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«Lévy Processes and Applications in Finance»

Master Thesis

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SAMOS

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Περίληψη

Η παρούσα διπλωματική εργασία πραγματεύεται τις διαδικασίες Lévy για τιμολόγηση δικαιωμάτων προαίρεσης, καθώς αποτελεί πεδίο έντονου ερευνητικού ενδιαφέροντος σχετικά με την εφαρμογή των εμπλεκόμενων μοντέλων στα χρηματοοικονομικά, ιδιαίτερα κατά την τελευταία δεκαετία. Στόχος μας ήταν να αποσαφηνίσουμε τα κύρια μαθηματικά χαρακτηριστικά πέντε διάσημων μοντέλων (Black – Scholes, Merton, Heston, Kou και Generalized Hyperbolic) και να παρέχουμε εργαλεία μοντελοποίησης, και όχι να παρουσιάσουμε μια συστηματική ανασκόπηση της βιβλιογραφίας αναφορικά με όλα τα μοντέλα στοχαστικών διαδικασιών Lévy που έχουν περιγραφεί από ερευνητές.

Πιο συγκεκριμένα, στο πρώτο κεφάλαιο ξεκινά μια σύντομη εισαγωγή, στην οποία περιγράφονται βασικές έννοιες των διαδικασιών Lévy ούτως ώστε να παρέχουμε μια ολοκληρωμένη εικόνα του ορισμού, των χαρακτηριστικών και των ιδιοτήτων τους, που θα βοηθήσει τον αναγνώστη στην κατανόηση του περιεχομένου της διπλωματικής.

Στο επόμενο κεφάλαιο, εμβαθύνουμε στις περιπλοκές των διαδικασιών Lévy με άλματα, εξετάζοντας το υπόδειγμα «toy» και διευκρινίζοντας τις απείρως διαιρετές κατανομές τους. Παρουσιάζονται οι τύποι Lévy – Khintchine και Lévy – Itô και ακολούθως, αναλύονται διάφορες υποκατηγορίες των διαδικασιών Lévy, συμπεριλαμβανομένων των διεργασιών με άλματα πεπερασμένης μεταβολής, των φασματικών μονόπλευρων διεργασιών και εκείνων με πεπερασμένες πρώτες ροπές. Τέλος, διερευνούμε υποδειγματικές διεργασίες Lévy όπως διαδικασίες Poisson, σύνθετες διαδικασίες Poisson, γραμμική κίνηση Brown και σταθερές διεργασίες.

Στο Κεφάλαιο 3, εστιάζουμε στις εφαρμογές των διαδικασιών Lévy στον χρηματοοικονομικό κλάδο. Συζητούνται κανόνες τιμολόγησης, ισοδύναμα μέτρα martingale, μέτρα ουδέτερου κινδύνου και η έννοια της μη πληρότητας της αγοράς. Γίνεται μια σχολαστική διερεύνηση της ισοδυναμίας των μέτρων στο πλαίσιο των διαδικασιών Lévy.

Τέλος, στο Κεφάλαιο 4 περιγράφονται πέντε δημοφιλή μοντέλα τιμολόγησης στη βιβλιογραφία των μαθηματικών χρηματοοικονομικών. Εξετάζεται το θεμελιώδες μοντέλο Black – Scholes και ακολουθεί μια εις βάθος ανάλυση του μοντέλου Merton Jump-Diffusion Model, Heston Stochastic Volatility Model, Kou Double Exponential

Jump-Diffusion Model και του Generalized Hyperbolic Model. Τα μαθηματικά χαρακτηριστικά, οι εφαρμογές και οι επιπτώσεις του καθενός διευκρινίζονται διεξοδικά.

Λέξεις-κλειδιά: Βασικές έννοιες των διαδικασιών Lévy, Διαδικασιές Lévy με άλματα, υποδειγματικές διεργασίες Lévy, Διαδικασιές Lévy στον χρηματοοικονομικό κλάδο, μοντέλα τιμολόγησης.

Abstract

The present thesis deals with the study of Lévy processes for option pricing, since it is a field of intense research interest regarding the application in finance, especially over the last decade. Our aim was to elucidate the primary mathematical characteristics of five renowned models (Black – Scholes, Merton, Heston, Kou and Generalized Hyperbolic) and provide modelling tools, rather than displaying an exhaustive overview of all Lévy models described in the literature or delving into their intricate mathematical properties.

More specifically, the first chapter embarks on a brief description of Lévy processes, providing a comprehensive understanding of their definition, characteristics, and properties.

In the next chapter, we delve into the intricacies of Lévy processes, examining a 'toy' example of jump-diffusion Lévy processes and elucidating their infinitely divisible distributions. The Lévy – Khintchine formula and the Lévy – Itô decomposition are presented, followed by an exploration of various subclasses of Lévy processes, including subordinators, jumps of finite variation, spectrally one-sided processes, and those with finite first moments. Finally, we investigate exemplary Lévy processes such as Poisson processes, compound Poisson processes, linear Brownian motion, and stable processes.

In Chapter 3, we shift our focus to the applications of Lévy processes in finance. Pricing rules, equivalent martingale measures, risk-neutral measures, and the concept of market incompleteness are discussed. A meticulous exploration of the equivalence of measures within the context of Lévy processes is undertaken.

Chapter 4 navigates through popular pricing models in the mathematical finance literature. The foundational Black – Scholes model is examined, followed by an in-depth analysis of the Merton Jump-Diffusion Model, Heston Stochastic Volatility Model, Kou Double Exponential Jump-Diffusion Model, and the Generalized Hyperbolic Model. Each model's mathematical characteristics, applications, and implications are thoroughly elucidated.

Keyword: Description of Lévy processes, the intricacies of Lévy processes, exemplary Lévy processes, Lévy processes in finance, pricing models.

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Chapter 2

Chapter 1: Introduction

Lévy processes is a remarkable class of stochastic processes that have gained substantial prominence in the field of finance and beyond. Their unique characteristics and properties make them a powerful tool for modelling complex phenomena.

The history of Lévy processes can be traced back to the early 20th century. The concept was introduced by the French mathematician Paul Lévy in his groundbreaking work, "Théorie de l' Addition des Variables Aléatoires", published in 1937 [1]. Lévy's exploration of these processes marked a significant departure from traditional stochastic modelling. He laid the foundation for understanding random phenomena characterized by sudden and discontinuous changes, which had profound implications in various domains, especially in finance. The evolution of Lévy processes continued with the contributions of other notable mathematicians. For instance, Andrey Kolmogorov and Bruno de Finetti played key roles in the development of the theory. Their work expanded the understanding of Lévy processes and their applications, particularly in probability and statistical theory [2, 3]. This rich history underscores the enduring significance of Lévy processes in modern mathematics and finance.

The application of Lévy processes in finance is multifaceted and has yielded invaluable insights and tools for financial modelling and risk management. Lévy processes can effectively capture both continuous and discontinuous movements in financial time series, making them highly suitable for modelling asset returns and price dynamics. Lévy-driven models, such as the Merton jump-diffusion model and the Variance Gamma model, have been instrumental in improving the accuracy of financial pricing and risk assessment. These models incorporate jumps, which represent sudden market events, making them essential for capturing the unpredictability of financial markets [4].

Furthermore, Lévy processes have found applications in risk management, including the calculation of value-at-risk (VaR) and conditional value-at-risk (CVaR). Their ability to model extreme events and discontinuities in financial time series is essential for estimating tail risk, which is crucial for risk mitigation and regulatory compliance. In addition, Lévy processes are extensively used in derivative pricing, where complex financial instruments such as options and structured products can be valued with greater accuracy, considering the impact of jumps and other stochastic elements [5].

The growing significance of Lévy processes in finance continues to influence how we perceive and manage risk in the ever-evolving landscape of global financial markets. Their flexibility and adaptability provide financial practitioners with valuable tools to navigate the complexities of modern finance. Lévy processes represent a fascinating and indispensable topic in the realms of probability theory and finance. Their historical development and subsequent applications in financial modelling and risk management have transformed the way we understand and approach randomness in the financial world. This thesis will delve into the intricacies of Lévy processes and their far-reaching implications in the field of finance, exploring various models, characteristics, and applications.

Chapter 2: Lévy Processes

2.1 Definition of Lévy Processes

Lévy processes, named after the eminent French mathematician Paul Lévy, constitute a class of stochastic processes with distinctive features that render them indispensable in diverse fields, particularly in finance. A Lévy process is defined as a continuous-time stochastic process $\{X_{(t)}, t \geq 0\}$ with $X_0 = 0$ a.s., characterized by three core properties [6]:

- Independent increments: Lévy processes boast the property of independent increments. This signifies that the increments of the process over non-overlapping time intervals are statistically independent. Consequently, Lévy processes do not display any form of memory, a trait that is particularly advantageous in modelling random jumps.
  i.e., $X_t - X_s$ is independent of $\mathscr{F}_s$ for any $0 \leq s < t \leq T$.
- Stationarity of Increments: Lévy processes exhibit stationarity, meaning their statistical characteristics remain invariant over time. This property implies that the distribution of increments remains consistent irrespective of when the observation takes place.
  i.e., for any $0 \leq s,t \leq T$ the distribution of $X_{t+s} - X_t$ does not depend on t.
- Stochastically continuous: The distribution of increments in a Lévy process is solely determined by the length of the time interval and is independent of the starting point. This attribute is often expressed through the characteristic function, a crucial component of Lévy processes.
  i.e., for any $0 \leq t \leq T$ and $\varepsilon > 0$: $\lim_{s \to t} P(|X_t - X_s| > \varepsilon) = 0$.

2.2 Characteristics and Properties of Lévy Processes

Lévy processes possess a set of defining characteristics and properties that set them apart from other stochastic processes [5, 6]:

- Jump Components: A hallmark of Lévy processes is their capacity to exhibit jumps, abrupt and discontinuous changes in their values. These jumps play a fundamental role in modelling unexpected events across various domains, including financial markets.
- Drift and Diffusion: Lévy processes can encompass both jump components and drift and diffusion components, similar to traditional continuous-time stochastic processes like Brownian motion. This feature enables the modelling of gradual changes over time.
- Stability: Lévy processes are stable, signifying that the summation of independent Lévy-distributed random variables remains Lévy-distributed. This stability property is especially relevant in finance, where stable distributions are frequently observed in asset returns.
- Infinite Divisibility: The distribution of a Lévy process at any fixed time is infinitely divisible, meaning it can be expressed as the sum of a large number of independent and identically distributed (i.i.d.) random variables. This property proves invaluable in modelling aggregated data and complex systems.
- Self-Similarity: Lévy processes often exhibit self-similarity, implying that they manifest similarities at different time scales. This characteristic makes them well-suited for modelling phenomena displaying fractal-like behaviour.

The simplest Lévy process is the deterministic process of a linear drift while Brownian motion (which describes random movements of particles) is the only non-deterministic model with continuous sample paths (**Figure 1**). Other examples of Lévy processes are the Poisson and compound Poisson distributions. A mixture of a linear drift, Brownian motion and compound Poisson processes is again a Lévy process, which is often called a jump-diffusion process even though not all jump-diffusion processes are Lévy ones [7].

