of this conserved quantity with respect to various values of the non-ignorable coordinates.

In cases where the separation of the Hamilton-Jacobi equation is not possible, the Killing tensor emerges as the only method, providing an expression that encapsulates the constant, the canonical momenta, and the Killing tensor within a single formula (7.189). The second method was

[5] Recall that λ_2 is a constant double eigenvalue of Killing tensor.

developed by Walker and Penrose [106], now the correlation between the existence of Killing tensor and the fourth constant of motion becomes evident.

The discover and the interpretation of hidden symmetries of a Hamiltonian system is not trivial since it demands invertible coordinate transformations and a bit of luck. Moreover, in the last decades, computational methods have been developed where automated hidden symmetries can be discovered adding machine learning methods in our line of attack [107].

Chapter 8

Analysis of Carter's Case [D]

In this chapter our interest focuses on Carter's Case [D]. The following metric describes a non-expanding spacetime in vacuum[1] with cosmological constant which is a direct product of 2 two-dimensional spaces of constant curvature,

$$ds^2 = 2\left[\tilde{V} - (12\Lambda)(x - C_x)^2\right] d\tilde{t}^2 - \frac{2dx^2}{\tilde{V} - (12\Lambda)(x - C_x)^2} - 2\left[\tilde{K} - (12\Lambda)(y - C_y)^2\right] d\tilde{z}^2 - \frac{2dy^2}{\tilde{K} - (12\Lambda)(y - C_y)^2} \quad (8.1)$$

This metric has already been discovered by Plebański as Case C [58], by Carter as case [D] [55], by Hauser and Malhiot as Case (0,0) with two Killing vectors ∂_3, ∂_4 for $\epsilon = +1$ or ∂_1, ∂_2 for $\epsilon = -1$ [13], by Kasner independently [59], and by us. This metric is a general family of metrics since it includes Plebański-Hacyan metric [42], Bertotti-Robinson [39], [40] and Nariai spacetimes ($\Lambda > 0$) [108], [41], [45].

We shall proceed now to the analysis of this solution. Assuming that the cosmological constant is positive throughout the procedure, the constant of integration $\tilde{K}$ must be always positive and the metric must satisfies the following constraint (otherwise the Lorentzian signature maintenance will be violated),

$$\tilde{K} - (12\Lambda)(y - C_y)^2 > 0$$

In case where the constant $\tilde{V}$ is positive, the metric (8.1) holds. The functions $M^2(x)$ and $P^2(y)$ are inverted parabolas, their peak take the values of the constants $\tilde{V}, \tilde{K}$ accordingly, where $x = C_x$ and $y = C_y$. Also the squared character of these functions dictates the constraint

$$\tilde{V} - (12\Lambda)(x - C_x)^2 > 0$$

[1] Originally, Carter's Case [D] was found by Carter in electrovacuum where the electromagnetic field is present. Although, in case where the charge is zero the metric is equal to metric (8.1), since the charge term e is located at the component of the squared term of the polynomial, as one could observe in relations (3.26) and (3.27).

The roots of our functions are $x_{\pm} = C_x \pm \sqrt{\frac{\tilde{V}}{12\Lambda}}$ and $y_{\pm} = C_y \pm \sqrt{\frac{\tilde{K}}{12\Lambda}}$. The annihilation of the denominators creates coordinate singularities, thus the coordinates x, y lie between the roots, i.e. inside the positive area of the parabola.

8.1 Reduction to Flat Spacetime

In this section we present two different ways to gain a flat spacetime from metric (8.1). The annihilation of Weyl components

$$\Psi_0 = \Psi_4^* = -3\Psi_2 = -3\Lambda = 0 \tag{8.2}$$

is the standard way to apply this reduction. Indeed, with the appropriate choice of constants we take

$$ds^2 = 2\left(d\tilde{t}^2 - d\tilde{z}^2 - dx^2 - dy^2\right) \tag{8.3}$$

where the constants satisfy the relations

$$\tilde{V} = 1 \tag{8.4}$$

$$\tilde{K} = 1 \tag{8.5}$$

A different way to achieve the same reduction is to consider the point $x_M = C_x$ and $y_M = C_y$, which describes the top values of the parabolas. Since the parentheses in (8.1) vanish, at this point all the metric functions are constants and equal with

$$M^2(x_M) = \frac{1}{S^2(x_M)} = \tilde{V} \tag{8.6}$$

$$P^2(y_M) = \frac{1}{R^2(x_M)} = \tilde{K} \tag{8.7}$$

Consequently, our spacetime is proved to be asymptotically flat at these points.

8.2 Geodesics

In this section we are going to compute the geodesics for this spacetime. We already know the general relations for geodesics with respect to the functions of metric.

Considering the expressions for $S^2(x)$ and $R^2(y)$ the relations (7.164) and (7.165) can be treated as follows.

$$\frac{dx}{d\lambda} = \frac{1}{2}\left[E^2 - \mathcal{K}_+\left[\tilde{V} - 12\Lambda(x - C_x)^2\right]\right]^{\frac{1}{2}} \tag{8.8}$$

$$\rightarrow x(\lambda) = \sqrt{\frac{E^2 - \tilde{V}\mathcal{K}_+}{12\Lambda\mathcal{K}_+}}\sinh\left(\sqrt{12\Lambda\mathcal{K}_+}\frac{\lambda}{2} + C_{Gx}\right) + C_x, \tag{8.9}$$

where the integration gave birth to constant C_{Gx}. The geodesic for $y(\lambda)$ is obtained with similar manner,

$$y(\lambda) = \sqrt{\frac{\tilde{K}\mathcal{K} - L^2}{12\Lambda\mathcal{K}}} \sin\left(\sqrt{12\Lambda\mathcal{K}}\frac{\lambda}{2} + C_{Gy}\right) + C_y \tag{8.10}$$

We have already expressed the coordinates x, y with respect to the affine parameter λ. We shall continue by plugging the last expressions into the geodesics for $\tilde{t}, \tilde{z}$,

$$\dot{\tilde{t}} = \frac{E}{2M^2(x)} = \frac{E}{2\left[\tilde{V} - 12\Lambda(x - C_x)^2\right]}$$

$$\rightarrow \tilde{t}(\lambda) = \frac{1}{\sqrt{12\Lambda\tilde{V}}} \operatorname{arctanh}\left[\frac{E}{\sqrt{\tilde{V}\mathcal{K}_+}} \tanh[\sqrt{12\Lambda\mathcal{K}_+}\frac{\lambda}{2} + C_{Gx}]\right] + t_0 \tag{8.11}$$

$$\dot{\tilde{z}} = -\frac{L}{2P^2(y)} = \frac{-L}{2\left[\tilde{K} - 12\Lambda(y - C_y)^2\right]}$$

$$\rightarrow \tilde{z}(\lambda) = -\frac{1}{\sqrt{12\Lambda\tilde{K}}} \arctan\left[\frac{L}{\sqrt{\tilde{K}\mathcal{K}}} \tan[\sqrt{12\Lambda\mathcal{K}}\frac{\lambda}{2} + C_{Gy}]\right] + z_0 \tag{8.12}$$

In order to find the final relations for the geodesic lines we assumed that

$$\tilde{K}, \tilde{V}, \mathcal{K} > 0 \tag{8.13}$$

It worths to note that if we try to make $\tilde{t}$ to behave as the affine parameter, we must dictate the following relations,

$$E^2 - \tilde{V}\mathcal{K}_+ = 0$$

$$\tilde{V} = \frac{R_c^2}{2}$$

$$\mathcal{K}_+ = 2R_c^2$$

In this case, as we can see from equation (8.10), $x = C_x$. Consequently, the left product space remains Minkowskian, and our primary concern shifts to the right product space, which appears to operate independently. In this scenario, the coordinate $y(t)$ follows a sinusoidal function of time, while the coordinate $\tilde{z}(t) \propto rt$, where $r > 0$, since the cosmological constant is of a very small value. At this moment, we lack a physical explanation for this behavior.