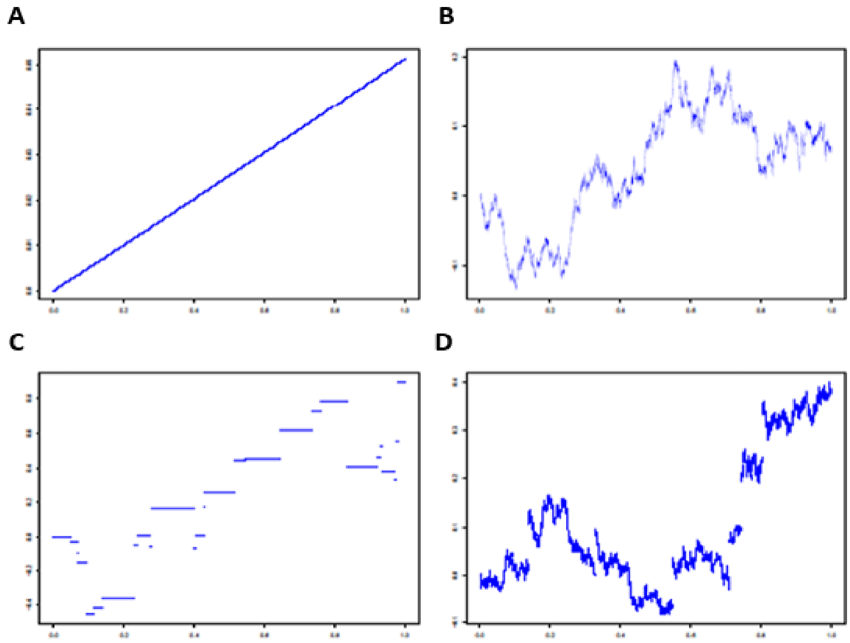

Figure 1: *Examples of Lévy processes: (A) Linear drift, (B) Brownian motion, (C) Compound Poisson process, (D) Lévy jump-diffusion. Source: Papapantoleon, Antonis. (2008). An introduction to Lévy processes with applications in finance.*

2.3 A 'Toy' Example of Jump-diffusion Lévy Processes

To grasp the essence of Lévy jump-diffusion processes, we turn to a simplified illustrative example, often referred to as the 'Toy' model. This example helps us understand the interplay between continuous diffusion and discrete jump components within Lévy processes. While the 'Toy' model is deliberately simplistic, it captures the fundamental characteristics and properties of Lévy jump-diffusion processes, making it an ideal starting point for in-depth comprehension [8, 9].

In the 'Toy' model, we consider a stochastic process $\{X_{(t)}, t \geq 0\}$ that evolves over time. This process incorporates two fundamental components, each contributing to its overall behaviour. The continuous diffusion component ($dB_{(t)}$) resembles the smooth, continuous paths of traditional Brownian motion. This component follows a stochastic differential equation of the form:

$$dB_{(t)} = \mu d_t + \sigma dW_{(t)},$$

where:

μ represents the drift, indicating the expected change in X over time. It accounts for the tendency of X to move in a particular direction, such as an expected rate of return.

σ denotes the volatility, measuring the randomness or dispersion of the process. It quantifies the degree of uncertainty or fluctuations.

$dW_{(t)}$ is the increment of a Wiener process, representing the continuous stochastic behaviour. It accounts for the continuous, random movements in the process.

In contrast to the smooth dynamics of the continuous diffusion, the 'Toy' model introduces jump events at random time intervals. These jump events can be modelled using a Poisson process and typically follow a distribution characterized by two main parameters, size (J) and frequency (λ). The former represents the magnitude of the jump, and the latter represents the average number of jump events occurring in a given time interval. The increments $dN_{(t)}$ represent the impact of these jumps on the overall process X(t), which can be described as:

$$dX_{(t)} = \mu d_t + \sigma dW_{(t)} + dN_{(t)}.$$

Based on the above, and given that all sources of randomness are independent, the characteristic function of $X_{(t)}$ is [5, 6]:

$$E[e^{iuX_t}] = E[\exp(iu(b_t + \sigma W_t + \sum_{k=1}^{N_t} J_k - t\lambda\beta))]$$

whereas, recalling that the characteristic functions of the normal and compound Poisson distributions are:

$$E\left[e^{iu\sigma W_t}\right] = e^{-\frac{1}{2}\sigma^2 u^2 t}, W_t \sim N(0,t)$$

$$E\left[e^{iu\sum_{k=1}^{N_t} J_k}\right] = e^{\lambda_t(E[e^{iuJ_k-1}])}, N_t \sim Poisson(\lambda t)$$

as well as the fact that the distribution of J_k is F, and t is a common factor, the Lévy jump-diffusion equation can be rewritten as:

$$E\left[e^{iuXt}\right] = \exp\left[t(iub - \frac{u^2\sigma^2}{2} + \int_R \left(e^{iux} - 1 - iux\right)\lambda F(dx)\right]$$

The 'Toy' model is a foundational representation of Lévy jump-diffusion processes that are frequently utilized in the realm of finance to model asset prices and returns. The jumps in the model can be interpreted as sudden, real-word market events, such as earnings announcements, economic news, or geopolitical shocks, which can lead to significant price movements. This example serves as a stepping stone for financial practitioners to grasp the nuanced interplay between continuous and discontinuous price changes, providing insights that are pivotal for pricing complex financial derivatives and managing risk in volatile markets. Thus, the 'Toy' model turns out to be a valuable tool for risk assessment, option pricing, and understanding the dynamics of financial markets [8].

2.4 Infinitely Divisible Distributions and the Lévy Processes

Lévy processes are closely related to the infinite divisible distributions, which are probability distributions that can be obtained by infinite convolutions of simpler probability distributions. The infinite divisible distributions are a broad class of probability distributions that exhibit the property that if you sum any number of independent random variables with the same distribution, the resulting distribution is of the same type. This property is known as stability, and it is a defining characteristic of the infinite divisible distributions [10, 11].

De Finetti (1929) was a pioneer in introducing the concept of infinitely divisible distributions and illuminating their profound connection to Lévy processes, shedding light on the vast diversity within the realm of Lévy processes [12]. This linkage not only underlines the versatility of Lévy processes but also underscores their rich mathematical and stochastic properties, making them a powerful tool for modelling various phenomena, particularly in the field of finance.

According to the definition of infinite divisibility, a random variable X is infinitely divisible only if for all $n \in \mathbb{N}$, there are i.i.d. random variables $X_1^{(n)}, \dots, X_n^{(n)}$ such that:

$$X \overset{(law)}{=} X_1^{(n)} + X_2^{(n)} + \dots + X_n^{(n)}.$$

Equivalently, a probability measure ρ is considered infinitely divisible only if for all $n \in \mathbb{N}$, there is another probability measure ρ_n such that:

$$\rho = \underbrace{\rho_n * \rho_n \dots * \rho_n}_{n-times}$$

Alternatively, the characteristic function of the infinitely divisible random variable could be used. Consequently, a probability measure ρ is considered infinitely divisible only if for all $n \in \mathbb{N}$, there is another probability measure ρ_n such that:

$$\hat{\rho}(u) = \left(\widehat{\rho_n}(u)\right)^n$$

The following theorem gives a complete characterization of random variables with infinitely divisible distributions via their characteristic functions. This is the celebrated Lévy – Khintchine formula which will be described in detail in the next section. For now, we will use the preparatory result below (Sato 1999, Lemma 7.8):

If (ρ_k) $where\ k \geq 0$ is a sequence of infinitely divisible distributions and $\rho_k \xrightarrow{w} \rho$, then ρ is also infinitely divisible.

Let $X = (X_t)$ $where\ t \geq 0$ be a Lévy process. Then, for $n \in \mathbb{N}$ and X_t, $t > 0$, the following will apply:

$$X_t = X_{t/n} + (X_{2t/n} - X_{\frac{t}{n}}) + \dots + (X_t - X_{\frac{(n-1)t}{n}})$$

And if it is assumed that each increment in a Lévy process follows a stationary distribution,

$$X_{\frac{tk}{n}} - X_{\frac{t(k-1)}{n}} \triangleq X_{\frac{t}{n}}\ for\ any\ k \geq 1, where \triangleq implies\ distribution\ equality$$

Considering the independence of the increments, the random variables are independent of each other as well; therefore, $\left(X_{\frac{tk}{n}} - X_{\frac{t(k-1)}{n}}\right)$ for k ≥ 1 is an i.i.d. sequence of random variables. In this case and based on the definition, the random variable X_t must be infinitely divisible.

Infinitely divisible distributions include the Normal, Poisson, Exponential, Geometric, Negative Binomial, Cauchy as well as the strictly stable distributions. All these are in contrast to the Uniform Distribution and Binomial distribution which are not infinitely divisible [5, 10].

2.5 The Lévy – Khintchine Formula

As already mentioned, the celebrated Lévy – Khintchine formula offers a comprehensive characterization of infinitely divisible distributions, primarily expressed in terms of their characteristic functions. It is a fundamental result in probability theory and stochastic processes [13,14]. Early contributions in proving versions of this representation came from renowned mathematicians B. de Finetti and A. Kolmogorov, who established it under specific assumptions [15]. Subsequently, P. Lévy and A. Khintchine independently demonstrated the formula in its general form. Lévy approached the proof by analyzing the sample paths of the stochastic process, while Khintchine employed direct analytic methods to establish the result [1, 16].

To validate the Lévy – Khintchine Formula, the stochastic process must meet the Lévy condition, which is expressed as follows:

For every Lévy process $X = X^{(1)} + X^{(2)} + X^{(3)}$, the characteristic exponent must be of the form: $\psi(u) = \exp\,(iu\mu - \frac{\sigma^2 u^2}{2} + iu \int_{\{|x|<1\}} \left(e^{ix} - 1 - ix\right) v(dx))$

where:

- $\Psi(u)$ is the characteristic function of the Lévy process.
- μ is the mean of the process (drift term).
- σ^2 is the variance of the diffusion term.
- v is the measure of the jumps of the process in the interval [-1,1].

This condition incorporating the drift term, diffusion term and the jump component ensures that the characteristic function has the specific form required for the use of the Lévy – Khintchine Formula. The Lévy – Khintchine theorem establishes that a probability measure ρ is infinitely divisible if and only if it can be expressed in terms of a triplet (b, c, v), called as the Lévy or characteristic triplet, where:

- $b \in \mathbb{R}$ and it is called the drift term.
- $c \in \mathbb{R} \geq 0$ and it is the Gaussian or diffusion coefficient (symmetric, non-negative definite d×d matrix).

- v represents the Lévy measure.

The probability measure *ρ(u)* is characterized by the following expression:

$$\hat{\rho}(u) = \exp\left(i\langle u, b\rangle - \frac{\langle u, cu\rangle}{2} + \int_{R^d} (e^{i\langle u,x\rangle} - 1 - i\langle u, x\rangle 1_D)\right) v\,(dx)$$

where D refers to a closed ball in d-dimensional real space, denoted as $R^{\mathrm{d}}\supset$, i.e., $D := \{|x \leq 1|\}$.