8.3 The eigenvalues of Killing tensor

In chapter 7.5 we acquired any necessary relation that we need in order to characterize the eigenvalues of our Killing tensor. Now we have to adjust the expressions for the functions of the metric. For instance the relations (7.184) and (7.185) are reformed as follows,

$$\lambda_0 + \lambda_1 = \lambda_+ M^2(x) = \lambda_+[\tilde{V} - 12\Lambda(x - C_x)^2] \tag{8.14}$$

$$\lambda_0 - \lambda_1 = \lambda_- P^2(y) = \lambda_-[\tilde{K} - 12\Lambda(y - C_y)^2] \tag{8.15}$$

Let us recall the relation (7.188) of chapter 7, which gives a relation between the eigenvalues of the Killing tensor with respect to the constants of motion and the metric functions.

$$\frac{1}{2}\left[\frac{(E+L)^2}{\lambda_0-\lambda_1}+\frac{(E-L)^2}{\lambda_0+\lambda_1}\right]-\frac{1}{\lambda_2}\sqrt{\left(\frac{E^2}{M^4(x)}-\frac{\mathcal{K}_+}{M^2(x)}\right)\left(\frac{\mathcal{K}}{P^2(y)}-\frac{L^2}{P^4(y)}\right)}=\mathcal{K} \tag{8.16}$$

A brief comment can be made regarding the non-constant eigenvalues aligning with the non-negligible coordinates individually. This separation arises due to the entanglement of the Killing tensor. It is easy to verify that when our coordinates approach a flat region (with an appropriate selection of constants) as $x \to C_x$ and $y \to C_y$, then the eigenvalues remain constant, as expected.

Finally, with the following relation, combined with the geodesic equations, we can conduct a study of an observer's trajectories, which can provide insights into the stability and instability issues of our spacetime,

$$K^{2\mu}{}_{\nu}=\begin{pmatrix}\lambda_+[\tilde{V}-12\Lambda(x-C_x)^2] & 0 & 0 & 0\\ 0 & \lambda_-[\tilde{K}-12\Lambda(y-C_y)^2] & 0 & 0\\ 0 & 0 & -\lambda_2 & 0\\ 0 & 0 & 0 & -\lambda_2\end{pmatrix} \tag{8.17}$$

8.4 Reduction to Nariai Metric

The characterization of coordinates in an arbitrary metric is not an easy task. There are various approaches to discern the nature of each coordinate. As a matter of fact, reduction techniques can prove to be fruitful in determining the generality of solutions and also in the characterization of constants of integration.

Furthermore, reductions are employed to gain a deeper understanding of the physical significance of the resulting metric and the coordinate system within each metric. In our case, prior literature indicates that our metric is quite general, as it can be reduced to various spacetimes. In this subsection we aim to obtain the form of equation (3.14). In order to achieve it, we proceed by applying the subsequent transformation,

$$\sqrt{2\tilde{V}}\tilde{t}=\hat{t} \qquad \sqrt{2}\frac{x-C_x}{\sqrt{\tilde{V}}}=\hat{x} \tag{8.18}$$

$$\sqrt{2\tilde{K}}\tilde{z}=\hat{z} \qquad \sqrt{2}\frac{y-C_y}{\sqrt{\tilde{K}}}=\hat{y} \tag{8.19}$$

With the latter we take a more understandable structure of our spacetime,

$$ds^2 = \left(1 - 6\Lambda\hat{x}^2\right) d\hat{t}^2 - \frac{d\hat{x}^2}{1 - 6\Lambda\hat{x}^2} - \left(1 - 6\Lambda\hat{y}^2\right) d\hat{z}^2 - \frac{d\hat{y}^2}{1 - 6\Lambda\hat{y}^2} \tag{8.20}$$

The first two-dimensional part of our metric is the two dimensional de-Sitter spacetime of radius $\frac{1}{\sqrt{6\Lambda}}$. Thus the relation (8.20) represents a hyperboloid and a 2-sphere, both of them have radius $\frac{1}{\sqrt{6\Lambda}}$. As we know, the product $dS_2 \times S^2$ gives the Nariai spacetime.

This metric is a direct product of a hyperboloid with a sphere, with constant radius [109], [110]. Thus, as Ginsparg and Perry showed in [111] the Nariai metric could be obtained from the limit of de-Sitter-Schwarzschild black hole as we see in Chapter 3, equation (3.3). In this limit the cosmological horizon coincides with the black hole horizon.

However, it is worth noting that the radius of the metric is related with the cosmological constant, which is correlated with the Weyl components and by extension with the tidal forces of the gravitational field. Next, we can apply an additional coordinate transformation

$$\sqrt{6\Lambda}\hat{z} = \phi \tag{8.21}$$

$$\sqrt{6\Lambda}\hat{y} = \cos\theta \tag{8.22}$$

to our second 2-dimensional metric providing a more appropriate form.

$$\frac{1}{6\Lambda}\left[d\theta^2 + \sin^2\theta d\phi^2\right] \tag{8.23}$$

The reduced metric takes the form

$$ds^2 = \left(1 - 6\Lambda\hat{x}^2\right) d\hat{t}^2 - \frac{d\hat{x}^2}{1 - 6\Lambda\hat{x}^2} - \frac{1}{6\Lambda}\left(d\theta^2 + \sin^2\theta d\phi^2\right) \tag{8.24}$$

Considering the relation above it is obvious that as $\hat{x}$ tends to zero the first part of metric reduces to two-dimensional flat spacetime, or differently, when $\hat{x} \to \infty$ the metric goes to infinity. This spacetime is homogeneous with the same causal structure as de-Sitter spacetime, but when the observer approaches to infinity the Nariai radius is constant and equal to R_c, which means that Nariai spacetime is not flat asymptotically. In this system of coordinates, $\hat{x}$ plays the role of the radius, hence it lies between $(0, R_c)$, meanwhile $\theta \in (0, \pi)$ and $\phi \in (0, 2\pi)$.

In our quest for a physical interpretation of the fourth constant of motion, we have the metric (8.24) at our disposal. This metric is ideal for this analysis because the hat components play the role of time and radius as well, where the other two coordinates are the angles of S^2. A standard way to test the validity of manner that the Carter's constant is defined during the separation of HJ equation, is to compare it with the well-established version of the relation

$$K^{\mu\nu} p_\mu p_\nu \equiv \mathcal{K}$$

This comparison could be performed in a convenient sector, for instance at the angle $\theta = \frac{\pi}{2}$. The next step contains the definition of the fourth constant of motion in both ways as we did in the previous chapter. The corresponding metric in this case is

$$ds^2 = 2\left[\frac{1-6\Lambda\hat{x}^2}{2}d\hat{t}^2 - \frac{d\hat{x}^2}{2\left(1-6\Lambda\hat{x}^2\right)} - \frac{1}{12\Lambda}\left(d\theta^2 + \sin^2\theta d\phi^2\right)\right] \tag{8.25}$$

If we reproduce the same derivations, as before we will obtain the corresponding Carter's constant for this metric defined as

$$\mathcal{K} \equiv p_\theta^2 + \frac{\hat{L}^2}{sin^2\theta} = \frac{1}{6\Lambda}\left[-\hat{\bar{m}}^2 + \frac{\hat{E}^2}{1-6\Lambda\hat{x}^2} - p_{\hat{x}}^2(1-6\Lambda\hat{x}^2)\right] \tag{8.26}$$

Hence, using the left part of the latter, we can conclude that for $\theta = \frac{\pi}{2}$ the Carter's constant coincides with the already known value of the angular momentum [104], [112]. A noteworthy comment is that the energy and angular momentum is not the same as in the previous paragraph of Carter's Case [D], but these quantities have changed during the transformation. We could prove the aforementioned correlation of Carter's constant with angular momentum easily, comparing the previous geodesics (8.9)-(8.12) with the new ones at $\theta = \frac{\pi}{2}$. We will exhibit this only, for the geodesics. So, we start by applying the transformation in the previous geodesics and then integrate the new geodesics. Next, we will compare the forms of the geodesics. Thus, for $\theta = \frac{\pi}{2}$ we have that

$$\mathcal{K} = \hat{L}^2 = R_c^2\left[-\hat{\bar{m}}^2 + \frac{\hat{E}^2}{1-\frac{\hat{x}^2}{R_c^2}} - p_{\hat{x}}^2(1-\frac{\hat{x}^2}{R_c^2})\right], \tag{8.27}$$

where the canonical momentum of $\hat{x}$ is

$$p_{\hat{x}}^2 = \frac{\dot{\hat{x}}}{2(1-\frac{\hat{x}^2}{R_c^2})} \tag{8.28}$$

Using the latter to integrate the geodesic we take

$$\hat{x}(\lambda) = \sqrt{\frac{R_c^2\left[R_c^2\hat{E}^2 - 2R_c^2\hat{\bar{m}}^2 - \hat{L}^2\right]}{\hat{L}^2 + 2\hat{\bar{m}}^2R_c^2}}\sinh\left[\sqrt{\frac{\hat{L}^2}{R_c^2} + 2\hat{\bar{m}}^2}(2\lambda) + \hat{C}_Gx\right] \tag{8.29}$$