This theorem provides a fundamental connection between infinitely divisible measures and the components of the triplet (b, c, v), enabling the representation of such measures in a specific mathematical form. The truncation functions as well as the uniqueness of this representation is extensively discussed below.

A truncation function h: $R^d \to R^d$ is defined by a bounded function satisfying $h(x) = x$ in a neighbourhood of zero.

Alternatively, a truncation function h': $\mathbb{R}^d \to \mathbb{R}$ refers to a bounded and measurable function that satisfies the following:

$$h'(x) = 1 + o(|x|), as\ |x| \to 0$$

$$h'(x) = O(1/|x|), as\ |x| \to \infty$$

The above two definitions are linked through $h(x) = x \times h'(x)$.

Some well-known examples of the truncation functions are given below:

i. $h(x) = x1_D(x)$, also called as the canonical truncation function (**Figure 2**).

ii. $h(x) = \frac{x}{1+|x|^2}$, also called as the continuous truncation function (**Figure 2**).

iii. $h(x) \equiv 0$ and $h(x) \equiv x$; despite being commonly used, they are not always permissible choices.

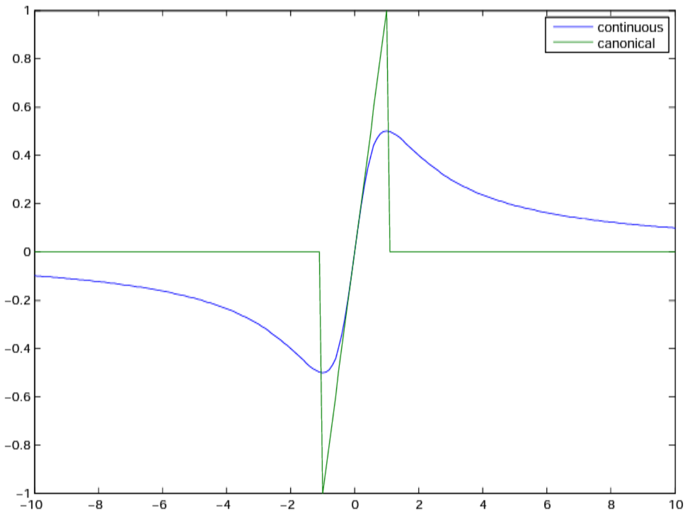

Figure 2: *Illustration of the canonical and the continuous truncation functions. Source: Tsitisvili, M. (2020). Lévy processes and applications.*

In fact, the Lévy – Khintchine representation of $\hat{\rho}$ depends on the choice of truncation function, which means that in case of another truncation function h rather than the canonical one, the equation becomes:

$$\hat{\rho}(u) = exp\left(i\langle u, b_h\rangle - \frac{\langle u, cu\rangle}{2} + \int_{\mathbb{R}^d}\left(e^{i\langle u,x\rangle} - 1 - i\langle u, h(x)\rangle\right)v(dx)\right)$$

where: $b_h = b + \int_{\mathbb{R}^d}(h(x) - x1_D(x))v(dx)$

If we aim to emphasize the influence of the truncation function on the Lévy triplet, we will represent it as (b_h, c, v) or $(b, c, v)_h$. It's important to note, though, that the diffusion characteristic c and the Lévy measure v remain consistent, regardless of the selected truncation function.

One approach to determine if a given random variable follows an infinitely divisible distribution is by examining its characteristic exponent. Let's denote this characteristic exponent as Θ, given by $\psi(u) := -log\mathbb{E}(e^{iu\Theta})$ for all $u \in \mathbb{R}$. Then, Θ exhibits an infinitely divisible distribution if for all $n \geq 1$, there exists a characteristic exponent of a probability distribution, denoted as ψ_n, satisfying the relationship $\psi(u) = n\psi_n(u)\ for\ all\ u \in \mathbb{R}$ [15]. The complete characterization of infinitely divisible distributions is achieved through the characteristic exponent ψ and the utilization of the Lévy-Khintchine formula. Then, the Lévy exponent ψ of X is defined as:

$$\psi(u) = i\langle u, b\rangle - \frac{\langle u, cu\rangle}{2} + \int_R \left(e^{i\langle u,x\rangle} - 1 - i\langle u, x\rangle 1_D(x) v(dx)\right), \text{ where: } \left[e^{i\langle u,x_1\rangle}\right] = e^{\psi(u)}.$$

2.6 The Lévy - Itô Decomposition

In the previous section, it was established that for any Lévy process denoted as $X = (X_t)_{t\geq 0}$, the random variables $X_t, t \geq 0$ exhibit an infinitely divisible distribution, which we characterized using the Lévy – Khintchine representation. In this section, we aim to demonstrate an 'inverse' result. Starting from an infinitely divisible distribution ρ or equivalently from a Lévy triplet (b, c, v), our objective is to construct a Lévy process, denoted as $X = (X_t)_{t\geq 0}$ in such a way that $P(X_1) = \rho$. This process will establish a connection between the distribution and the Lévy triplet, demonstrating their interdependence [17, 18].

In the context of the Lévy – Itô decomposition, the Lévy measure must satisfies the condition: $\lim_{\|u\|\to 0} v(|u|) = 0$, which guarantees the continuity of the Lévy measure as well as the condition: $\int_{\mathbb{R}^d} (|x|^2 \wedge 1) v(dx) < \infty$, ensuring exponential boundedness.

Theorem: Let ρ be an infinitely divisible distribution with Lévy triplet (b, c, v), where $b \in \mathbb{R}^d, c \in \mathbb{S}^d \geq 0$ and v is a Lévy measure. Then, there exists a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ on which four independent Lévy processes exist, $X^{(1)}, X^{(2)}, X^{(3)}, X^{(4)}$, where: $X^{(1)}$ is a constant drift, $X^{(2)}$ is a Brownian motion, $X^{(3)}$ is a compound Poisson process and $X^{(4)}$ is a square integrable, pure jump martingale with a.s. countable number of jumps of magnitude less than 1 in each finite time interval. Setting $X = X^{(1)} + \cdots + X^{(4)}$, we have that there exists a probability space on which a Lévy process $X = (X_t)_{t\geq 0}$ is defined, with Lévy exponent:

$$\psi(u) = i\langle u, b\rangle - \frac{\langle u, cu\rangle}{2} + \int_{\mathbb{R}^d} (e^{i\langle u,x\rangle} - 1 - i\langle u, x\rangle 1_D(x)) v(dx), \text{ for all } u \in \mathbb{R}^d,$$

and path or Lévy – Itô decomposition:

$$X_t = bt + \sqrt{c}W_t + \int_0^t \int_{D^c} x\mu^X(ds, dx) + \int_0^t \int_D x(\mu^X - v^X)(ds, dx), \text{ where } v^X = Leb \otimes v.$$

As shown, the Lévy – Itô decomposition provides an insightful framework for elucidating the composition of a generic Lévy process by breaking it down into three independent auxiliary Lévy processes, each exhibiting distinct path characteristics. Proficiency in comprehending the Lévy – Itô decomposition empowers us to differentiate various essential subclasses of Lévy processes based on their path behaviours. To delve into this subject, we will briefly explore the theory of Poisson random measures and the associated square-integrable martingales. This background is necessary for a more thorough understanding of Lévy processes and their diverse attributes.

2.7 Subclasses of Lévy Processes

We are already familiar with the fact that Brownian motion, compound Poisson processes, and Lévy jump-diffusion processes fall under the category of Lévy processes. We've explored their Lévy-Itô decomposition and characteristic functions. In this section, we will introduce additional subclasses of Lévy processes that hold particular significance.

2.7.1 Subordinator

A 'subordinator' is an a.s. increasing in t Lévy process. For X to be a subordinator, the triplet must satisfy the following:

- $v(-\infty, 0) = 0$.
- $c = 0$.
- $\int_{(0,1)} xv(dx) < \infty$.
- $\gamma = b - \int_{(0,1)} xv(dx) > 0$.

The Lévy – Itô decomposition of a subordinator is:

$$X_t = \gamma t + \int_0^t \int_{(0,\infty)} x\mu^L(ds, dx)$$

while the Lévy – Khintchine formula takes the form:

$$E[e^{iuX_t}] = exp\left[t\left(iu\gamma + \int_{(0,\infty)}(e^{iux} - 1)v(dx)\right)\right].$$

Two characteristic examples of this subclass are the Poisson and the inverse Gaussian processes (**Figure 3**).

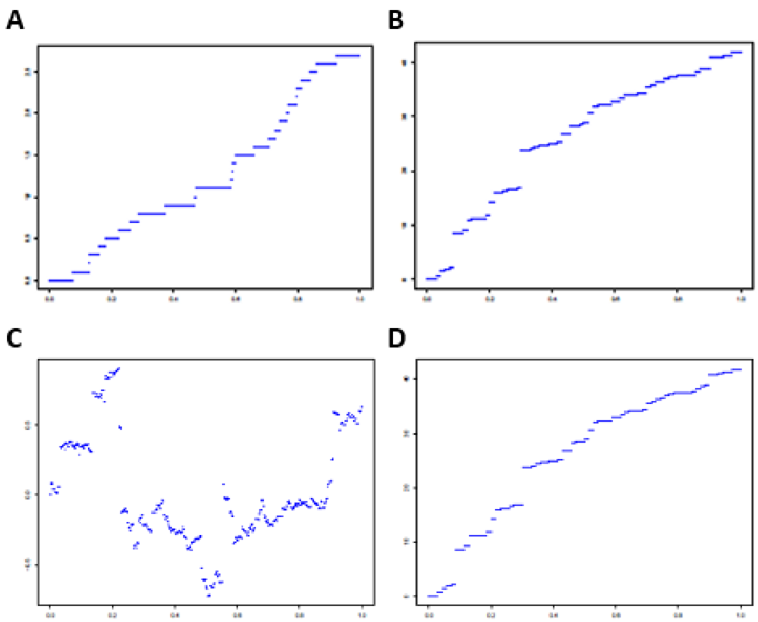

Figure 3: *Simulated paths of (A) a finite activity, (B) an infinite activity subordinator, (C) a normal inverse Gaussian and (D) an inverse Gaussian process. Modified image from: Papapantoleon, Antonis. (2008). An introduction to Lévy processes with applications in finance.*

2.7.2 Jumps of finite variation

A Lévy process has jumps of finite variation if and only if:

$$\int_{|x|\leq 1} |x| v(dx) < \infty$$

In this case, the Lévy – Itô decomposition of X resumes the form:

$$X_t = \gamma t + \sqrt{c}W_t + \int_0^t \int_{\mathbb{R}} x\mu^X(ds, dx)$$

while the Lévy – Khintchine formula takes the form:

$$E[e^{iuX}] = exp\left[t\left(iu\gamma - \frac{u^2 c}{2} + \int_{\mathbb{R}} (e^{iux} - 1)v(dx)\right)\right]$$

where γ is defined as in the 'Subordinator' section.

Moreover, if $v(|-1,1|) < \infty$, which means that $v(\mathbb{R}) < \infty$, then the jumps of X correspond to a compound Poisson process.

2.7.3 Spectrally one-sided

A Lévy process is called 'spectrally negative' if $v(0, \infty) = 0$, which means it has only negative jumps. The Lévy – Itô decomposition of a spectrally negative Lévy process has the form:

$$X_t = bt + \sqrt{c}W_t + \int_0^t \int_{x<-1} x\mu^X(ds, dx) + \int_0^t \int_{-1<x<0} x(\mu^X - v^X)(ds, dx)$$

Accordingly, the Lévy – Khintchine formula takes the form:

$$E[e^{iuX_t}] = exp\left[t\left(iub - \frac{u^2 c}{2} + \int_{(-\infty,0)} (e^{iux} - 1 - iu1_{\{x.-1\}})v(dx)\right)\right]$$

Similarly, a Lévy process is called 'spectrally positive' if $-X$ is spectrally negative.

2.7.4 Finite first moment

As we have seen already, a Lévy process has a finite first moment, if and only if:

$$\int_{|x|\geq 1} |x|v(dx) < \infty$$

Therefore, we can also compensate the big jumps to form a martingale hence the Lévy – Itô decomposition of X resumes the form:

$$X_t = b't + \sqrt{c}W_t + \int_0^t \int_{\mathbb{R}} x(\mu^X - v^X)(ds, dx)$$

Accordingly, the Lévy – Khintchine formula takes the form:

$$E\left[e^{iuX_t}\right] = exp\left[t\left(iub' - \frac{u^2c}{2} + \int_{\mathbb{R}} \left(e^{iux} - 1 - iux\right)v(dx)\right)\right]$$

where $b' = b + \int_{|x|\geq 1} xv(dx)$.

2.8 Examples of Lévy Processes

Some profound examples of Lévy Processes are given below for a better comprehension.

2.8.1 Poison Processes

For each $\lambda > 0$ consider a probability distribution μ_λ which is concentrated on $k = 0,1,2, \ldots$ such that: $\mu_\lambda = (\{k\}) = e^{-\lambda} \lambda^k / k!$. After appropriate calculations with regards to the Poisson distribution, its characteristic function has the form:

$$\sum_{k\geq 0} e^{i\theta k} \mu_\lambda(\{k\}) = e^{-\lambda(1-e^{i\theta})} = \left[e^{-\frac{\lambda(1-e^{i\theta})}{n}}\right]^n .$$

The right side is the characteristic function of the sum of n independent Poisson processes, each of which with parameter λ/n. In the Lévy – Khintchine decomposition we see that $b = c = 0$ and $v = \lambda\delta_1$, the Dirac measure supported on $\{1\}$. Also recall that a Poisson process $\{N_t : t \geq 0\}$ is a Lévy process with distribution at time $t > 0$, which is Poisson wit parameter λt.

From the above calculations we have: $E\left(e^{i\theta N_t}\right) = e^{-\lambda t(1-e^{i\theta})}$ and thus, its characteristic exponent is given by $\psi(\theta) = \lambda(1 - e^{i\theta})$ for $\theta \in \mathbb{R}$.

2.8.2 Compound Poison Processes

Let's suppose that N is a Poisson random variable with parameter $\lambda > 0$ and that $\{\xi_i : i \geq 1\}$ is an i.i.d. sequence of random variables with common law F having no atom at zero. Then, by first conditioning on $N \in \mathbb{R}$, we have:

$$E\left(e^{i\theta}\sum_{i=1}^{N}\xi_{\iota}\right) = \sum_{n\geq 0} E(e^{i\theta\sum_{i=1}^{N}\xi_i})e^{-\lambda}\frac{\lambda^n}{n!}$$

$$= \sum_{n\geq 0}\left(\int_{\mathbb{R}} e^{i\theta x}F(dx)\right)^n e^{-\lambda}\frac{\lambda^n}{n!} = e^{-\lambda\int_{\mathbb{R}}(1-e^{i\theta x})F(dx)}$$

We notice that the distributions of the form $\sum_{i=1}^{N}\xi_{\iota}$ are infinitely divisible with the triplet components:

- $b = -\lambda\int_{0<|\chi|<1} xF(dx)$.
- $c = 0$.
- $v(dx) = \lambda F(dx)$.

When F has an atom of unit mass at 1, a simple Poisson distribution occurs.

Now, suppose that $\{N_t : t \geq 0\}$ is a Poisson process with parameter λ and consider a compound Poisson process $\{X_t : t \geq 0\}$ defined by $X_t = \sum_{i=0}^{N_i}\xi_{\iota}, t \geq 0$. Using the fact that N has stationary independent increments together with the mutual independence of random variables $\{\xi_i : i \geq 1\}$ for $0 \leq s < t < \infty$, it is clear that by writing: $X_t = X_s + \sum_{i=N_{s+1}}^{N_t}\xi_i$, X_t refers to the sum of X_s and to an independent copy of X_{t-s}. The right continuity and left limits of the process N ensure the right continuity and left limits of X, thus establishing that compound Poisson processes belong to the category of Lévy Processes. Building upon the calculations from the preceding section, for each $t \geq 0$, we can replace N_t with the variable N to derive the Lévy – Khintchine formula for a compound Poisson process, which takes the form:

$$\Psi(\theta) = \lambda(1 - e^{i\theta x})F(dx)$$

Notably, it's worth mentioning that the Lévy measure of a compound Poisson process is consistently finite, and its total mass corresponds to the rate λ of the underlying process N.

Compound Poisson processes establish a direct connection between Lévy processes and random walks. In essence, they are discrete-time processes represented as $S = \{S_n : n \geq 0\}$, where $S_0 = 0$, and S_n is calculated as the sum of independent random variables ξi, for $n \geq 1$, with i ranging from 1 to n. Essentially, a compound Poisson process can be viewed as a variation of a random walk, where the jumps between points are spaced out by independent and exponentially distributed time intervals.

2.8.3 Linear Brownian Motion

Based on the probability law:

$$\mu_{s,\gamma} := \frac{1}{\sqrt{2\pi s^2}} e^{\frac{-(x-\gamma)^2}{2s^2}} dx$$

where $\gamma \in \mathbb{R}$ and $s > 0$, we reconstruct the equation:

$$\int_{\mathbb{R}} e^{i\theta x} \mu_{s,\gamma}(dx) = e^{-\frac{s^2\theta^2 + i\theta\gamma}{2}} = \left[e^{-\frac{1}{2}\left(\frac{s}{\sqrt{n}}\right)^2 \theta^2 + i\theta\frac{\gamma}{n}} \right]^n$$

with $b = -\gamma, c = s, v = 0$.

Furthermore, the characteristic exponent: $\psi(\theta) = \frac{s^2\theta^2}{2} - i\theta\gamma$ is immediately recognizable as that of the scale Brownian motion with linear drift, $X_t := sB_t + \gamma t, t \geq 0$, where $B = \{B_t : t \geq 0\}$ is a Standard Brownian motion; therefore, it’s about a linear Brownian motion with parameters $c = 1, \gamma = 0$.

2.8.4 Stable Processes

Stable processes constitute a category of Lévy processes characterized by their characteristic exponent, which aligns with the properties of stable distributions. Stable distributions were initially introduced by Lévy in 1924 and 1925 as a significant addition to the realm of infinitely divisible distributions, joining the ranks of Gaussian and Poisson distributions [19, 20].

A random variable, denoted as Y, is considered to have a stable distribution when it adheres to a distributional equality for all $n \geq 1$: $Y_1 + \cdots + Y_n \triangleq a_n Y + b_n$, where Y_1, Y_2, …, Y_n are independent replicas of Y, with $a_n > 0$ and $b_n \in \mathbb{R}$. This definition implies that any stable random variable is inherently infinitely divisible. Notably, it is essential to acknowledge that a_n must satisfy the following: $a_n = n^{1/a}$ for $a \in (0,2]$, where the parameter α indicates an index.

In specific instances where $b_n = 0$, the distribution falls into the category of strictly stable distributions. Then, we necessarily have: $Y_1 + \cdots + Y_n \triangleq n^{1/a} Y$, while the case $\alpha = 2$ strictly corresponds to zero mean Gaussian random variables.

Stable random variables for $\alpha \in (0,1) \cup (1,2)$, have characteristic exponents of the form:

$$\psi(\theta) = c|\theta|^{\alpha} \left(1 - i\beta tan\frac{\pi\alpha}{2} sgn\theta\right) + i\theta\eta$$

where $\beta \in [-1,1], \eta \in \mathbb{R}, c > 0$.

Accordingly, stable random variables for $\alpha = 1$, have characteristic exponents of the form:

$$\psi(\theta) = c|\theta| \left(1 - i\beta \frac{2}{\pi} sgn\theta \log|\theta|\right) + i\theta\eta$$

where $\beta \in [-1,1], \eta \in \mathbb{R}, c > 0$.

In contrast to the previous illustrations, the distributions associated with these characteristic exponents exhibit heavy tails. This implies that the tails of their distributions decay at a rate slow enough that they possess moments strictly less than α. The parameter β plays a crucial role in quantifying the asymmetry within the Lévy measure and, by extension, the distribution's asymmetry. Notably, the probability density functions of stable processes are explicitly known and can be expressed as convergent power series.

Chapter 3: Applications of Lévy Processes in Finance

3.1 Pricing rules and equivalent martingale measures

In this Chapter, we follow the presentation of Ole E. Barndorff-Nielsen, Thomas Mikosch & Sidney Resnick (2001) combined with the doctoral dissertation of Antonis Papapantoleon (2006) [21, 22]. The current brief subsection summarizes the arbitrage theory for semi-martingale models. Consider a market composed of underlying assets described by an adapted semi-martingale:

$S_t = (S_t^0, S_t^1, \dots, S_t^d),\ t \in [0, T]$, where S^0 serves as the numeraire (e.g., $S_t^0 = exp(rt)$), and a discount factor is represented by $B(t,T) = S_t^0 / S_T^0$. A contingent claim, denoted by its terminal payoff H, is an $\mathcal{F}_T$-measurable random variable and the set of relevant contingent claims is denoted as $\mathcal{H}$. A pricing rule is a method assigning a value $\Pi_t(H)$ to each $H \in \mathcal{H}$ at each time, subject to the following conditions:

- Adaptivity: $\Pi_t(H)$ is an adapted process and a semi-martingale.
- Positiveness: $H \geq 0 \Rightarrow \Pi_t(H) \geq 0$.
- Linearity: not valid for large portfolios in practice.

For any event $A \in \mathcal{F}$, the random variable 1_A represents the payoff of a contingent claim paying 1 at T if A occurs and zero otherwise, essentially a bet on A (also known as a lottery). We assume that $1_A \in \mathcal{H}$, indicating that such contingent claims are priced in the market. Notably, $1_\Omega = 1$ is equivalent to a zero-coupon bond paying 1 at T. Its value, $\Pi_t(1)$, signifies the present value of 1 unit of currency paid at T, i.e., the discount factor: $\Pi_t(1) = e^{-r(T-t)}$.

Now, let's define $\mathbb{Q}: \mathcal{F} \to \mathbb{R}$ as $\mathbb{Q}(A) = \frac{\Pi^0(1_A)}{\Pi^0(1)} = e^{rT}\Pi_0(1_A)$. In other words, $\mathbb{Q}(A)$ will represent the value of a bet with a nominal amount of $exp(rT)$ on the occurrence of event A. The linearity and positiveness of Π entail the following properties for $\mathbb{Q}$:

- $1 \geq \mathbb{Q}(A) \geq 0$: This is evident since $1 \geq 1_A \geq 0$.