After the comparison with the previous geodesics with $\mathcal{K}_+ \equiv \mathcal{K} + 2\bar{m}^2$,

$$\hat{x}(\lambda) = \sqrt{\frac{E^2 - \tilde{V}\mathcal{K}_+}{\tilde{V}6\Lambda\mathcal{K}_+}}\sinh\left[\sqrt{12\Lambda\mathcal{K}_+}\frac{\lambda}{2} + C_{Gx}\right], \tag{8.30}$$

we take that

$$L^2 + 2\bar{m}^2 = 2(\hat{L}^2 + 2\hat{\bar{m}}^2R_c^2)$$
$$E^2 = 4\tilde{V}R_c^2\hat{E}^2$$

These relations show that $\tilde{V} \propto R_c^{-2}$ as expected, since the following fraction is in square meters,

$$\frac{dx^2}{\tilde{V} - 12\Lambda(x - C_x)^2}$$

Nariai as 6-dimensional Spacetime

Moving forward, we apply another transformation that brings to surface the scheme of a hyperboloid combined with a sphere, more clearly. Another metric form of Nariai arises from the embedding in a 6-dimensional spacetime that we present in Chapter 3, eq. (3.7) with $\epsilon_1 = \epsilon_2 = 1$. The M_6 metric, along with the transformations, are presented as follows,

$$ds^2 = dZ_0^2 - dZ_1^2 - dZ_2^2 - dZ_3^2 - dZ_4^2 - dZ_5^2 \tag{8.31}$$

$$Z_0 = \sqrt{R_c^2 - \hat{x}^2}\sinh\left(\frac{\hat{t}}{R_c}\right)$$
$$Z_1 = \sqrt{R_c^2 - \hat{x}^2}\cosh\left(\frac{\hat{t}}{R_c}\right)$$
$$Z_2 = \hat{x}$$
$$Z_3 = R_c \sin\theta\cos\phi$$
$$Z_4 = R_c \sin\theta\sin\phi$$
$$Z_5 = R_c \cos\theta$$

The two constraints of our coordinates describe the characteristic surfaces of our 2-product space. The first one describes the hyperboloid of the de-Sitter part and the second one is the equation of sphere. Both have radius $R_c = \frac{1}{\sqrt{6\Lambda}}$,

$$\begin{aligned} -Z_0^2 + Z_1^2 + Z_2^2 &= R_c^2 \\ Z_3^2 + Z_4^2 + Z_5^2 &= R_c^2 \end{aligned} \tag{8.32}$$

Nariai as Chargeless Robinson-Bertotti Universe

In order to present the metric in the same form as equation (3.14) we must first apply the same transformation for the de-Sitter part of metric (8.24)

$$\sqrt{6\Lambda}\hat{t} = \tau \tag{8.33}$$

$$\sqrt{6\Lambda}\hat{x} = \cos r, \tag{8.34}$$

after that, we obtain the following form for the metric where $R_c = 1/\sqrt{6\Lambda}$,

$$ds^2 = R_c^2\left(\sin^2 r d\tau^2 - dr^2\right) - R_c^2\left(d\theta^2 + \sin^2\theta d\phi^2\right) \tag{8.35}$$

At last, we imply the coordinate transformation that will take us to the convenient form of the metric (3.14).

$$u = \sqrt{2}R_c \tan\left(\frac{r}{2}\right)e^{\tau} \tag{8.36}$$

$$v = -\sqrt{2}R_c \tan\left(\frac{r}{2}\right)e^{-\tau} \tag{8.37}$$

$$\zeta = \sqrt{2}R_c \tan\left(\frac{\theta}{2}\right)e^{i\phi} \tag{8.38}$$

After these transformations the Nariai metric could be rewritten as

$$ds^2 = \frac{2dudv}{(1-\frac{uv}{2R_c^2})^2} - \frac{2d\zeta d\bar{\zeta}}{(1+\frac{\zeta\bar{\zeta}}{2R_c^2})^2}. \tag{8.39}$$

Finally, we present the Nariai spacetime as in relation (3.14) with $a = b = R_c$ and $\epsilon_1 = \epsilon_2 = +1$. In this form we can safely conclude that the metric depend on the timelike coordinate that is embedded in u, v coordinates.

The Nariai metric is a chargeless Robinson - Bertotti universe. **Regarding this, we ought to remark that our metric represents a cosmological model with radius $R_c = 13.654$ billion light years. This is only 3.4 times smaller than the real estimate of the observable universe which is 46.5 billion light years.** Besides, the radius of the universe turned out to be a conformal factor of Nariai metric, which becomes evident in equation (8.35). Along these lines, we can study cosmological models such as this one by adjusting different radius.

Chapter 9

Discussion and Conclusions

This work marks the initial phase of a study aimed at establishing a comprehensive understanding of vacuum spacetimes with cosmological constant that admit the canonical forms of the Killing Tensor. The incorporation of canonical forms into the resolution process of Einstein's Field Equations has been proved fruitful yielding a wide range of solutions spanning various Petrov types and a new exact solution.

The primary concept of this thesis was that the general substance of the canonical forms of Killing tensor would lend us with more general families of spacetimes. Such spacetimes could ideally admit one spacelike Killing vector. Regarding this, we would be able to obtain a vast variety of solutions including cosmological models and non-stationary spacetimes as well.

Our paradigms were basically the works of Hauser-Malhiot [113], [114] on vacuum and of Papakostas on interior solutions with perfect fluids [14], [9]. Both of these works incorporated a reduced form of the canonical forms of a Killing Tensor which characterized by two double eigenvalues. Our work meant to follow a similar methodology with our paradigms with more general Killing tensors though.

The first part of this dissertation focuses on obtaining the canonical forms of a random Killing tensor through geometric methods, as an algebraic approach is not applicable. We followed the method was applied by Churchill in [16], adapting it to our general metric. Churchill obtained all these forms that we present in **Chapter 5**. However, he also gives one more form similar to our K^1 which does not contribute at all. The Killing equations of these two forms are the same, since our metric is symmetric in interchanges between two real null vectors $n^\mu \longleftrightarrow l^\mu$. This kind of symmetry concerns the concept of the *symmetric null tetrad*. Its significance arises by Debever et. al [115].

After obtaining the four canonical forms of the Killing tensor, we needed to decide which of these forms would be the primary focus of our study. We chose to concentrate on the K^1, K^2 and K^2 forms in vacuum with cosmological constant[1] driven by a specific concept.

We considered the possibility that our Killing tensor, as a more general form, could serve as an initial premise in order to obtain more general canonical forms of the Weyl tensor (Petrov types).

[1] As we explained the similarities between K^2 and K^3 led us to consider them simultaneously.

The Study of the $K^1_{\mu\nu}$ form

In order to investigate the validity of the latter consideration we studied the K^1 form. This is obvious in K^1 form, where we consider λ_0 to be a constant and we annihilated λ_7 in order to make this concept more applicable. Hence, the only difference between the following forms resulted to be the constant $q = \pm 1, 0$.

$$\tilde{K}^{HM}{}_{\mu\nu} = \begin{pmatrix} 0 & \tilde{\lambda_1} & 0 & 0 \\ \tilde{\lambda_1} & 0 & 0 & 0 \\ 0 & 0 & 0 & \tilde{\lambda_2} \\ 0 & 0 & \tilde{\lambda_2} & 0 \end{pmatrix} \qquad K^1_{\mu\nu} = \begin{pmatrix} q & \lambda_1 & 0 & 0 \\ \lambda_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \lambda_2 \\ 0 & 0 & \lambda_2 & 0 \end{pmatrix}$$

Our aim was to align our approach with this paradigm. However, we discovered that this parallelism was not feasible. The reason lies in the Killing equation (6.4).

$$q(\epsilon + \bar{\epsilon}) = 0 \tag{6.4}$$

Therein we annihilated the real part of $\epsilon + \bar{\epsilon}$ in order to study the K^1 form. Differently, the annihilation of factor q reduces the $K^1_{\mu\nu}$ into $\tilde{K}^{HM}$. With this manner we proved that we cannot consider simultaneously spacetimes that belong to the family of our paradigm. Besides, it is widely known that the Jordan forms concern all the vector space that can be described by a similar matrix with the same eigenvalues except the diagonalized case, which is the exceptional case of this family [89].

About the general methodology that we followed, we should denote that the rotation we used is applicable only if (at least) one of λ_7 or λ_0 is absent. In case where the terms $\tilde{\boldsymbol{\theta}}^3 \otimes \tilde{\boldsymbol{\theta}}^3$ and $\tilde{\boldsymbol{\theta}}^4 \otimes \tilde{\boldsymbol{\theta}}^4$ or $\tilde{\boldsymbol{\theta}}^1 \otimes \tilde{\boldsymbol{\theta}}^1$ and $\tilde{\boldsymbol{\theta}}^2 \otimes \tilde{\boldsymbol{\theta}}^2$ were present the invariant character of the Killing tensor would annihilate the remaining free parameter of the rotation, namely the parameter b. Strictly speaking, the capitalization of the parameter b manages to gain the **key relations** which basically determines our solutions. Thus, a rotation around a null tetrad frame could provide multiple solutions only if the reduced forms of the Canonical forms of the Killing Tensor are considered. This specific choice yielded three classes of solutions for every Killing tensor initiated by the Key relation $\mu\tau = 0$.