- If A, B are disjoint events expressed as $1_{A \cup B} = 1_A + 1_B$, the linearity of the valuation operator implies that $\mathbb{Q}(A \cup B) = \mathbb{Q}(A) + \mathbb{Q}(B)$.

If we expand the linearity condition to encompass infinite sums, the measure $\mathbb{Q}$ is a probability measure on a family of events $\mathcal{F}$ in an event space $(\Omega, \mathcal{F})$. Therefore, by initiating from a valuation rule Π, we have effectively established a probability measure $\mathbb{Q}$ across scenarios space. Conversely, Π can be recovered from $\mathbb{Q}$ through the following process:
For random payoffs in the form of $H = \sum_i c_i 1_{A_i}$, indicating portfolios of cash-or-nothing options in financial terms, while the linearity of Π implies $\Pi_0(H) = \mathbb{E}^{\mathbb{Q}}[H]$.

Now, if Π adheres to an additional continuity property (i.e., if a dominated convergence theorem applies to $\mathcal{H}$), we can deduce that for any random payoff $H \in \mathcal{H}$: $\Pi_0(H) = e^{-rT}\mathbb{E}^{\mathbb{Q}}[H]$. Hence, there exists a one-to-one correspondence between linear valuation rules Π that satisfy the aforementioned properties and probability measures $\mathbb{Q}$ on event scenarios:

$$\Pi_0(H) = e^{-rT}\mathbb{E}^{\mathbb{Q}}[H]$$

and

$$\mathbb{Q}(A) = e^{rT}\Pi_0(1_A).$$

The above relation is often called as *'risk-neutral pricing formula'* where the discounted expectation under $\mathbb{Q}$ determines the value of a random payoff.

3.2 Risk-neutral measure

As demonstrated earlier, any linear valuation rule Π adhering to the specified properties is essentially a 'risk-neutral' pricing rule – there are no other alternatives! It's crucial to recognize that $\mathbb{Q}$ is not linked to the actual/objective probabilities of scenario occurrences. In fact, no objective probability measures on scenarios have been defined yet. In terms of mathematics, $\mathbb{Q}$, referred as a risk-neutral measure or pricing measure, is a probability measure on the set of scenarios and $\mathbb{Q}(A)$ should not be interpreted as the probability of

A happening in the real world, since it represents the value of a bet on A. A risk-neutral measure serves as a convenient representation of the pricing rule Π and is derived by examining contingent claim prices at $t = 0$, rather than through an econometric analysis of time series or similar methods.

Similarly, for each t, the mapping $A \mapsto A = e^{rt}\Pi_t(1_A)$ defines a probability measure over scenarios between 0 and t, denoted as $\mathbb{Q}_t$ on $(\Omega, \mathcal{F}_t)$. Assuming that the pricing rule Π is time-consistent (i.e., the value at 0 of the payoff H at T is the same as the value at 0 of the payoff $\Pi_t(H)$ at t), then $\mathbb{Q}_t$ is the restriction of Q, defined above, to $\mathcal{F}_t$. Additionally, $\Pi_t(H)$ is determined by the discounted conditional expectation with respect to $\mathbb{Q}$: $\Pi_t(H) = e^{-r(T-t)}\mathbb{E}^{\mathbb{Q}}[H|\mathcal{F}t]$.

Consequently, any time-consistent linear pricing rule Π, satisfying certain continuity properties, will be expressed as a discounted conditional expectation with respect to some probability measure $\mathbb{Q}$. In the subsequent analysis, the implications and restrictions imposed on $\mathbb{Q}$ by the absence of arbitrage will be further explored.

Now, let's consider that, apart from the market scenarios $(\Omega, \mathcal{F})$ and the information flow $\mathcal{F}_t$, we possess additional information about the probability of these scenarios occurring, denoted by a probability measure $\mathbb{P}$. In this context, $\mathbb{P}$ can signify either the objective probability of future scenarios or an investor's subjective viewpoint. In any of the cases, the pricing rule must adhere to specific constraints in order to align with this statistical perspective on the market's future evolution. A pivotal condition for a pricing rule is its ability to prevent the emergence of arbitrage opportunities. An arbitrage opportunity refers to a self-financing strategy φ that can yield a positive terminal gain without any probability of an intermediate loss:

$$\mathbb{P}(\forall t \in [0,T], V_t(\varphi) \geq 0) = 1$$

$$\mathbb{P}(V_T(\varphi) > V_0(\varphi)) \neq 0.$$

Certainly, such strategies must align with reality, taking the form of simple processes to be practically applicable. It's noteworthy that the definition of an arbitrage opportunity involves the probability measure $\mathbb{P}$, yet $\mathbb{P}$ is solely utilized to determine whether the profit is feasible or unattainable, not to calculate the probability of its occurrence. This definition only involves events with probabilities of 0 or 1, so the subsequent reasoning does not necessitate precise knowledge of scenario probabilities. The self-financing property is crucial. It's easy to showcase non-self-financing strategies that satisfy the aforementioned property by infusing cash into the portfolio just before maturity. An implication of the absence of arbitrage is *'the law of one price'*: two self-financing strategies with identical terminal payoffs must always have the same value; otherwise, the disparity would create an arbitrage opportunity.

We will now consider a market where prices are determined by a pricing rule represented by a probability measure $\mathbb{Q}$, as mentioned above. Let's take an event A with $\mathbb{P}(A) = 0$ and an option that pays the holder 1 (unit of currency) if event A occurs. Given that the event A is deemed impossible, this option holds no value for the investor. However, the pricing rule defined by $\mathbb{Q}$ assigns to this option a value at $t = 0$ equal to:

$$\Pi_0(1_A) = e^{-rT}\mathbb{E}^{\mathbb{Q}}[1_A] = e^{-rT}\mathbb{Q}(A).$$

Therefore, the pricing rule $\mathbb{Q}$ aligns with the investor's perspectives only when $\mathbb{Q}(A) = 0$. Conversely, if $\mathbb{Q}(A) = 0$, then the option with a payoff of $1_A \geq 0$ is considered worthless; if $P(A) \neq 0$, acquiring this option (for free) would result in an arbitrage opportunity. The harmony between the pricing rule $\mathbb{Q}$ and the stochastic model $\mathbb{P}$ implies that $\mathbb{Q}$ and $\mathbb{P}$ are equivalent probability measures, defining the same set of (im)possible events:

$$\mathbb{P} \sim \mathbb{Q}: \forall A \in \mathcal{F}, \mathbb{P}(A) = 0 \Leftrightarrow \mathbb{Q}(A) = 0.$$

3.3 Equivalence of martingale measure

Now, let's contemplate an asset S^i traded at price S_t^i. This asset can either be retained until T, yielding a terminal payoff of S_T^i, or be sold for S_t^i, where the resulting sum invested at the interest rate r would generate a terminal wealth of $e^{r(T-t)}S_t^i$. Both of these buy-and-hold strategies are self-financing and share the same terminal payoff. Therefore, they should hold the same value at time t as following:

$$\mathbb{E}^{\mathbb{Q}}(S_T^i|\mathcal{F}_t) = \mathbb{E}^{\mathbb{Q}}(e^{r(T-t)}S_t^i|\mathcal{F}_t) = e^{r(T-t)}S_t^i.$$

Then, dividing by $S_T^0 = e^{rT}$, the equitation is turned into:

$$\mathbb{E}^{\mathbb{Q}}\left(\frac{S_T^i}{S_T^0}|\mathcal{F}_t\right) = \frac{S_t^i}{S_t^0} \Leftrightarrow \mathbb{E}^{\mathbb{Q}}(\tilde{S}_T^i|\mathcal{F}_t) = \tilde{S}_t^i$$

Hence, the absence of arbitrage indicates that the discounted values $\tilde{S}_t^i = e^{-rt}S_t^i$ of all traded assets serve as martingales under the probability measure $\mathbb{Q}$. A probability measure satisfying the aforementioned conditions is termed an *'equivalent martingale measure'*. Therefore, it has been demonstrated that any pricing rule free from arbitrage is characterized by an equivalent martingale measure. Conversely, it is evident that any equivalent martingale measure $\mathbb{Q}$ defines a pricing rule free from arbitrage through $\Pi_t(H) = e^{-r(T-t)}\mathbb{E}^{\mathbb{Q}}[H|\mathcal{F}t]$.

Consider a self-financing strategy φ. Naturally, a practical strategy must be represented by a simple (piecewise constant) predictable process. As $\mathbb{Q}$ is a martingale measure, $\tilde{S}$ becomes a martingale under $\mathbb{Q}$. Therefore, the value of the portfolio $V_t(\varphi) = V_0 + \int_0^t \varphi_u d\tilde{S}_u$ is a martingale, and more specifically, $\mathbb{E}^{\mathbb{Q}}\left[\int_0^t \varphi_u d\tilde{S}_u\right] = 0$. Subsequently, the random variable $\int \varphi d\tilde{S}$ may take both positive and negative values:

$$\mathbb{Q}(V_T(\varphi) - V_0 \geq 0) \neq 1.$$

Since $\mathbb{P} \sim \mathbb{Q}$, this implies $\mathbb{P}(\int \varphi_t d\tilde{S}_t \geq 0) \neq 1$: φ cannot be an arbitrage strategy. Thus, there exists a direct correspondence between arbitrage-free pricing rules and equivalent martingale measures.

Proposition: *In a market described by a probability measure $\mathbb{P}$ on scenarios, any arbitrage-free linear pricing rule Π can be represented as $\Pi_t(H) = e^{-r(T-t)}\mathbb{E}^{\mathbb{Q}}[H|\mathcal{F}_t]$, where $\mathbb{Q}$ is an equivalent martingale measure, meaning a probability measure on the market scenarios such that: $\mathbb{Q} \sim \mathbb{P}$, $\mathbb{E}^{\mathbb{Q}}(\tilde{S}_T^i|\mathcal{F}_t) = \tilde{S}_t^i$.*

3.4 On market incompleteness

So far, our assumption has been the existence of an arbitrage-free pricing rule or equivalent martingale measures, which may not be evident in a given model. The preceding arguments establish that if there exists an equivalent martingale measure, the market is free from arbitrage. Demonstrating the converse, a more intricate task, is at times referred to as the *'Fundamental Theorem of Asset Pricing'*: *The market model defined by* $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P})$ *and asset prices* $(S_t)_{t\in[0,T]}$ *is arbitrage-free if and only if there exists a probability measure* $\mathbb{Q} \sim \mathbb{P}$ *such that the discounted assets* $(\tilde{S}_t)_{t\in[0,T]}$ *are martingales with respect to* $\mathbb{Q}$.

To this end, it can be specified that: *The market model is complete if and only if there exists a unique martingale measure* $\mathbb{Q} \sim \mathbb{P}$.