For the study of the K^1 form we did not find the metric in full detail yet, but we already proved that it is only admitted by type N solutions. This was demonstrated twice. The first method involved the standard approach of applying a rotation. The other method was to consider a suitable choice where $\pi = \tau$. This idea emerged by the work of Debever et. al. where they showed that their key equations (4.19) and (4.20) in [17] are invariant under rotations.

$$\pi\bar{\tau} = \bar{\pi}\tau \qquad \mu\bar{\rho} = \bar{\mu}\rho$$

Following this, we attempted to simplify our equations to satisfy the relation mentioned above. Ultimately, both methods only led to Type N solutions. Although, the most general solution of Hauser-Malhiot spaces is of Type I [13]. **This clarifies that there is no discernible relationship between the Killing tensor forms and the Petrov types (Canonical forms of Weyl tensor)**.

This becomes also evident by studying the $K^1_{\mu\nu}$ form, where in **Chapter 6** we proved that the solutions which admit a more general Killing form, namely the K^1 form, do not present any similarity with those of our paradigm. **In this point we answer to the first research question: the K^1 form does not encompass solutions or Petrov types of the reduced form of K^{HM}. Concequently, we cannot obtain more general Petrov types of our paradigm considering the Jordan canonical forms of Killing tensor.**

The Study of the $K^2_{\mu\nu}$ and $K^3_{\mu\nu}$ form

In **Chapter 7** the study of the 2nd and 3rd canonical forms were proved quite generous. Using the acquired simplifications due to a rotation around the null tetrad frame, we obtained three classes of solutions. The first class $\mu = 0 \neq \tau$ contains a type D solution and eight type N solutions. The second class $\mu = 0 = \tau$ is a subset of the type N solutions of the first class, while the third class $\mu \neq 0 = \tau$ yields two type III solutions.

In **Chapter 7** we put under the spotlight the type D solution of the first class. After we made a suitable choice $\kappa + \bar{\nu} = \bar{\pi} + \tau = 0$ we were able to apply the integrability theorem of Frobenius, defining our local coordinate system (t, z, x, y). **The novelty in this work is the discovery of a new type D solution** which does not belong either to Kinnersley's solutions [49] or to Debever - Plebański - Demiański [2]. As we proved in Section 7.3, the main characteristics that cannot classify our solution in the above families are:

- The existence of cosmological constant which is equal to Weyl component Ψ_2.
- Goldberg-Sachs theorem is not applicable due to the combination $\kappa \neq 0 = \sigma$ of spin coefficients.
- Our solution is a non-geodesic $(\kappa, \nu \neq 0)$, a shearfree $(\sigma = \lambda = 0)$ solution without expansion $(\mu = \rho = 0)$ [18].

It is well-documented that Kinnersley found all type D solutions in vacuum **without a cosmological constant** [49]. The inclusion of the cosmological constant in our solution implies that our solution does not belong to Kinnersley's solutions. The DPD solution is considered as the most general type D solution, and one of its intrinsic characteristics is its adherence to the Goldberg-Sachs theorem. Scoping to prove that our solution is not part of DPD family we ought to show that the two real principal vectors, n^μ and l^μ, must be non-geodesic and shear-free. Indeed, this condition holds for our solution, as we have proven in Appendix F. **Moreover, one could claim that our solution would possibly belong to Plebański-Hacyan family as a chargeless case since this family is the only solution, at our knowledge, that the Goldberg-Sachs theorem is not applicable. Although, in this family of solutions the existence of the electromagnetic field is correlated with the only non-zero Weyl component and with the negative cosmological constant as well. Along these lines, the annihilation of the electromagnetic field nullifies**

[2] Mostly known as Plebański - Demianski. [50], [53] (3.16 in **Chapter 3**).

the negative cosmological constant and the only component of Weyl tensor giving a conformally flat spacetime [100], [42].

The most general spacetime of this solution is encompassed in the general system of equations (7.80)-(7.88) along with the BI (I), (VI). We were not able to retrieve it without simplifications. By applying the separation of variables in Hamilton-Jacobi equation, the system of equations eventually yielded a metric. The tremendous system of equations finally reduced to two equations, namely, relations (7.106) and (7.107), where as a matter of fact they have similar form.

There are various methods for solving a second order nonlinear ordinary differential equation. In our case, equation (7.108) can be tackled by considering that $P = P(R)$, where $R = R(y)$. This approach is fundamental for autonomous equations like ours. However, this method led to elliptic integrals with hypergeometric functions as results, which became unmanageable for us. Ultimately, we applied a different general method in which one can make three choices to establish a correlation between $P(y)$ and $R(y)$. All these choices provided us with multiple solutions, which we present in the subsection 7.3 titled as **New Exact Solutions of Type D**.

The form of the differential equation allows for a variety of possible solutions because there are numerous ways to correlate two general functions in an autonomous differential equation. The main differences that characterize the obtained metrics are located in the correlation of $P^2(y)$ with respect to $R(y)$ and in the corresponding correlation of $M^2(x)$ with respect to $S(x)$.

Additionally, We found new metrics which can be reduced to the general equation of a 2-product spaces with constant curvature assuming a positive cosmological constant $\Lambda > 0$,as we mentioned in 7.3. We should also note that in all these metrics the Weyl components are

$$\Psi_0 = \Psi_4^* = -3\Psi_2 = -3\Lambda$$

The latter makes clear that the annihilation of the cosmological constant leads to conformally flat spacetime as we proved in case $F = 0$ and to Carter's Case [D]. We manage to give a specific form to our metric where the annihilation of the cosmological constant should give conformally flat spacetimes with appropriate selection of constants of integration. It should be also indicated that the first two choices yielded to the same solution ($\tilde{F} = 0$).

$$\begin{aligned} ds^2 = & \frac{8\tilde{V}(\tau_3 x + \tau_4)^2}{\tilde{V}^2(\sqrt{48\Lambda}x - C_x)^2 + 4} d\tilde{t}^2 - \frac{32\tilde{V}^2}{\left[\tilde{V}^2(\sqrt{48\Lambda}x - C_x)^2 + 4\right]^2} dx^2 \\ & - \frac{8\tilde{K}(\tau_1 y + \tau_2)^2}{\tilde{K}^2(\sqrt{48\Lambda}y - C_y)^2 + 4} d\tilde{z}^2 - \frac{32\tilde{K}^2}{\left[\tilde{K}^2(\sqrt{48\Lambda}y - C_y)^2 + 4\right]^2} dy^2 \end{aligned}$$

Another interesting solution is the Carter's Case [D] [55] solution that emerged during our investigation. This solution emerged with a unique correlation between $P(y)$ with $R(y)$ as we proved making the **Choice 3**.

$$ds^2 = 2\left[\left[\tilde{V} - (12\Lambda)(x - C_x)^2\right] d\tilde{t}^2 - \frac{dx^2}{\tilde{V} - (12\Lambda)(x - C_x)^2}\right]$$

$$-2\left[\left[\tilde{K}-(12\Lambda)(y-C_y)^2\right]d\tilde{z}^2+\frac{dy^2}{\tilde{K}-(12\Lambda)(y-C_y)^2}\right]$$

One more point that we ought to comment is that: **since the differential equation for x part and y part were solved separately our metrics could be any possible combination of functions $M^2(x)$, $S^2(x)$ with any $P^2(y)$, $R^2(y)$** accordingly. Hence, all these combinations are parts of the family creating new 2-product spacetimes.

Aside from extracting solutions, in the remaining sections of **Chapter 7**, we obtained the general forms for geodesics. Geodesic equations are solvable due to the separation of variables in the Hamilton-Jacobi equation. Moreover, we defined the fourth constant of motion both ways: 1) Firstly we took advantage of the separation of the HJ equation and thereafter 2) we employed the Killing tensor. In case where the HJ cannot be separated the Killing tensor is the only way to acquire the fourth constant of motion. Importantly, these operations can be applied to any of our spacetimes, as all the relationships are expressed in terms of the general metric functions.