This theorem establishes the link between the financial concept of market completeness which involves the perfect hedging of any contingent claim, and the unique equivalent martingale measure, which is a mathematical attribute of the underlying stochastic model. In discrete time models, the theorem stands as outlined. However, in continuous time models, defining admissible strategies, contingent claims, and the concept of a 'martingale measure' requires careful consideration. Especially when dealing with unbounded jumps in S, common in exponential-Lévy models, formulating a rigorous representation becomes challenging. While many stochastic models used in option pricing are arbitrage-free, only a select few, such as stochastic volatility models, exponential-Lévy models and jump-diffusion models, fall into the category of complete models.

Completeness in this context implies that any random variable $H \in \mathcal{H}$, contingent on the history of S between 0 and T, can be expressed as the sum of a constant and a stochastic integral of a predictable process with respect to $\tilde{S}$. If this holds for all terminal payoffs with finite variance, i.e., any $H \in L^2(\mathcal{F}_T, \mathbb{Q})$ can be represented as $H = \mathbb{E}[H] + \int_0^T \varphi_s d\widetilde{S_t}$ for some predictable process φ, the martingale $(\tilde{S}_t)_{t\in[0,T]}$ is said to have the predictable representation property. Thus, market completeness is often equated with the predictable representation property, extensively studied for many classical martingales.

The predictable representation property is demonstrated for (geometric) Brownian motion or a Brownian stochastic integral but fails for most discontinuous models used in finance. For instance, it is known to fail for all non-Gaussian Lévy processes except the (compensated) Poisson process. Even when the predictable representation property holds, it does not guarantee market completeness. The interpretation of predictable processes like φ as trading strategies requires the ability to approximate their value processes using implementable (piecewise constant in time) portfolios. Predictable processes that can be reasonably interpreted as 'trading strategies' usually fall into the categories of simple predictable processes or caglad processes.

Lastly, it's noteworthy that we seek a representation of H in terms of a stochastic integral with respect to $\tilde{S}$. Another theorem reveals that when randomness stems from a Brownian motion W and a Poisson random measure M, a random variable with finite variance can be represented as a stochastic integral:

$$H = \mathbb{E}[H] + \int_0^t \varphi_s dW_s + \int_0^t \int_{\mathbb{R}^d} \psi\,(s,y)\,\widetilde{M}(dsdy)$$

Although often termed a predictable representation property by many authors, this property has no correlation with market completeness. Even when S is driven by the same sources of randomness W and M, and $M = J_s$ represents the jump measure of the process S, the mentioned expression cannot be represented as a stochastic integral with respect to S. Nonetheless, such representations prove useful for discussions on hedging strategies.

3.5 Equivalence of measures in the context of Lévy Processes

The previous subsection illustrates that when employing jump-diffusion processes to model market prices, it is imperative to understand the process of changing the probability measure. In order for the measures to be equivalent in the context of a compound Poisson process, let's assume that:

- N is a Poisson process on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ relative to a filtration $\{\mathcal{F}_t\}$.
- λ represents its intensity.
- The process M_t, defined as $N_t - \lambda t$, represents the associated compensated Poisson process, while it is essential to also note that under the probability measure $\mathbb{P}$, M_t is a martingale.
- $\tilde{\lambda}$ represents any positive real number.

The process Z defined by $Z(t) = exp\left(\left(\lambda - \tilde{\lambda}\right)t\right)\left(\frac{\tilde{\lambda}}{\lambda}\right)^{N_t}$ satisfies:

$$dZ(t) = \frac{\tilde{\lambda} - \lambda}{\lambda} Z(t-) dM_t$$

Hence, Z is a martingale under the probability measure $\mathbb{P}$, and for all $t \geq 0$, the expected value $\mathbb{E}\big(Z(t)\big)$ equals 1.

<u>*Proof*</u>: Let $X_t = \frac{\tilde{\lambda}-\lambda}{\lambda} M_t$.

The continuous part of X is: $X_t^c = \big(\lambda - \tilde{\lambda}\big)t \Rightarrow [X]_t^c = 0$.

The jump part is: $J_t = \frac{\tilde{\lambda}-\lambda}{\lambda} N_t$.

Therefore, $1 + \Delta X_t = \left(\frac{\tilde{\lambda}}{\lambda}\right)^{\Delta N_t}$.

By using a Lévy process X of finite variations, we obtain:

$$Z_t = \mathcal{E}(X)_t = exp\left(\big(\lambda - \tilde{\lambda}\big)t\right) \prod_{0<s\leq t} (1 + \Delta X_s) = exp\left(\big(\lambda - \tilde{\lambda}\big)t\right) \left(\frac{\tilde{\lambda}}{\lambda}\right)^{N_t}$$

Since M is a martingale, Z is also a martingale.

Furthermore, for $T > 0$: $\mathbb{P}(A) = \mathbb{E}\big(1_A Z(T)\big)\ where\ A \in \mathcal{F}_T$.

Then, the process N is a Poisson process with intensity $\tilde{\lambda}$, under the probability $\mathbb{P}$.

<u>*Proof*</u>: The Laplace transform of N under $\mathbb{P}$ is applied, $\forall A \in \mathcal{F}_T$:

$$\mathbb{E}^{\mathbb{P}}(e^{uN_t}) = \mathbb{E}(e^{uN_t} Z_t) = e^{(\lambda-\tilde{\lambda})t} \mathbb{E}\left[e^{uN_t} \left(\frac{\tilde{\lambda}}{\lambda}\right)^{N_t}\right] = e^{(\lambda-\tilde{\lambda})t} \mathbb{E}\left[exp\left[\left(u + \ln\left(\frac{\tilde{\lambda}}{\lambda}\right)\right) N_t\right]\right]$$

$$= e^{(\lambda-\tilde{\lambda})t} exp\left[\lambda t \left(e^{u+\ln\left(\frac{\tilde{\lambda}}{\lambda}\right)} - 1\right)\right] = exp\left(\tilde{\lambda}(e^u - 1)\right).$$

Then, the process N is a Poisson process with intensity $\tilde{\lambda}$, under the probability $\mathbb{P}$.

If: $\lambda > 0, \sigma > -1, \sigma \neq 0, \alpha \in \mathbb{R}$, and the process S, defined as:

$$S_t = S_0 exp[\alpha t + N_t ln(1 + \sigma) - \lambda \sigma t] = S_0 e^{(\alpha - \sigma\lambda)t} (\sigma + 1)^{Nt},$$

represents the price of an asset, then S satisfies the following:

$$dS_t = \alpha S_t dt + \sigma S(t^-) dM_t = \alpha S_t dt + \sigma S(t^-) d(N_t - \lambda t).$$

The above can be derived using the Itô formula. S is referred to as a 'geometric Poisson process'.

Now, let's assume that under a probability measure $\mathbb{P}$, N is a Poisson process with intensity $\tilde{\lambda} > 0$. This probability measure is risk-neutral if, under $\mathbb{P}$, S satisfies:

$$dS_t = rS_t dt + \sigma S(t^-)d(N_t - \tilde{\lambda}t)$$

where r is the riskless interest rate.

Therefore, $dS_t = aS_t dt + \sigma S(t^-)d(N_t - \lambda t) = rS_t dt + \sigma S(t^-)d\big(N_t - \tilde{\lambda}t\big)$,

which is possible if and only if $\alpha - \sigma\lambda = r - \sigma\tilde{\lambda} \Leftrightarrow \tilde{\lambda} = \lambda - \frac{\alpha - r}{\sigma}$.

When a Brownian component is included to the compound Poisson process, we recall the Girsanov theorem: Let $(X, \mathbb{P})$ and $(X, \mathbb{Q})$ be Brownian motions on $(\Omega, \mathcal{F}_T)$ with volatilities $\sigma^{\mathbb{P}} > 0$ and $\sigma^{\mathbb{Q}} > 0$, and drifts $\mu^{\mathbb{P}}$ and $\mu^{\mathbb{Q}}$. $\mathbb{P}$ and $\mathbb{Q}$ are equivalent if and only if $\sigma^{\mathbb{P}} = \sigma^{\mathbb{Q}}$.

In this case, the density is:

$$exp\left[\frac{\mu^{\mathbb{P}} - \mu^{\mathbb{Q}}}{\sigma^2} X_T - \frac{1}{2}\frac{(\mu^{\mathbb{Q}})^2 - (\mu^{\mathbb{P}})^2}{\sigma^2} T\right].$$

Now, we assume that on the same space $(\Omega, \mathcal{F}, \mathbb{P})$:

- W represents a Brownian motion.
- $Q_t = \sum_{i=1}^{N_t} Y_i$ represents a compound Poisson process with N as a Poisson process with intensity λ.
- $\tilde{\lambda} > 0$.
- $Y_i, i \in \mathbb{N}$ are i.i.d. random variables with density f.
- $\tilde{f}$ is defined as $f(y) = 0 \Rightarrow \tilde{f}(y) = 0$.
- Θ represents an adapted process.

Then,

$$Z_t^1 = exp\left(-\int_0^t \Theta_u \, dW_u - \frac{1}{2}\int_0^t \Theta_u^2 du\right)$$

$$Z_t^2 = e^{(\lambda-\tilde{\lambda})t} \prod_{i=1}^{N(t)} \frac{\tilde{\lambda}\tilde{f}(Y_i)}{\lambda f(Y_i)}$$

$$Z_t = Z_t^1 Z_t^2,$$

where Z is a martingale under $\mathbb{P}$; in particular, $\mathbb{E}(Z_t) = 1$, for all $t \geq 0$.

Proof:: The proof of the above is straightforward when Θ depends solely on W. Given that the processes W and Q are independent, Z^1 and Z^2 are two independent martingales and subsequently, Z itself is a martingale. Thus, based on the Itô formula, the equation has the following form:

$$Z_t = 1 + \int_0^t Z_{s}^2 - dZ_s^1 + \int_0^t Z_{s}^1 - dZ_s^2 + [Z^1, Z^2]_t.$$

So, since Z^1 is continuous and Z^2 is a pure jump quadratic martingale, we have $[Z^1, Z^2]_t = 0$.

Chapter 4: Popular Pricing Models

4.1 Black – Scholes Model

There are plenty of models that showcase the versality of Lévy processes in capturing complex dynamics observed in financial markets, including both continuous and discontinuous movements. Overall, they play a crucial role in option pricing, risk management as well as understanding of the market behaviour. In this chapter we review some of the most popular models in mathematical finance literature, focusing on their connection to Lévy processes.

Although it is not explicitly based on Lévy processes, the most famous asset price model is that of Samuelson (1965), Black and Scholes (1973) and Merton (1973) [23-25]. The Black – Scholes model can be seen as a special case where the underlying process has continuous paths. It corresponds to a geometric Brownian motion without jumps for the underlying asset's price, assuming that financial markets are efficient and there are no arbitrage opportunities. The log-returns exhibit a normal distribution characterized by a mean (μ) and variance (σ^2). In other words, L_1 follows a normal distribution with parameters μ and σ^2, denoted as $L_1 \sim Normal(\mu, \sigma^2)$. The probability density function of the long returns is subsequently expressed as:

$$f_{L_1}(x) = \frac{1}{\sigma\sqrt{2\pi}}\, exp\left[-\frac{(x-\mu)^2}{2\sigma^2}\right]$$

The characteristic function is given by:

$$\varphi_{L_1}(u) = exp\left[i\mu u - \frac{\sigma^2 u^2}{2}\right].$$

The first and second moments are $E[L_1] = \mu$ and $Var[L_1] = \sigma^2$, respectively.