Chapter 8 focuses on Carter's Case [D]. In this chapter, we establish that Carter's Case [D] reduces to Minkowski spacetime. Following this reduction, we explore the possibility of further reduction to the Nariai metric. We present coordinate transformations where there is a dependence on the timelike coordinate. **We also demonstrated that this solution describes a cosmological model with a radius 3.4 times smaller than the radius of the observable universe to date.**. Subsequently, we derive the geodesic equations using the general forms that were introduced in the previous chapter. With these relations it is easy for someone to investigate the stability or instability of the solution around unique points but also to check if the separation of HJ equation was achieved properly. In order to test this we studied the behaviour of the Carter's constant at the section where $\theta = \frac{\pi}{2}$. With this manner we proved that the Carter's constant is related with the squared angular momentum as expected [104], [112].

Future prospects

This study yields a lot of queries that need to be studied. For instance, we are aware that the suitable choice, that follows,

$$\kappa + q\bar{\nu} = 0 = \pi + \bar{\tau}$$

constrains the possible solutions and it also dictated that this solution must admits only the K^2 form. Another limitation emerged due to our incapability to solve the general system of equations (7.80)-(7.88) along with the BI (I), (VI). Hence, the necessary work that must be done is the resolving of the general system of equations that we avoided to solve considering the separation of HJ equation. The solution of this system of equation could provide us with a general solution which would not possibly be a 2-product spacetime.

Furthermore, we must confront the exact same problem considering though the case where the electromagnetic field is present. In order to do this, we must obtain new constraints for the electromagnetic field based on the Killing equation as demonstrated by Carter and McLenaghan [116], [60].

One thing that remained to be studied is the exact same problem with an initial annihilation of λ_0 instead of λ_7 for the same forms. In this case all the non-singular cases (K^1, K^2, K^3) coincide. [3].

In conclusion, exploring more general forms of the Killing Tensor has proven to be fruitful, answering to the open questions that we set, leading to the discovery of new type D exact solution and a wide variety of type N solutions, as well as two solutions of type III. We operated this study with the usage of the complex vectorial formalism is based on null tetrads creating an indispensable structure of the vector basis. At last, as we have shown the basic methodology we developed can be also applied to any Killing Tensor, offering a path to avoid potential deadlocks.

[3]Assuming the existence of a 2-dimensional Abelian group of motion, Prof. Papakostas managed to find solutions of Type I in his Philosophical Doctorate.

Chapter 10

Appendices

10.1 Appendix I

We array the NPEs and the ICs considering that μ, γ, ρ are real, ϵ is imaginary and $\Psi_3 = \Psi_4 = 0$. **At last, we choose** $\tau = 0$.

$$D\rho - \bar{\delta}\kappa = \rho^2 - \kappa\left[2(\alpha + \bar{\beta}) + (\alpha - \bar{\beta})\right] \quad \text{(a)}$$

$$\delta\kappa = \kappa\left[2(\bar{\alpha} + \beta) - (\bar{\alpha} - \beta)\right] - \Psi_o \quad \text{(b)}$$

$$-\Delta\kappa = -4\kappa\gamma + \Psi_1 \quad \text{(c)}$$

$$D\mu = \mu\rho + \Psi_2 + 2\Lambda \quad \text{(h)}$$

$$\Delta\mu = -\mu(\mu + 2\gamma) \quad \text{(n)}$$

$$\Delta\rho = -\rho(\mu - 2\gamma) - \Psi_2 - 2\Lambda \quad \text{(q)}$$

$$\delta\rho = \rho(\bar{\alpha} + \beta) - \Psi_1 \quad \text{(k)}$$

$$\bar{\delta}\mu = -\mu(\alpha + \bar{\beta}) \quad \text{(m)}$$

$$D\alpha - \bar{\delta}\epsilon = \alpha(\rho - 3\epsilon) - \bar{\beta}\epsilon - \bar{\kappa}\gamma \quad \text{(d)}$$

$$D\beta - \delta\epsilon = \beta(\rho + \epsilon) - \kappa(\mu + \gamma) - \epsilon\bar{\alpha} + \Psi_1 \quad \text{(e)}$$

$$\Delta\alpha - \bar{\delta}\gamma = \gamma(\alpha + \bar{\beta}) - \mu\alpha \quad \text{(r)}$$

$$-\Delta\beta + \delta\gamma = -\gamma(\bar{\alpha} + \beta) + \mu\beta \quad \text{(o)}$$

$$\delta\alpha - \bar{\delta}\beta = \mu\rho + \alpha(\bar{\alpha} - \beta) - \beta(\alpha - \bar{\beta}) - \Psi_2 + \Lambda \quad \text{(l)}$$

$$D\gamma - \Delta\epsilon = -\epsilon(\gamma + \bar{\gamma}) \quad \text{(f)}$$

Integrability Conditions of Eigenvalue λ_1

$$\delta\gamma - \gamma(\bar{\alpha} + \beta) = 0 \quad (CR1 : \lambda_1)$$

$$\Delta\gamma - 2\gamma^2 = 0 \quad (CR2 : \lambda_1)$$

Integrability Conditions of Eigenvalue λ_2

$$Q[\delta\mu - 2\mu(\bar{\alpha} + \beta)] = \delta\rho - \rho(\bar{\alpha} + \beta) \quad (CR1 : \lambda_2)$$

$$Q[\Delta\mu - 2\mu(\mu + 2\gamma)] = D\mu + \Delta\rho - 2\rho\gamma \qquad (CR2:\lambda_2)$$

$$\delta\mu + \mu(\bar{\alpha} + \beta) = 0 \qquad (CR3:\lambda_2)$$

Next,we try to solve $(CR2:\lambda_2)$ using the (n), along with the summation of (h) and (q). Then, we take the expression below considering though that $\mu \neq 0$.

$$\mu + 2\gamma = 0 \qquad (10.1)$$

The latter along with (n) yields that $\Delta\mu = \Delta\gamma = 0$. This result is true only when

$$\mu = \gamma = 0 \qquad (10.2)$$

The last relation is impossible for Case II since the real part of μ vanishes. Nevertheless, the last relation determines the annihilation of $d\lambda_1$ in the same fashion with Case I. If we try to investigate a little further this case and we set μ, γ to be equal to zero we will see that (r), (o) along with (c) yield the following,

$$\Delta(\bar{\alpha} + \beta) = 0 \to \Delta\kappa = 0 = \Psi_1$$

Hence, BI (II) brings to surface the following

$$\rho\Psi_2 = 0. \qquad (10.3)$$

Between these two choices only Ψ_2 is acceptable since the other choice $\rho = 0$ yields $d\lambda_2 = 0$ as an immediate impact. This choice does not allow us to solve the problem with the usage of a Killing tensor since the Killing tensor depends only by two constant eigenvalues. The second choice yields a Type N solution since Ψ_0 is the only non-zero Weyl component. **The other subcase where $\alpha - \bar{\alpha} = 0 \neq \tau$ gives the same result**. Also, considering that

$$\gamma - \bar{\gamma} = \tau - \bar{\tau} = \mu - \bar{\mu} = 0$$

the imaginary part of NPE (r) is

$$(\delta - \bar{\delta})\gamma = -\gamma(\beta - \bar{\beta}), \qquad (r_-)$$

while the corresponding imaginary part of $(CR1:\lambda_1)$ is

$$(\delta - \bar{\delta})\gamma = \gamma(\beta - \bar{\beta}). \qquad ((CR1:\lambda_1))$$

Hence, the above results to

$$\beta - \bar{\beta} = \kappa - \bar{\kappa} = 0 \qquad (10.4)$$

The imaginary part of NPEs (c) and (e) is proved to be

$$0 = 2\epsilon(\tau + \bar{\tau}) + \Psi_1 - \Psi_1^* \qquad (c_-)$$

$$0 = (\delta + \bar{\delta})\epsilon - 2\epsilon(\alpha - \beta - \tau) + \Psi_1 - \Psi_1^* \qquad (e_-)$$

Trying to eliminate the imaginary part of Ψ_1 we obtain

$$(\delta + \bar{\delta})\epsilon = 2\epsilon(\alpha - \beta + \tau) \qquad (10.5)$$

In the same fashion the imaginary part of (d) gives

$$(\delta + \bar{\delta})\epsilon = 2\epsilon(3\alpha + \beta - \tau) \qquad (d_-)$$

The last two relations provide us with

$$\tau = \alpha + \beta \qquad (10.6)$$

Hence, the Killing equation (6.5) can be rewritten as follows

$$\kappa = 2Q\tau \qquad (10.7)$$

So, if we act upon the latter with the derivative Δ we acquire that

$$\Delta\kappa = -4\kappa\mu \qquad (10.8)$$

The substitution of the relation (10.8) into (c) yields

$$D\tau = -4\kappa(\gamma + \mu) + 2\tau(\rho + \epsilon) + \Psi_1 \qquad \text{(c)}$$

Scoping to deploy the relation (10.6) we are led to the deduction of (d) and (e).

$$D(\alpha + \beta) = -\kappa(2\gamma + \mu) + 2\tau(\rho + \epsilon) + \Psi_1 \qquad \text{(d)-(e)}$$

If we combine the last two relations we get to the following result

$$\gamma = -\frac{3}{2}\mu \qquad (10.9)$$

Next, if we act upon the latter with the derivative Δ considering though the NPE (n) and $(CR1 : \lambda_1)$ we take

$$\Delta\gamma = -\frac{3}{2}\Delta\mu \rightarrow \mu = \gamma = 0 \qquad (10.10)$$

At last, we proved that the **Case II** is impossible for any subcase of $\tau(\alpha - \bar{\alpha}) = 0$ since we cannot reach a solution where the below relation holds.

$$\mu + \bar{\mu} \neq 0 = \mu - \bar{\mu}$$

10.2 Appendix II

In this appendix we are going to prove that

$$\Omega = 2\frac{D\rho - \rho^2}{Q^2(\gamma + \bar{\gamma})^2} \not\ni x, y \qquad (10.11)$$

From covariant derivatives of Q we aware that $\delta Q = 0$, the latter is equivalent with $Q \not\ni x, y$. The proof is based on the following equations

$$D\gamma = 0 \qquad \text{(f)}$$

$$\rho + \bar{\rho} = r(t, z)(\gamma + \bar{\gamma}) \qquad (6.78)$$

$$\delta(\gamma+\bar{\gamma}) = (\gamma+\bar{\gamma})(\bar{\alpha}+\beta) \qquad \text{(CR1:}\lambda_1\text{)}$$

where the spin coefficients are

$$\gamma+\bar{\gamma} = \frac{1}{(M-P)}\frac{M_t-L_z}{AM-L} \tag{10.12}$$

$$\bar{\alpha}+\beta = -\frac{\delta(M-P)}{M-P} \tag{10.13}$$

If we substitute the equations (10.12) and (10.13) into $(CR1:\lambda_1)$ we take that

$$\frac{M_t-L_z}{AM-L} \not\ni x,y \tag{10.14}$$

also, the action of D into ρ gives

$$Dr(\gamma+\bar{\gamma}) = \frac{(M+P)r_t-(L+N)r_z}{(M-P)\left[A(M+P)-(L+N)\right]}(\gamma+\bar{\gamma}) \tag{10.15}$$

If we consider that $\Delta r = 0 \to r_t = Ar_z$, the equation (10.16) could be rewritten as

$$Dr(\gamma+\bar{\gamma}) = \frac{r_z}{(M-P)^2}\frac{M_t-L_z}{AM-L} \tag{10.16}$$

Recombining now all the previous relations we obtain

$$\Omega = \frac{2}{Q^2}\frac{r_z\frac{1}{(M-P)^2}\frac{M_t-L_z}{AM-L}-r^2\frac{1}{(M-P)^2}\left[\frac{M_t-L_z}{AM-L}\right]^2}{\frac{1}{(M-P)^2}\left[\frac{M_t-L_z}{AM-L}\right]^2} = \frac{2}{Q^2}\left[\frac{r_z}{\left[\frac{M_t-L_z}{AM-L}\right]}-r^2\right] \not\ni x,y \tag{10.17}$$

10.3 Appendix A

In this Appendix we are going to analyze the outcomes of the other cases that the following equation yields.

$$\frac{C_{Mx}}{C_M}(3\Psi_2+\Psi_0) = 0$$

The latter yields two cases.

Case I: $C_{Mx} = 0 \neq 3\Psi_2+\Psi_0$

Case II: $C_{Mx} = 0 = 3\Psi_2+\Psi_0$

Let us remind to the reader that we already are aware that P,R depends only on y since with $\Psi_0 = \Psi_0^*$ we take the annihilation of $\Phi(x)$. Next, the other choice of the relation (7.102) implies that $M(x,y) \to M(y)$. Hence, the contribution that one could gain from NPEs and BI (VI) is the following

$$12\Psi_2 = -\frac{1}{PR}\left[\frac{P_y}{R}\right]_y - \frac{1}{MR}\left[\frac{M_y}{R}\right]_y. \tag{A.1}$$

$$12\Psi_2 = -\frac{S_y}{RS}\left[\frac{P_y}{PR}+\frac{M_y}{MR}\right] \tag{A.2}$$

$$4\Psi_0 = \frac{1}{PR}\left[\frac{P_y}{R}\right]_y - \frac{1}{MR}\left[\frac{M_y}{R}\right]_y \tag{A.3}$$

$$4\Psi_0 = \frac{S_y}{RS}\left[\frac{P_y}{PR} + \frac{M_y}{MR}\right] \tag{A.4}$$

$$\left[\frac{S_y}{R}\right]_y = 0 \tag{A.5}$$

$$C_{Mx} = 0 \tag{A.6}$$

$$2\Psi_0\frac{\Omega_y}{\Omega} - \frac{C_{Py}}{C_P}[3\Psi_2 + \Psi_0] = 0 \tag{A.7}$$

If we add (A.1) with (A.3) and (A.2) with (A.4) accordingly we take

$$3\Psi_2 + \Psi_0 = 0 = \left[\frac{M_y}{R}\right]_y \tag{A.8}$$

The last expression clarifies that the Weyl component Ψ_0 is also constant so the Case I is impossible. Hence, we continue the analysis only for Case II.

The imaginary part of BI (VI) which is expressed by relation (7.103) along with the latest annihilation dictate that $\Omega_y = 0$, which yields that the metric function $M(y)$ is constant. In addition, for the metric function $S(x, y)$ we obtain that $S(x, y) \to S(x)$, since the only contribution in respect to y is vanished along with Ω_y. According to this, the relation (A.2) makes our spacetime conformally flat resulting to

$$\Psi_2 = \Psi_0 = \Psi_4 = 0 \tag{A.9}$$

10.4 Appendix B

The only equations that we have to confront are the following

$$12\Psi_2 = -4\Psi_0 = -\frac{1}{MS}\left[\frac{M_x}{S}\right]_x \tag{10.18}$$

$$12\Psi_2 = -4\Psi_0 = -\frac{1}{PR}\left[\frac{P_y}{R}\right]_y \tag{10.19}$$

One could observe that these two equations are the same if we substitute $M \rightarrow P$ and $S \rightarrow R$. We may now continue with the treatment only of (10.19). Let's present the non-linear differential equation of second order in a most proper form.

$$\frac{P_{yy}}{P} - \frac{P_y}{P}\frac{R_y}{R} + 12\Lambda R^2 = 0 \tag{10.20}$$

We can choose to correlate the two unknown functions with the next relation, where Π is a constant of integration.

$$\frac{P_y}{P} = -\frac{R_y}{R} \rightarrow P(y) = \frac{\Pi}{R(y)} \tag{10.21}$$

Thus, our equation is a non-linear differential equation of second order

$$\frac{R_{yy}}{R} - 3\left(\frac{R_y}{R}\right)^2 - 12\Lambda R^2 = 0$$

in order to solve it we have to make the following definition.

$$k \equiv \frac{dR(y)}{dy}$$

Then, the derivative of k with respect to R could be obtained by the first derivative with respect to y

$$\frac{dk}{dy} = \frac{dk}{dR}\frac{dR}{dy} \rightarrow k_R k = R_{yy}$$

then the differential equation could be rewritten as follows

$$(k^2)_R - 6\frac{k^2}{R} - 24\Lambda R^3 = 0$$

Moving forward, we can divide our function to a homogeneous solution and to a partial solution. In this case these indices do not indicate derivation.

$$k^2 = k_0^2 + k_P^2$$

Homogeneous Solution: $(k_0^2)_R - 6\frac{(k_0)^2}{R} = 0 \rightarrow k_0^2 = KR^6$ where K is constant.

Partial Solution: $(k_P^2) = \tilde{U}R^4$ where $\tilde{U}$ is also a constant.