Accordingly, the skewness and kurtosis are $skew[L_1] = 0$ and $kurt[L_1] = 3$.

Then, the canonical decomposition of L is $L_t = \mu t + \sigma W_t$, and the Lévy triplet is represented as $(\mu, \sigma^2, 0)$.

Traders often use the Black – Scholes formula to calculate the implied volatility, which is the one that makes the theoretical option price equal to the market price. The primary use, though, is to determine the theoretical fair value of European call and put options.

Regarding a European call option, the Black – Scholes formula is given by:

$$C = S_0 N(d_1) - Ke^{-rT} N(d_2)$$

where:

C represents the call option price,

S_0 refers to the current price of the underlying asset,

$N(\cdot)$ is the cumulative distribution function of the standard normal distribution,

K is the strike price of the option,

r is a risk-free interest rate,

T represents the time to expiration of the option (in years),

d_1, d_2 are used to adjust the formula based on the current state of the market, and $d_1 = \frac{\ln(S_0/K)+\left(r+\frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}$ while $d_2 = d_1 - \sigma\sqrt{T}$.

On the other hand, for a European put option, is given by:

$$P = Ke^{-rT} N(-d_2) - S_0 N(-d_1),$$

where P represents the put option price.

The Black – Scholes model is characterised by three main limitations. It assumes constant volatility that may not hold in reality, as well as continuous price movements, ignoring jumps. Finally, the European options which the model is designed for, can only be exercised at expiration. Despite its limitations, however, it laid the foundation for the field of financial derivatives pricing and remains a valuable tool in option pricing and risk management.

4.2 Merton Jump-Diffusion Model

An extension of the Black-Scholes model that incorporates jumps in asset prices is the Merton model named by Robert Merton who introduced it in 1976 to account for sudden, discontinuous movements [26]. This model explicitly uses a Levy process, combining a geometric Brownian motion for continuous movements similar to the Black – Scholes model, with a jump component represented by a compound Poisson process.

The dynamics of the asset price (S_t) are given by the following stochastic differential equation:

$$dS_t = \mu S_t dt + \sigma S_t dW_t + S_t dJ_t,$$

where:

μ is the expected continuous return,

σ is the volatility of continuous returns,

W_t is a Wiener process (Brownian motion), and

J_t is a jump process with independent and identically distributed jumps.

To this end, the standard decomposition of the underlying process is:

$$L_t = \mu t + \sigma W_t + \sum_{k=1}^{N_t} J_k$$

where J_k follows a normal distribution $\left(J_k \sim Normal(\mu_J, \sigma_J^2)\right)$ for $k = 1, \ldots, N_t$. As a result, the jump size exhibits a probability density function of:

$$f_J(x) = \frac{1}{\sigma_J \sqrt{2\pi}} exp\left[-\frac{(x-\mu_J)^2}{2\sigma_J^2}\right].$$

The characteristic function of L_1 is:

$$\varphi_{L_1}(u) = exp\left[i\mu u - \frac{\sigma^2 u^2}{2} + \lambda\left(e^{i\mu_J u - \sigma_J^2 u^2/2} - 1\right)\right]$$

where the Lévy triplet is $(\mu, \sigma^2, \lambda \times f_J)$ and the intensity of jumps (λ) represents the average number of jumps per unit time.

Similar to the Black-Scholes model, the Merton model can be used for option pricing and risk management under the risk-neutral measure. It allows for a more realistic representation of market behaviour by capturing sudden, unexpected events that can significantly impact asset prices. The introduction of jumps results in a broader range of possible price movements, making the model more flexible and suitable for describing certain market phenomena.

However, the Merton model assumes constant parameters over time, which may not hold in real-world scenarios, while another significant limitation is the fact that even if the model captures jumps, the distributional assumption for jump sizes might not perfectly represent extreme events.

4.3 Heston Stochastic Volatility Model

Another fundamental option pricing model is the Heston stochastic volatility model, proposed by Steven Heston in 1993 to derive a closed-form solution for the price of a European call option on an asset with stochastic volatility [27]. This model is considered an extension of the Black – Scholes model that encompasses and treats it as a special case. Heston’s framework incorporates features such as a non-lognormal distribution of asset returns, the leverage effect, and a significant mean-reverting property of volatility while maintaining analytical tractability. The volatility surfaces generated by Heston's model exhibit a resemblance to empirical implied volatility surfaces from the Black – Scholes model. However, the complication arises from the risk-neutral valuation concept. It becomes challenging to construct a riskless portfolio when asserting that the asset's volatility undergoes stochastic variations, primarily due to the fact that volatility is not a tradable security [28].

Here, we outline Heston's stochastic volatility model and provide some details on computing option prices. The following notations are used:

- $S(t)$: equity spot price or financial index.
- $V(t)$: variance.

- C: European call option price.
- K: strike price.
- $W_{1,2}$: standard Brownian movements.
- r: interest rate.
- q: dividend yield.
- κ: mean reversion rate.
- θ: long-run variance.
- V_0: initial variance.
- σ: the volatility of variance.
- ρ: the correlation parameter.
- t_0: the current date.
- T: the maturity date.

The Heston stochastic volatility model is specified as follows:

$$\frac{dS(t)}{S(t)} = \mu dt + \sqrt{V(t)}dW_1$$

$$dV(t) = \kappa\big(\vartheta - V(t)\big)dt + \sigma\sqrt{V(t)}dW_2.$$

In order to incorporate the leverage effect, the Wiener stochastic processes W_1 and W_2 should exhibit correlation, expressed as $dW_1 \cdot dW_2 = \rho dt$. The stochastic model governing the variance is associated with the square-root process introduced by Feller (1951) and Cox, Ingersoll, and Ross (1985) [29, 30].In this square-root process, the variance is always positive, and if $2\kappa\theta > \sigma^2$, then it can only approach zero but never reach it. Furthermore, it is worth noting that the deterministic part of the above process is asymptotically stable when $\kappa > 0$. The equilibrium point is clearly $V_t = \theta$.

Utilizing Itô's lemma and employing standard arbitrage arguments, we reach the partial differential equation formulated by Garman:

$$\frac{\partial C}{\partial t}+\frac{S^2V}{2}\frac{\partial^2 C}{\partial S^2}+(r-q)S\frac{\partial C}{\partial S}-(r-q)C+[\kappa(\theta-V)-\lambda V]\frac{\partial C}{\partial V}+\frac{\sigma^2 V}{2}\frac{\partial^2 C}{\partial V^2}+\rho\sigma SV\frac{\partial^2 C}{\partial S\partial V}=0$$

where λ refers to the market price of volatility risk.

Heston constructs this solution to the partial differential equation not through direct methods but by employing the characteristic functions, seeking the solution in such a form that corresponds to the Black – Scholes model. Therefore, we have:

$$C(S_0,K,V_0,t,T)=SP_1-Ke^{-(r-q)(T-t)}P_2,$$

where P_1 represents the delta of the European call option, while P_2 denotes the conditional risk-neutral probability of the asset price exceeding K at maturity. Both probabilities, P_1 and P_2, also adhere to the partial differential equation mentioned above. Heston also employs the characteristic functions $\varphi_1(u)$ and $\varphi_2(u)$ to describe the risk-neutral probabilities associated with the asset price and the variance. Assuming that they are known, the terms P_1 and P_2 are determined through the inverse Fourier transformation as follows:

$$P_j=\frac{1}{2}+\frac{1}{\pi}\int_0^{\infty}Re\left[\frac{e^{-iu\ln K}\varphi_j(S_0,V_0,t,T,u)}{iu}\right]du,$$

where $j=1,2$.

Subsequently, Heston postulates that the characteristic functions φ_1 and φ_2 take on the following structure:

$$\varphi_j(S_0,V_0,\tau;\ \phi)=exp\{C_j(\tau;\ \phi)+D_j(\tau;\ \phi)V_0+i\phi S_0\},$$

where $\tau=T-t$.

Upon substituting φ_1 and φ_2 into the aforementioned Garman equation, we obtain the subsequent ordinary differential equations for the unknown functions $C_j(\tau;\phi)$ and $D_j(\tau;\phi)$, accordingly:

$$\frac{dC_j(\tau;\phi)}{d\tau}-\kappa\theta D_j(\tau;\phi)-(r-q)\phi i=0,$$

$$\frac{dD_j(\tau;\phi)}{d\tau} - \frac{\sigma^2 D_j^2(\tau;\phi)}{2} + (b_j - \rho\sigma\phi i)D_j(\tau;\phi) - u_j\phi i + \frac{\phi^2}{2} = 0$$

and in case of zero initial conditions, then we have:

$$C_j(0,\phi) = D_j(0,\phi) = 0.$$

Finally, the solution that arises from this system is the following:

$$C(\tau;\phi) = (r-q)\phi i\tau + \frac{\kappa\theta}{\sigma^2}\left\{(b_j - \rho\sigma\phi i + d)\tau - 2\ln\left[\frac{1-ge^{d\tau}}{1-g}\right]\right\}$$

and

$$D(\tau;\phi) = \frac{b_j - \rho\sigma\phi i + d}{\sigma^2}\left[\frac{1-e^{d\tau}}{1-ge^{d\tau}}\right]$$

for $C(\tau;\phi)$ and $D(\tau;\phi)$, accordingly.

To the aforementioned equations the following apply:

$$g = \frac{b_j - \rho\sigma\phi i + d}{b_j - \rho\sigma\phi i - d}$$

$$d = \sqrt{(\rho\sigma\phi i - b_j)^2 - \sigma^2(2u_j\phi i - \phi^2)}$$

$$u_1 = 0.5$$

$$u_2 = -0.5$$

$$\alpha = \kappa\theta$$

$$b_1 = \kappa + \lambda - \rho\sigma$$

$$b_2 = \kappa + \lambda.$$

To sum up, the Heston model allows for stochastic volatility, addressing one of the limitations of the Black – Scholes model, which assumes constant volatility. The correlation term ρ introduces the leverage effect, capturing the phenomenon where volatility tends to increase when asset prices decrease. While Heston model does not provide closed-form solutions for option prices, it allows for efficient numerical methods for pricing and risk management. Since it involves multiple parameters and stochastic processes, it is of course more complex and the calibration to market data can be very challenging.

4.4 Kou Double Exponential Jump-Diffusion Model

The Kou double exponential jump-diffusion model is a mathematical model similar to Merton's that incorporates a double exponential distribution for jump sizes, allowing for both upward and downward jumps. It was primarily introduced by Yi-Kang Kou (2002) while more recently, it was expanded by Ren-Raw Chen [31, 32]. Its use in finance aims to describe the dynamics of asset prices from the aspect of both stochastic volatility and jumps.

Therefore, the Kou model extends the geometric Brownian motion and the stochastic differential equation for the asset price $S(t)$ is given by:

$$dS(t) = rS(t)dt + \sqrt{V(t)}S(t)dW_1(t) + S(t)dJ(t),$$

where:

- r is the risk-free interest rate.
- $V(t)$ is the instantaneous variance, modelled as a Cox-Ingersoll-Ross (CIR)[1] process [33].
- $W_1(t)$ is a standard Brownian motion representing stochastic volatility.
- $J(t)$ is a jump process.