If we substitute our solution into the differential equation we take

$$k^2 = KR^6 - 12\Lambda R^4 \rightarrow k = -eR^2\sqrt{KR^2 - 12\Lambda}$$

In this point we define $e \equiv \pm$. In order to express R as a function of y we have to proceed backwards considering that $k \equiv \frac{dR(y)}{dy}$. Afterwards, we take the following integral

$$\frac{dR}{R^2\sqrt{KR^2-12\Lambda}} = -edy.$$

Applying the following transformation to the left part of the integral we take

$$\sqrt{\frac{K}{12\Lambda}}R \equiv \cos w$$

Hence, we take

$$\sqrt{\frac{K}{12\Lambda}}\frac{dw}{\cos^2 w} = e\sqrt{12\Lambda}dy$$

After the integration the result is

$$\sqrt{\frac{K}{12\Lambda}}\tan w = e\sqrt{12\Lambda}y - D_y$$

with the usage of $\sqrt{\frac{K}{12\Lambda}}R = \cos w$ we finaly take

$$R^2(y) = \frac{-12\Lambda}{(\sqrt{12\Lambda}y - eD_y)^2 - K}$$

In this point we make the following choice.

$$D_y = \sqrt{12\Lambda}C_y$$

Hence, the $R^2(y)$ takes the form

$$R^2(y) = \frac{-12\Lambda}{(12\Lambda)(y-eC_y)^2 - K} \to R^2(y) = \frac{1}{\tilde{K} - (12\Lambda)(y-C_y)^2}$$

As one could observe e doesn't have any special contribution since its existence equivalently means just a shift on the x axis. Hence, we consider it to be equal $+1$. At last, we obtained the corresponding solution for $M^2(x)$ with the same manner since the initial differential equations are the same. The integration constant is multiplied by 12Λ. With this choice of constants of integration the annihilation of the cosmological constant reduces our spacetime to Minkowski spacetime with the appropriate choice of the remains constants.

10.5 Appendix C

In this appendix we present the four different results of the integral (7.117). There are four different results are depended by the sign of the constant $\tilde{F}$ and the sign of the discriminant $\Delta = -[16\tilde{F} + \tilde{K}^2]$.

$\underline{\tilde{F} > 0}$

In this case where $\tilde{F} > 0$ we have only one option since the discriminant can only be negative $\Delta < 0$.

$$R(y) = \frac{8\tilde{F}}{(16\tilde{F} + \tilde{K}^2)\cosh\Big(\sqrt{4\tilde{F}}(\sqrt{48\Lambda}y - C_y)\Big) - \tilde{K}} \tag{10.22}$$

$$T(y) = \tau_1 e^{\sqrt{F}y} + \tau_2 e^{-\sqrt{F}y}$$

Considering the last two results we can construct now the form of $P^2(y)$ and the corresponding metric functions which are depended by x.

$$ds^2 = M^2(x)\left(Adt + dz\right)^2 - P^2(y)\left(Bdt + dz\right)^2 - S^2(x)dx^2 - R^2(y)dy^2 \tag{10.23}$$

$$R^2(y) = \frac{(8\tilde{F})^2}{\left[\left(16\tilde{F} + \tilde{K}^2\right)\cosh\left(\sqrt{4\tilde{F}}(\sqrt{48\Lambda}y - C_y)\right) - \tilde{K}\right]^2} \tag{10.24}$$

$$P^2(y) = \frac{8\tilde{F}G^2\left[\tau_1 e^{\sqrt{F}y} + \tau_2 e^{-\sqrt{F}y}\right]^2}{\left(16\tilde{F} + \tilde{K}^2\right)\cosh\Big(\sqrt{4\tilde{F}}(\sqrt{48\Lambda}y - C_y)\Big) - \tilde{K}} \tag{10.25}$$

$$S^2(x) = \frac{(8\tilde{H})^2}{\left[\left(16\tilde{H} + \tilde{V}^2\right)\cosh\Big(\sqrt{4\tilde{H}}(\sqrt{48\Lambda}x - C_x)\Big) - \tilde{V}\right]^2} \tag{10.26}$$

$$M^2(x) = \frac{8\tilde{H}Y^2\left[\tau_3 e^{\sqrt{H}x} + \tau_4 e^{-\sqrt{H}x}\right]^2}{\left(16\tilde{H} + \tilde{V}^2\right)\cosh\Big(\sqrt{4\tilde{H}}(\sqrt{48\Lambda}x - C_x)\Big) - \tilde{V}} \tag{10.27}$$

where the constants $G, Y, F, K, H, V, \tau_1, \tau_2, \tau_3, \tau_4, C_x, C_y$ are constants of integration while the $\tilde{H}, \tilde{V}$ are defined by

$$\tilde{H} = \frac{H}{48\Lambda} \qquad \tilde{V} = \frac{V}{48\Lambda}$$

$\underline{\tilde{F} = 0}$

The second result concerns the case where the constant F is equal to zero and the constant K have to be non-zero. Although, the discriminant is negative.

$$R(y) = \frac{4\tilde{K}}{\tilde{K}^2(\sqrt{48\Lambda}y - C_y)^2 + 4} \tag{10.28}$$
$$T(y) = \tau_1 y + \tau_2$$

Considering the last two results we can construct the form of $P^2(y)$ and the corresponding metric functions which are depended by x.

$$ds^2 = M^2(x)\left(Adt + dz\right)^2 - P^2(y)\left(Bdt + dz\right)^2 - S^2(x)dx^2 - R^2(y)dy^2 \tag{10.29}$$

$$R^2(y) = \frac{16\tilde{K}^2}{\left[\tilde{K}^2(\sqrt{48\Lambda}y - C_y)^2 + 4\right]^2} \tag{10.30}$$

$$P^2(y) = \frac{4\tilde{K}G^2(\tau_1 y + \tau_2)^2}{\tilde{K}^2(\sqrt{48\Lambda}y - C_y)^2 + 4} \tag{10.31}$$

$$S^2(x) = \frac{16\tilde{V}^2}{\left[\tilde{V}^2(\sqrt{48\Lambda}x - C_x)^2 + 4\right]^2} \tag{10.32}$$

$$M^2(x) = \frac{4\tilde{V}Y^2(\tau_3 x + \tau_4)^2}{\tilde{V}^2(\sqrt{48\Lambda}x - C_x)^2 + 4} \tag{10.33}$$

Where the constants $G, Y, F, K, H, V, \tau_1, \tau_2, \tau_3, \tau_4, C_x, C_y$ are constants of integration while the $\tilde{H}, \tilde{V}$ are defined by $\tilde{H} = \frac{H}{48\Lambda}, \tilde{V} = \frac{V}{48\Lambda}$.

If we apply the following transformations

$$d\tilde{t} = Y(Adt + dz) \tag{10.34}$$
$$d\tilde{z} = G(Bdt + dz) \tag{10.35}$$
$$d\tilde{x} = \frac{\tilde{V}\sqrt{48\Lambda}dx - C_x\tilde{V}}{2} \tag{10.36}$$
$$d\tilde{y} = \frac{\tilde{K}\sqrt{48\Lambda}dy - C_y\tilde{K}}{2} \tag{10.37}$$

we obtain

$$ds^2 = \frac{(\tau_3[\frac{2\tilde{x}+C_x\tilde{V}}{\tilde{V}\sqrt{48\Lambda}}] + \tau_4)^2}{1+\tilde{x}^2}d\tilde{t}^2 - \frac{d\tilde{x}^2}{[1+\tilde{x}^2]^2} - \frac{(\tau_1[\frac{2\tilde{y}+C_y\tilde{K}}{\tilde{K}\sqrt{48\Lambda}}] + \tau_2)^2}{1+\tilde{y}^2}d\tilde{z}^2 - \frac{d\tilde{y}^2}{[1+\tilde{y}^2]^2} \tag{10.38}$$

$\tilde{F} < 0$ and $\Delta \geq 0$

In this case the constant $\tilde{F}$ is negative and the discriminant could be either positive or zero. The integral though gives us a quite complicate result when we try to express $R(y)$ in terms of y, then it yeilds a second order polynomial of $R(y)$. Obtaining the roots of the polynomial we get the result below. Possible transformations are necessary since the metric for this expression are cumbersome. The factor TAN in the next relation defined by $TAN \equiv 8\tilde{F}\tan^2(\sqrt{-4\tilde{F}}(y - C_y)) \neq 0$.

$$R(y) = \frac{\tilde{K}\left[1 + 2TAN\right] \pm \sqrt{\tilde{K}^2\left[1 + 2TAN\right]^2 + 64\tilde{F}TAN\left[1 + TAN\right]}}{4TAN} \tag{10.39}$$

$$T(y) = \tau_1 e^{\sqrt{F}y} + \tau_2 e^{-\sqrt{F}y}$$

$\tilde{F} < 0$ and $\Delta < 0$

The final result is presented below. In this case the discriminant is negative which equivalently means that $\tilde{K}^2 > 16|\tilde{F}|$.

$$R(y) = \frac{8|\tilde{F}|}{\tilde{K} + \sqrt{\tilde{K}^2 - 16|\tilde{F}|}\sin\left(\sqrt{4|\tilde{F}|}(\sqrt{48\Lambda}y - C_y)\right)} \tag{10.40}$$

$$T(y) = \tau_1 e^{i\sqrt{|F|}y} + \tau_2 e^{-i\sqrt{|F|}y}$$

Considering the last two results we can construct the form of $P^2(y)$ and the corresponding metric functions which are depended by x.