Now, being more specific about the variance $V(t)$ which follows a CIR process, the following equation applies:

[1] The Cox-Ingersoll-Ross (CIR) mathematical process was developed by John C. Cox, Jonathan E. Ingersoll and Stephen A. Ross (1985) as an offshoot of the interest rate model and is based on a stochastic differential equation. It describes the interest rate movements driven by a sole source of market risk.

$$dV(t) = \kappa\big(\theta - V(t)\big)dt + \sigma\sqrt{V(t)}dW_2(t)$$

where:

- κ is the mean reversion rate.
- θ is the long-term variance.
- σ is the volatility.
- $W_2(t)$ is another standard Brownian motion independent of $W_1(t)$.

Accordingly, the jump process $J(t)$ is modeled as a compound Poisson process with double-exponential jump sizes. The probability density function of the jump sizes is given by:

$$f_J(x) = p_1\lambda_1 e^{-\lambda_1|x|} + p_2\lambda_2 e^{-\lambda_2|x|}$$

where:

- p_1 and p_2 are probabilities of small and large jumps, respectively, and
- λ_1 and λ_2 are jump intensities.

In the field of finance, the standard decomposition of the above process in the context of Kou model, is expressed as follows:

$$L_t = \mu_t + \sigma W_t + \sum_{k=1}^{N_t} J_k\,,$$

where each J_k follows a double-exponential distribution with parameters p, λ_1, λ_2.

The characteristic function of L_1 is denoted then as:

$$\varphi_{L_1}(u) = exp\left[i\mu u - \frac{\sigma^2 u^2}{2} + \lambda\left(\frac{p\theta_1}{\theta_1 - iu} - \frac{(1-p)\theta_2}{\theta_2 + iu} - 1\right)\right],$$

and the associated Lévy triplet is represented by $\big(\mu, \sigma^2, \lambda \times f_J\big)$.

The probability density function of L_1 does not have a closed-form expression, but the first two moments are given by:

$$\mathrm{E}[L_1] = \mu + \frac{\lambda p}{\theta_1} - \frac{\lambda(1-p)}{\theta_2}$$

and

$$\mathrm{Var}[L_1] = \sigma^2 + \frac{\lambda p}{\theta_1^2} + \frac{\lambda(1-p)}{\theta_2^2}.$$

As we observe, the Kou double exponential jump-diffusion model is more complicated than basic diffusion models like Black – Scholes due to the introduction of jumps and stochastic volatility, and calibration to market data may require sophisticated numerical techniques. However, it is proven a valuable tool for capturing the complex dynamics of financial markets. Providing a more realistic representation of market behaviour, the Kou model is often used in the pricing of financial derivatives, especially options, where the impact of jumps on option prices is significant.

4.5 Generalized Hyperbolic Model

In this subsection, we provide a brief description of the generalized hyperbolic model, which was introduced by Eberlein and Keller in 1995 but eventually refined by Eberlein and Prause in 2002 [34, 35]. The hyperbolic distributions were pioneered by O. E. Barndor-Nielsen (1977) in relation to the *'sand project'* and is constituted by a five-parameter $(\alpha, \beta, \delta, \lambda, \mu)$ class of Lebesgue-continuous, infinitely divisible distributions GH, i.e. $X \sim GH(\alpha, \beta, \delta, \lambda, \mu)$ [36]. The Lebesgue density is given by $f_{GH(\alpha,\beta,\delta,\lambda,\mu)}$ where:

$$\begin{aligned} f_{GH(\alpha,\beta,\delta,\lambda,\mu)}&(x+\mu) \\ &= (2\pi)^{-1/2}\delta^{-1/2}\alpha^{-\lambda+1/2}(\alpha^2-\beta^2)^{\lambda/2}K_\lambda\left(\delta\sqrt{\alpha^2-\beta^2}\right)^{-1} \\ &\times \sqrt{1+\frac{x^2}{\delta^2}}^{\lambda-1/2} K_{\lambda-1/2}\left(\delta\alpha\sqrt{1+\frac{x^2}{\delta^2}}\right)\exp(\beta x) \\ &= \frac{e^{\beta x}}{\sqrt{2\pi}\alpha^{2\lambda-1}\delta^{2\lambda}} \times \frac{\left(\delta\sqrt{\alpha^2-\beta^2}\right)^\lambda}{K_\lambda\left(\delta\sqrt{\alpha^2-\beta^2}\right)} \\ &\times \left(\alpha\sqrt{\delta^2+x^2}\right)^{\lambda-1/2} K_{\lambda-1/2}\left(\alpha\sqrt{\delta^2+x^2}\right). \end{aligned}$$

The parameter domain is defined as follows:

- $\lambda \in \mathbb{R}$ affects the heaviness of the tails and allows the navigation through different subclasses [e.g., if $\lambda = 1$, we get the hyperbolic distribution whereas if $\lambda = -\frac{1}{2}$, we get the normal inverse Gaussian (NIG) distribution].
- $\alpha > 0$ determines the shape.
- $\beta \in (-\alpha, \alpha)$ determines the skewness.
- $\delta > 0$ represents a scaling parameter.
- $\mu \in \mathbb{R}$ determines the location.

The functions $K_\lambda, K_{\lambda-1/2}$ refer to the modified Bessel functions of the third kind with index λ and $\lambda - 1/2$, respectively [37].

The characteristic function of $GH(\alpha, \beta, \delta, \lambda, \mu)$ is the following:

$$X_{GH}(u) = e^{iu\mu} \frac{\left(\delta\sqrt{\alpha^2 - \beta^2}\right)^\lambda}{K_\lambda\left(\delta\sqrt{\alpha^2 - \beta^2}\right)} \times \frac{K_\lambda\left(\delta\sqrt{\alpha^2 - (\beta + iu)^2}\right)}{\left(\delta\sqrt{\alpha^2 - (\beta + iu)^2}\right)^\lambda}.$$

This is a real-analytic one and can be expanded to a holomorphic function on the strip:

$$S := \{\mathcal{Z}: -\alpha < \beta - \mathrm{Im}(\mathcal{Z}) < \alpha\}.$$

The structure of the characteristic function expression remains unchanged during the extension, as all involved functions are constrained by analytic extensions. Consequently, computing the extended characteristic function at a point in S involves substituting $\mathcal{Z}$ instead of u in the aforementioned expression. More specifically, this yields the moment-generating function $u \longmapsto X_{(\alpha,\beta,\delta,\lambda,\mu)}(-iu)$. Then, taking derivatives at $u = 0$, the first and second algebraic moments of a random variable $X \sim GH(\alpha, \beta, \delta, \lambda, \mu)$ are as follows:

$$E[X] = \mu + \frac{\beta\delta^2}{\zeta} \frac{K_{\lambda+1}(\zeta)}{K_\lambda(\zeta)}$$

and

$$Var[X] = \frac{\delta^2}{\zeta} \frac{K_{\lambda+1}(\zeta)}{K_\lambda(\zeta)} + \frac{\beta^2\delta^4}{\zeta^2} \left(\frac{K_{\lambda+2}(\zeta)}{K_\lambda(\zeta)} - \frac{K_{\lambda+1}^2(\zeta)}{K_\lambda^2(\zeta)} \right),$$

where the new parameter $\zeta := \delta\sqrt{\alpha^2 - \beta^2}$ has been used.

The canonical decomposition of a Lévy process governed by a generalized hyperbolic distribution (i.e., $X \sim GH$) is:

$$X = t\mathrm{E}[X] + \int_0^t \int_{\mathbb{R}} x(\mu^L - v^{GH})(\mathrm{d}s, \mathrm{d}x)$$

and the Lévy triplet is $(\mathrm{E}[X], 0, v^{GH})$. To this end, the Lévy measure of the GH distribution takes the following form:

$$v^{GH}(dx) = \frac{e^{\beta x}}{|x|}\left(\int_0^\infty \frac{exp\left(-\sqrt{2y+\alpha^2}|x|\right)}{\pi^2 y\left(J_{|\lambda|}^2\left(\delta\sqrt{2y}\right) + Y_{|\lambda|}^2\left(\delta\sqrt{2y}\right)\right)} dy + \lambda e^{-\alpha|x|} 1_{\{\lambda \geq 0\}}\right),$$

where J_λ and Y_λ denote the Bessel functions of the first and second kind with index λ.

A more detailed analysis of this Lévy measure is provided in Raible (2000) [38]. In general, the described GH distribution encompasses various well-known distributions as special or limiting cases, including the normal, exponential, gamma, variance gamma, hyperbolic, and normal inverse Gaussian distributions [39].

Chapter 5: Conclusions

Since the publication of Black and Scholes' article on option pricing in 1973, there has been an intense interest of theoretical and empirical research on the topic. Over the past three decades, numerous pricing models have emerged as alternatives to the classical Black – Scholes approach. The Black – Scholes model, relying on lognormal stock diffusion with constant volatility, has faced growing criticism for its limitations.

A significant factor contributing to this criticism is the extraordinary deviations of stock index option prices from the benchmark Black – Scholes model since the market crash on October 19, 1987. In practice, to reconcile the Black – Scholes formula with quoted prices of European calls and puts, it is often necessary to use varying volatilities, known as implied volatilities, for different option strikes and maturities. This is in contrast to the Black – Scholes model, which assumed a constant volatility based on historical volatility of the underlying asset. The need for different implied volatilities implies a substantially negatively skewed distribution, indicating leptokurtic behaviour with a fat tail on the negative side. The observed pattern of implied volatilities across strikes is commonly referred to as a ‘volatility smile’ or ‘skew’. This term is used because the implied volatility of in-the-money call options tends to be significantly higher than that of out-of-the-money options. Typically, the skew's steepness diminishes with increasing option maturities. The presence of the skew is often attributed to the market participants' fear of significant downward market movements. The quest for new models capable of incorporating the volatility smile has become one of the most active areas of study in modern quantitative finance.

To price derivatives using the Black – Scholes model, two essential assumptions are considered. Firstly, returns are influenced by a single source of uncertainty, and secondly, asset prices adhere to continuous sample paths resembling a Brownian motion. With these two assumptions in place, a continuously rebalanced portfolio becomes instrumental in perfectly hedging an options position. Consequently, this process establishes a singular price for the option. Extensions of the Black – Scholes model aiming to capture the volatility smile phenomenon can be broadly categorized into two groups, based on each one of the two fundamental assumptions. When the assumption of a unique source of uncertainty is relaxed, the result is the stochastic volatility family of models. In these models, the volatility parameter follows a distinct diffusion process, as introduced by Heston model. On the other hand, relaxing the assumption of continuous sample paths leads to jump models. In jump models, stock prices follow an exponential Lévy process of jump-diffusion type, where the evolution involves a diffusion process interspersed with jumps at random intervals, or pure jumps type. These jump models attribute deviations from the Black – Scholes model to concerns about potential stock market crashes. They interpret crashes as evidence that jumps can indeed occur, contrary to the continuous diffusion assumption. Upon examining a plot of a stock index time-series, it becomes evident that prices do not adhere strictly to a diffusion process and do exhibit jumps.

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