$$ds^2 = M^2(x)\left(Adt + dz\right)^2 - P^2(y)\left(Bdt + dz\right)^2 - S^2(x)dx^2 - R^2(y)dy^2 \tag{10.41}$$

$$R^2(y) = \frac{64|\tilde{F}|^2}{\left[\tilde{K} + \sqrt{\tilde{K}^2 - 16|\tilde{F}|}\sin\left(\sqrt{4|\tilde{F}|}(\sqrt{48\Lambda}y - C_y)\right)\right]^2} \tag{10.42}$$

$$P^2(y) = \frac{8|\tilde{F}|G^2\left[\tau_1 e^{i\sqrt{|F|}y} + \tau_2 e^{-i\sqrt{|F|}y}\right]}{\tilde{K} + \sqrt{\tilde{K}^2 - 16|\tilde{F}|}\sin\left(\sqrt{4|\tilde{F}|}(\sqrt{48\Lambda}y - C_y)\right)} \tag{10.43}$$

$$S^2(x) = \frac{64|\tilde{H}|^2}{\left[\tilde{V} + \sqrt{\tilde{V}^2 - 16|\tilde{H}|}\sin\left(\sqrt{4|\tilde{H}|}(\sqrt{48\Lambda}x - C_x)\right)\right]^2} \tag{10.44}$$

$$M^2(x) = \frac{8|\tilde{H}|Y^2\left[\tau_3 e^{i\sqrt{|H|}x} + \tau_4 e^{-i\sqrt{|H|}x}\right]}{\tilde{V} + \sqrt{\tilde{V}^2 - 16|\tilde{H}|}\sin\left(\sqrt{4|\tilde{H}|}(\sqrt{48\Lambda}x - C_x)\right)} \tag{10.45}$$

At last, we obtained the general solutions for every possible case of the assumed constant. In the next few pages we are going to annihilate the second square bracket premising the form of $g(y)$ in respect to $R(y)$. After this assumption the rest terms will give us the expression of $T(y)$ in respect to $R(y)$. Then we can construct $P(y)$ and the metric by extension.

10.6 Appendix D

We present the four cases that admit a manageable solution of the relation (7.123). The other cases for ζ give hypergeometric functions which cannot be used in order to express R with respect to y. The other relation that we will use in order to take the final result for $P(y)$ is the following

$$\frac{T_{yy}}{T_y} = (1 - 2\zeta)\frac{R_y}{R} \tag{10.46}$$

$$g(y) = GR^{\zeta}(y) \tag{10.47}$$

Possessing the final form of $R(y)$ along with the cases of ζ we can determine completely the function $P(y)$ since $P(y) = g(y)T(y)$.

$\zeta = +\frac{1}{2}$

In this case we have to confront the following integral which actually gives the same result with case $\tilde{F} = 0$ in **Choice 1**. This is the most general case for both choices.

$$\frac{dR}{R^2\sqrt{\frac{\tilde{K}}{R} - 1}} = \sqrt{48\Lambda}dy \tag{10.48}$$

which gives

$$R(y) = \frac{4\tilde{K}}{\tilde{K}^2(\sqrt{48\Lambda}y - C_y)^2 + 4} \tag{10.49}$$

Considering the relation (10.46) with $\zeta = \frac{1}{2}$ we get

$$T(y) = \tau_1 y + \tau_2 \tag{10.50}$$

the final relation for $P(y)$ is

$$P^2(y) = G^2 R(y) T^2(y) = \frac{4\tilde{K}G^2(\tau_1 y + \tau_2)^2}{\tilde{K}^2(\sqrt{48\Lambda}y - C_y)^2 + 4} \tag{10.51}$$

$\zeta = -\frac{1}{2}$

For this case the integral is presented below

$$\frac{dR}{R^2\sqrt{\tilde{K}R - 1}} = \sqrt{48\Lambda}dy \tag{10.52}$$

The result of this integral cannot provide us with a manageable result since we cannot express $R(y)$ with respect to coordinate y. However, the result is

$$\sqrt{\tilde{K}R - 1}\left[\frac{R\,\mathrm{arctanh}(\sqrt{1 - \tilde{K}R}) + \sqrt{1 - \tilde{K}R}}{R\sqrt{1 - \tilde{K}R}}\right] = \sqrt{48\Lambda}y - C_y \tag{10.53}$$

$\zeta = +1$

The result of the integral for this case is

$$\frac{dR}{R\sqrt{\tilde{K} - R^2}} = \sqrt{12\Lambda}dy \tag{10.54}$$

Hence, the expression for R(y) is given by

$$R^2(y) = \tilde{K}\left[1 - \tanh^2\left(\sqrt{\tilde{K}}(\sqrt{12\Lambda}y - C_y)\right)\right] \tag{10.55}$$

$$P^2(y) = G^2 R^2(y) T^2(y) = G^2\tilde{K}[1 - \tanh^2\tilde{y}]\left[\tilde{C}_y + \frac{\tilde{y}}{2} + \frac{\sinh(2\tilde{y})}{4}\right]^2 \tag{10.56}$$

$\zeta = -1$

The result of the integral for this case is given by

$$\frac{dR}{R^2\sqrt{\tilde{K}R^2 - 1}} = \sqrt{12\Lambda}dy \tag{10.57}$$

Hence, the expression for R(y) is the following.

$$R^2(y) = \frac{1}{\tilde{K} - (\sqrt{12\Lambda}y - C_y)^2} \tag{10.58}$$

$$P^2(y) = \frac{G^2T^2(y)}{R^2(y)} = \frac{G^2}{\tilde{K}^2}\left[\frac{12\Lambda y - C_y}{\tilde{K}} + C\sqrt{\tilde{K}}\sqrt{1 - \left(\frac{\sqrt{12\Lambda}y - C_y}{\tilde{K}}\right)^2}\right]^2 \tag{10.59}$$

10.7 Appendix F

Chandrasekhar and Xanthopoulos in [90] operated two classes of rotation around n^μ and l^μ in order to prove the Type D character of their solution which has the same characteristic relation for Weyl Components like ours which is $\Psi_0\Psi_4 = 9\Psi_2^2$ where $\Psi_1 = 0 = \Psi_3$.

These classes of rotation scoping to prove that the only non-zero Weyl component is Ψ_2. This is achieved by determining an appropriate choice of the rotation parameter $p = c{+}id$ which in our case that takes place at Section 4.4. In their publication this parameter has the values $\bar{a} = \pm\sqrt{\frac{-3\Psi_2}{\Psi_0}} = \pm 1$ and $b = \frac{a\bar{\Psi}_0}{6\Psi_2} = - \pm \frac{1}{2}$ for the two classes accordingly.

Following this, we operated the same rotation to our vectors for the case where $\bar{a} = +1$ and $b = -\frac{1}{2}$. So, our tetrads results to the following relations[1].

Class I, l^μ fixed

$$\tilde{n} = n + l + m + \bar{m} \tag{10.60}$$

[1] For reasons of simplifications we present the vectors without their indices.

$$\tilde{l} = l \tag{10.61}$$

$$\tilde{m} = m + l \tag{10.62}$$

Class II, n^μ fixed

$$\hat{n} = \tilde{n} \quad \rightarrow \quad \hat{n} = n + l + m + \bar{m} \tag{10.63}$$

$$\hat{l} = \tilde{l} - \frac{1}{2}\left(\tilde{m} + \bar{\tilde{m}} - \frac{1}{2}\tilde{n}\right) \quad \rightarrow \quad \hat{n} = \frac{n+l}{4} - \frac{m+\bar{m}}{4} \tag{10.64}$$

$$\hat{m} = \tilde{m} - \frac{\tilde{n}}{2} \quad \rightarrow \quad \hat{m} = \frac{l-n}{2} + \frac{m-\bar{m}}{2} \tag{10.65}$$

Using the relations above and the relations (4.30)-(4.32) we proved that the "new" null tetrads are not geodesic.

$$\hat{n}^\nu \hat{n}_{\mu;\nu} = -(\kappa + \bar{\kappa} + \tau + \bar{\tau})(n_\mu + l_\mu) - 2(\bar{\kappa} + \bar{\tau})m_\mu - 2(\kappa + \tau)\bar{m}_\mu \tag{10.66}$$

$$\hat{l}^\nu \hat{l}_{\mu;\nu} = -(\kappa + \bar{\kappa} + \tau + \bar{\tau})(n_\mu + l_\mu) - 2(\bar{\kappa} + \bar{\tau})m_\mu - 2(\kappa + \tau)\bar{m}_\mu \tag{10.67}$$

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